N Step Binomial Model.

Let the set of times be $\mathbb{T} = \{0, 1, 2, \dots, N\}$.

Assume at each time set the security price may move up or down by fixed factors:

$$\mathbb{P}\left(S(t+1) = S(t)(1+m_u)\right) = p_u$$

$$\mathbb{P} \; S(t+1) = S(t)(1+m_d) = p_d \,.$$

Assume fixed interest rate $r > 0$ so that $m_2 < r < m_1$

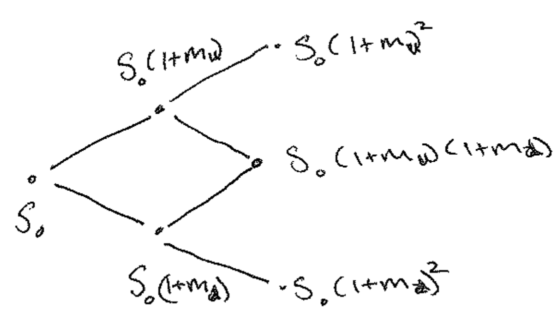
and there is a bond valued @ time t@:

$$A(t+1) = (1+r)\, A(t)$$

$$A(t) = (1+r)^t A(0) \,.$$

Illustration of stock price:

for $N=2$.



Sample space is sequences of u, d "paths"

$$\Omega = \{(\omega_1, \ldots, \omega_N) : \omega_i = u \text{ or } d \text{ for } i = 1, \ldots, N\}$$

for $\omega \in \Omega$

$$S^{\omega}(t) = S_0 (1 + m_{\omega_1}) \cdots (1 + m_{\omega_t})$$

At each time step $\mathcal{F}_t$ represents information filtered up to that time t.

$$\mathcal{F}_0 = \{\emptyset, \Omega\}; \quad \mathcal{F}_1 = \{\emptyset, A_u, A_d, \Omega\}$$

$$\mathcal{F}_2 = \{\emptyset, A_u, A_d, A_{ud} \cup A_{dd}, A_{du} \cup A_{uu}, A_{uu}, A_{uu}^c, A_{ud}, A_{ud}^c, A_{du}, A_{du}^c, A_{dd}, A_{dd}^c, \Omega\}$$

where

$$A_u = \{\omega \in \Omega : \omega_1 = u\}$$

$$A_{uu} = \{\omega \in \Omega : \omega_1 = u; \omega_2 = u\}$$

<u>etc</u>.

The probability measure obtains:

$$\mathbb{P}(A_u) = p_u \quad ; \quad \mathbb{P}(A_{uu}) = p_u^2 \quad \text{etc.}$$

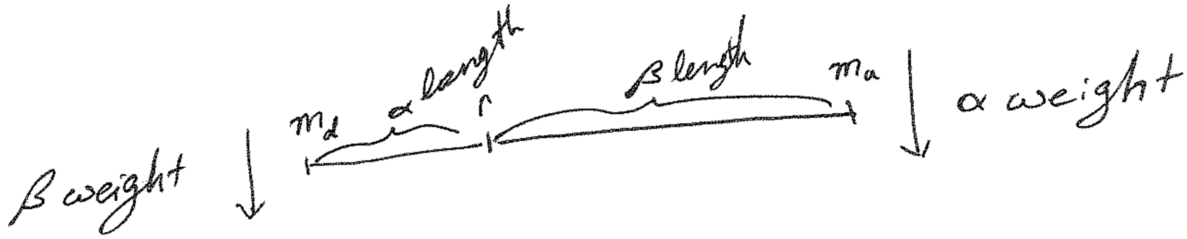
The Risk Neutral Measure.

This measure has the property that the weights of probability 'balance' the movement up or down of the stock price:

$$\tilde{p}_u = \frac{r - m_d}{m_u - m_d} \; ; \; \tilde{p}_d = \frac{m_u - r}{m_u - m_d}$$

Then $\mathbb{E}\, S(t+1) = (1+r)\, S(t)$

Anther way to view this, is that the probability weights balance the up/down movement.



$\beta \propto \tilde{p}_d$ $\qquad$ $\alpha \propto \tilde{p}_u$

. . . .

$\propto$ = "proportional to"

★ It bears repeating:

The validity of the valuation of the Option by the Risk Neutral Measure Is based on ~~the~~ the ability to create a <u>Replicating Portfolio</u> for the Binomial Model.

What is Replicating Portfolio in the N-step model? In this case the portfolio @ $t=n$ depends on outcome at time $t=n$:

Holdings @ $t=n$: $x \equiv$ security holding, $y \equiv$ bond holding.

$$x(\omega_1, \dots, \omega_k) = x(\underline{\omega}_k) = \frac{C^{\underline{\omega}_k, u} - C^{\underline{\omega}_k, d}}{S^{\underline{\omega}_k, u} - S^{\underline{\omega}_k d}}$$

$$y(\omega_1, \dots, \omega_k) = y(\underline{\omega}_k) = \frac{C^{\underline{\omega} d} S^{\underline{\omega} u} - C^{\underline{\omega} u} S^{\underline{\omega} d}}{A(k+1)\left(S^{\underline{\omega}_k u} - S^{\underline{\omega}_k d}\right)}$$

* holdings are <u>path dependent</u> if C is <u>path dependent</u>

PROPERTIES OF R.P.

Self Financing: Portfolio (x_k, y_k) is sufficient to purchase (x_{k+1}, y_{k+1}) ie at no time step is money added or removed:

$$V(k) = x_{k-1} S_k + y_{k-1} A_k = x_k S_k + y_k A_k .$$

Predictable: holdings @ $t=k$ only depend on info in $\mathcal{F}_k$
(Does not depend on future info)

Admissable: The value of the portfolio is never negative: $V(k) \geq 0$ for $k=0,\ldots,N$.

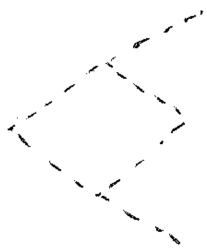
Given Option w/ payoff $G(N)$ @ expiry N,

~~What~~ what is the value of the option prior to payoff?

As discussed in the 1 step model:

$$G(0) = \frac{1}{1+r}\,\mathbb{E}(G(1)) = \frac{1}{1+r}\left\{G^u \tilde{p}_u + G^d \tilde{p}_d\right\}.$$

In the 2 step model:



Cover each step of the 2 step model —

$$G^u_{(1)} = \frac{1}{1+r}\,\mathbb{E}\left(G(2) \mid A_u\right) ; \quad G^d_{(1)} = \frac{1}{1+r}\,\mathbb{E}\left(G(2) \mid A_d\right)$$

$$G(0) = \frac{1}{1+r}\,\mathbb{E}\,G(1) = \frac{1}{1+r}\left\{G^u_{(1)}\tilde{p}_u + G^d_{(1)}\tilde{p}_d\right\}.$$

$$\therefore \quad G(0) = \frac{1}{1+r}\,\mathbb{E}\,G(1)$$

$$= \frac{1}{1+r}\left\{\tilde{p}_u\,\mathbb{E}\left(\tfrac{1}{1+r}G(2) \mid A_u\right) + \tilde{p}_d\,\mathbb{E}\left(\tfrac{1}{1+r}G(2) \mid A_d\right)\right\}$$

$$= \left(\frac{1}{1+r}\right)^2\left\{\tilde{p}_u^2\,G^{uu} + \tilde{p}_u\tilde{p}_d\left(G^{ud} + G^{du}\right) + \tilde{p}_d^2\,G^{dd}\right\}.$$

Eg European Call,

Consider Euro Call on security w/ $S(0) = 80$

return $m_u = .04$, $m_d = .01$ + bond rate $r = .02$

Call has expiry @ $T=2$ + strike price $X = 83$.

$$C(2) = (S(2) - X)^+ .$$

Value @ expiry:

$$C^{uu} = (80(1.04)^2 - 83)^+ = 3.5$$

$$C^{ud} = C^{du} = (80(1.04)(1.01) - 83)^+ = 1.03$$

$$C^{dd} = (80(1.01)^2 - 83)^+ = 0$$

Risk Neutral probability ~ $\tilde{p}_u = \dfrac{.02 - .01}{.04 - .01} = 1/3$; $\tilde{p}_d = \dfrac{.04 - .02}{.04 - .01} = 2/3$.

Value @ $t=0$

$$C(0) = \frac{1}{(1+r)^2} \left\{ \left(\tfrac{1}{3}\right)^2 3.5 + 2\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)(1.03) + \left(\tfrac{2}{3}\right)^2 0 \right\}$$

$$= .814.$$

EG Asian Call –

Consider an Asian option on the same stock over 2 time step w/ allows holder to purchase the security for $X = 83$ & sell it for the avg price.

$$C(2) = \left(\frac{S(1)+S(2)}{2} - X\right)^+$$

Value @ expiry:

$$C^{uu} = \left(\frac{80(1.04)^2 + 80(1.04)}{2} - 83\right)^+ = 1.86$$

$$C^{ud} = \left(\frac{80(1.04)(1.01) + 80(1.04)}{2} - 83\right)^+ = .62$$

$$C^{du} = \left(\frac{80(1.01)(1.04) + 80(1.01)}{2} - 83\right)^+ = 0$$

$$C^{dd} = \left(\frac{80(1.01)^2 + 80(1.01)}{2} - 83\right)^+ = 0$$

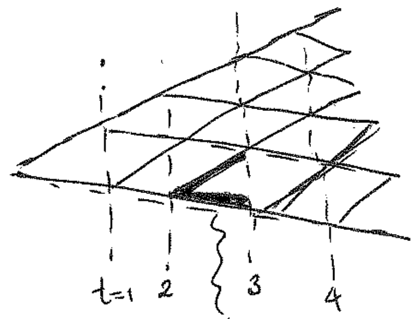
Value @ $t=0$:

$$C(0) = \frac{1}{1.02}\left\{\frac{1}{3^2}1.86 + \frac{1}{3}\frac{2}{3}(.62+0) + \left(\frac{2}{3}\right)^2 0\right\}.$$

$$= .34$$

Notice the Asian option has much lower variance & is less expensive.

General N step model... (7)



Each block is priced according to previously discussed rules...

$\therefore$ At time step $t=k$, first k steps are known.

let $\underline{\omega}_k = \omega_1, \ldots, \omega_k$ info known at $t=k$

In the language of σ-Alg -

$$A_{\underline{\omega}_k} = A_{\omega_1, \ldots, \omega_k} \in \mathcal{F}_k$$

$C^{\underline{\omega}_k} \equiv$ value of the option @ $t=k$ w/ outcome of 1st k steps.

what is the value?

$$C^{\underline{\omega}_k} = \frac{1}{1+r} E\left(C(k+1) \mid \underline{\omega}_k\right)$$

$$= \frac{1}{1+r}\left\{C^{\underline{\omega}_k, u}\, \tilde{p}_u + C^{\underline{\omega}_k, d}\, \tilde{p}_d\right\}$$

which we may iterate...

Thus for N steps ...

$$C_0 = \frac{1}{1+r}\{\tilde{p}_u C^u + \tilde{p}_d C^d\}.$$

$$= \frac{1}{1+r}\left\{\tilde{p}_u \frac{1}{1+r}\{\tilde{p}_u C^{uu} + \tilde{p}_d C^{ud}\} + \tilde{p}_d \frac{1}{1+r}\{\tilde{p}_u C^{du} + \tilde{p}_d C^{dd}\}\right\}$$

$\vdots$

$$C_0 = \frac{1}{(1+r)^N}\sum_{\omega\in\Omega} C^\omega \, \mathbb{P}(\omega).$$

Notice it is also true that ... for any $k<N$

$$C_0 = \frac{1}{(1+r)^k}\sum_{\underline{\omega}_k} C^{\underline{\omega}_k}\, \mathbb{P}(A_{\underline{\omega}_k})$$

the sum is over $\underline{\omega}_k \in \Omega_k = \{\omega_1, \dots, \omega_k : \omega_i = u \text{ or } d\}.$

<u>And</u> the value @ $t=k$ is

$$C^{\underline{\omega}_k} = \frac{1}{(1+r)^{N-k}} \sum_{\substack{\nu\in\Omega \\ :\nu_1,\dots,\nu_k = \underline{\omega}_k}} C^\nu \, \mathbb{P}(\nu).$$

These values hold in the

<u>path</u> <u>dependent</u> case (eg Asian Options)

We can simplify issues in the <u>path</u> independent case:

That is, let C be the value of an option so that C^{ω} only depends on total # of <u>up</u> & <u>down</u> moves... eg $N=3$ $C^{uud} = C^{udu}$

or, for any permutation σ:

$$C^{\omega_1, \ldots, \omega_N} = C^{\omega_{\sigma(1)}, \ldots, \omega_{\sigma(N)}}.$$

$$\{\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\} \text{ so that } \sigma(i) = \sigma(j) \Rightarrow i = j\}$$

The # of ω with k moves up & $N-k$ moves down is given by Binomial Coefficients $\binom{N}{k} = \frac{N!}{k!(N-k)!}$

(Path independent) (10)

Let $G(N;k) \equiv$ value of option @ $t=N$ w/ k steps <u>up</u>

then, from the formula for G_0 ...

$$G_0 = \frac{1}{(1+r)^N} \sum_{k=0}^{N} G(N;k)\, \mathbb{P}(\underbrace{u\cdots u}_{k}\underbrace{d\cdots d}_{N-k}) \binom{N}{k}$$

$$= \frac{1}{(1+r)^N} \sum_{k=0}^{N} \binom{N}{k} \tilde{p}_u^{\,k}\, \tilde{p}_d^{\,N-k}\, G(N;k)$$

A similar formula holds for avg over time $n<N$ values -

$G(n;l) \equiv$ value@ $t=n$ w/ l steps <u>up</u>

$$G_0 = \frac{1}{(1+r)^n} \sum_{\cancel{k}=0}^{n} G(n,l)\, \tilde{p}_u^{\,l}\, \tilde{p}_d^{\,n-l} \binom{n}{l}.$$

<u>And</u> value@ $t=n$:

$$G(n,l) = \frac{1}{(1+r)^{N-n}} \sum_{k=l}^{N-n+l} G(N,k)\, \tilde{p}_u^{\,k-l}\, \tilde{p}_d^{\,N-n-k} \binom{N-n}{k-l}$$

EUROPEAN CALL

IN PARTICULAR – The Value of the European Call may be written as . . .

$C_E(N) = C(N) = (S(N) - X)^+$ so it is path independent

$S(N;k) \equiv$ value of security @ $t=N$ if there are k steps up

$S(N;k) = S(0)\,(1+m_u)^k\,(1+m_d)^{N-k}$

$\therefore \quad C(N,k) = (S(N;k) - X)^+$

Let $k_0 \equiv$ min k so that $C(N,k) > 0$
$\equiv$ min k so that $S(N,k) > X$.

Then

$$(1+i)^N C_0 = \sum_{k=k_0}^{N} C(N;k)\,\tilde{p}_u^{\,k}\,\tilde{p}_d^{\,N-k}\binom{N}{k}$$

$$= \left\{\sum_{k=k_0}^{N} S(N;k)\,\tilde{p}_u^{\,k}\,\tilde{p}_d^{\,N-k}\binom{N}{k}\right\} - X\left\{\sum_{k=k_0}^{N}\tilde{p}_u^{\,k}\,\tilde{p}_d^{\,N-k}\binom{N}{k}\right\}$$

$$= \tilde{\mathbb{E}}\left[S(N)\,\chi_{\{S(N)>X\}}\right] - X\,\tilde{\mathbb{P}}\{S(N) > X\}.$$

European Call – RUBINSTEN FORMULA.

value of k_0,

$$S_0(1+m_u)^{k_0}(1+m_d)^{N-k_0} > X$$

$$\left(\frac{1+m_u}{1+m_d}\right)^{k_0} > \frac{X}{S_0(1+m_d)^N}$$

$$\therefore \quad k_0 = \left\lceil \frac{\log \frac{X}{S_0(1+m_d)^N}}{\log \frac{1+m_u}{1+m_d}} \right\rceil$$

re write the value of the Euro Call:

$$C_E(0) = \left(\frac{1}{1+r}\right)^N \left\{ \tilde{\mathbb{E}}\left[S(N)\, \chi_{\{S_N > X\}} \right] - X\, \tilde{\mathbb{P}}\{S_N > X\} \right\}$$

$$= \left(\frac{1}{1+r}\right)^N \sum_{k=k_0}^{N} S_0(1+m_u)^k(1+m_d)^{N-k}\, \tilde{p}_u^{\,k}\, \tilde{p}_d^{\,N-k} \binom{N}{k}$$

$$- \left(\frac{1}{1+r}\right)^N X \sum \tilde{p}_u^{\,k}\, \tilde{p}_d^{\,N-k} \binom{N}{k}$$

$$= S_0 \sum_{k=k_0}^{N} \tilde{q}_u^{\,k}\, \tilde{q}_d^{\,N-k} \binom{N}{k} - \frac{X}{(1+r)^N} \sum \tilde{p}_u^{\,k}\, \tilde{p}_d^{\,N-k} \binom{N}{k}$$

w/ $\tilde{q}_u = \tilde{p}_u \frac{1+m_u}{1+r}$; $\tilde{q}_d = \tilde{p}_d\left(\frac{1+m_d}{1+r}\right)$

$(\tilde{q}_u, \tilde{q}_d)$ is a probability measure.

EURO CALL RUBINSTEIN FORMULA:

Thus:

$$C_E(0) = S_0\, \mathbb{P}_{\tilde{q}}(S_N > X) - \frac{X}{(1+r)^N}\, \mathbb{P}_{\tilde{p}}(S_N > X)$$

or, if $K \equiv$ # of up steps

$$C_E(0) = S_0\, \mathbb{P}_{\tilde{q}}(K \geq k_0) - \frac{X}{(1+r)^N}\, \mathbb{P}_{\tilde{p}}(K \geq k_0)$$

$$= S_0\, \mathbb{P}_{\tilde{q}}\left(\frac{K - \tilde{q}_u N}{(N \tilde{q}_u \tilde{q}_d)^{1/2}} \geq \frac{k_0 - \tilde{q}_u N}{(N \tilde{q}_u \tilde{q}_d)^{1/2}}\right)$$

$$- \frac{X}{(1+r)^N}\, \mathbb{P}_{\tilde{p}}\left(\frac{K - \tilde{p}_u N}{(N \tilde{p}_u \tilde{p}_d)^{1/2}} \geq \frac{k_0 - \tilde{p}_u N}{(N \tilde{p}_u \tilde{p}_d)^{1/2}}\right)$$

$$\approx S_0\, \mathbb{P}\left(Z \geq \frac{k_0 - \tilde{q}_u N}{(N \tilde{q}_u \tilde{q}_d)^{1/2}}\right) - \frac{X}{(1+r)^N}\, \mathbb{P}\left(Z \geq \frac{k_0 - \tilde{p}_u N}{(N \tilde{p}_u \tilde{p}_d)^{1/2}}\right)$$

FOR $Z \sim N(0,1)$

a normal distribution w/ mean 0 + variance 1.

EG Consider an option on a stock w/ Expiry $T = 1$ year.

Divide 1 year into 100 segments

Compound Interest: @ $r = .05$; $\frac{r}{100} = .0005$

$S_0 = 100$; $m_u = .005$; $m_d = -.001$

What is the value of the Call w/ strike $X = 105$?

$$\tilde{p}_u = \frac{\frac{r}{100} + .001}{.005 + .001} = 1/4 \; ; \; \tilde{q}_u = \frac{1}{4}\left(\frac{1.005}{1.0005}\right) = .2511$$

$$k_0 = \left\lceil \frac{\log \frac{105}{100(.999)^{100}}}{\log \frac{1005}{999}} \right\rceil = \left\lceil \frac{\log 1.16}{\log 1.006} \right\rceil = \lceil 24.78 \rceil$$

$$C_E(0) = S_0\, \mathbb{P}\left(Z \geq \frac{-.11}{4.336}\right) - \frac{X}{(1.0005)^{100}}\, \mathbb{P}(Z \geq 0)$$

$$= 100\, \mathbb{P}(Z \geq -.02537) - (95.12)\, \mathbb{P}(Z \geq 0)$$

$$= (100)(.5101) - (95.12)(.5)$$

$$= 3.448$$