Risk Free Contracts

There are several contracts which pay fixed amounts at given times. For the sake of pricing these options, assume the interest rate r is fixed and the payments are risk free.

* BOND - Pays a fixed Face Value F at maturity time T.

* COUPON BOND - A standard Bond ~ paying a Face Value F at maturity T ~ which also yeilds a sequence of payments (Coupons) at regular intervals.
  ie payments of value C at times $\tau, 2\tau, \ldots, n\tau$.
  and payment of value F at time T.

*Annuity - a sequence of payments, of fixed value C, ~~ma~~ delivered at regular intervals, (usually once per year) over a fixed total payments.

Ie. payments of value C at times $\tau, 2\tau, \ldots, n\tau$.

*Perpetuity - a sequence of payments of fixed value C at regular intervals over an infinite number of intervals.

Ie payments of value C at times $\tau, 2\tau, \ldots, n\tau, \ldots$

Thus the value of a perpetuity is the value of an Annuity by taking $n \to \infty$.

Calculate value of Annuity.

Recall $\beta_\tau \equiv$ value today of \$1 at time τ.

$\therefore$ for risk free contract paying C at times $\{\tau, 2\tau, \ldots, n\tau\}$.

$$V(0) = \beta_\tau C + \beta_{\tau 2} C + \cdots + \beta_{\tau n} C$$

$$= C\left(\beta_\tau + \beta_{2\tau} + \cdots + \beta_{n\tau}\right).$$

Let us assume $\tau \equiv 1$ year, with yearly interest $\underline{r}$ we have

$$\beta_\tau = \frac{1}{1+r}\ ; \quad \beta_{k\tau} = \frac{1}{(1+r)^k}$$

$$\cancel{\beta_{k\tau} = \frac{1}{(1+r)^k}}$$

$\therefore$

$$V = C\left\{\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^n}\right\} = C\,\frac{1}{1+r}\left\{1 + \frac{1}{1+r} + \cdots + \frac{1}{(1+r)^{n-1}}\right\}$$

$$= C\,\frac{1}{1+r}\left\{\frac{1 - \frac{1}{(1+r)^n}}{1 - \frac{1}{1+r}}\right\} = C\;\frac{1 - \frac{1}{(1+r)^n}}{r}$$

PRESENT VALUE FACTOR FOR ANNUITY:

$$PA(r,n) = \frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right)$$

PERPETUITY Calculate value for contract paying C at times $\{\tau, 2\tau, \dots, n\tau, \dots\} = \tau\mathbb{N}$.

$$V(0) = C\beta_\tau + C\beta_{2\tau} + \dots + C\beta_{n\tau} + \dots$$
$$= C\left(\beta_\tau + \beta_{2\tau} + \dots + \beta_{n\tau} + \dots\right)$$

Again, assume $\tau = 1$ year

$$\therefore \quad V = C\left\{\frac{1}{1+r} + \frac{1}{(1+r)^2} + \dots + \frac{1}{(1+r)^n} + \dots\right\}$$
$$= C \lim_{n\to\infty} PA(r,n) = C \lim_{n\to\infty}\left\{\frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right)\right\}$$
$$= C/r$$

Notice, the value of an Annuity is equal to the value of Perpetuity less a perpetuity w/ first payment @ time <u>n+1</u> (ie 1 year after n years)

$$V_{Annuity} = V_{Perpetuity} - \frac{1}{(1+r)^n} V_{Perpetuity}$$
$$= C\frac{1}{r} - \frac{1}{(1+r)^n} C\frac{1}{r} = C\frac{1}{r}\left(1 - \frac{1}{(1+r)^n}\right) \quad \checkmark.$$

Given Annuity paying \$100 per year for 5 years

find value of annuity for $r = .01, .02$.

(1) $r = .01$

$$V(0) = 100 \frac{1}{.01}\left(1 - \left(\frac{1}{1.01}\right)^5\right) = (100)(.048534)$$

$$\approx 485$$

(2) $r = .02$

$$V(0) = 100 \frac{1}{.02}\left(1 - \left(\frac{1}{1.02}\right)^5\right) = (5000)(.0942692) \approx 471.$$

$\therefore$ Value is lower at higher interest.

Indeed for

$$V = C\frac{1}{1+r} + C\frac{1}{(1+r)^2} + \cdots + C\frac{1}{(1+r)^n}$$

Note that

$$\frac{d}{dr}\frac{1}{(1+r)^k} = -k\frac{1}{(1+r)^{k+1}} < 0 \qquad \therefore \; \cancel{\text{Value is}}$$

$\therefore$ Value of future payment decreases with increasing value of interest.

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* Notice this will have the effect of attracting more purchasers for bonds.

Example:

How much can you borrow if interest rate $r = .05$

and you can afford to pay \$1,000 per year

for the next 10 years?

~~[illegible]~~ This is simply the reverse of an Annuity, we are setting ourselves up as the lender in this model.

$\therefore$ Present value of Annuity is what we are able to borrow:

$$V_0 = (1000)(PA(r,n)) = (1{,}000)\left(\frac{1}{.05}\right)\left(1 - \left(\frac{1}{1.05}\right)^{10}\right)$$

$$\approx (20{,}000)(.386087) \approx 7721.73$$

~~If amount we are allowed to borrow L is less than V_0 : $L < V_0$ (on this payment schedule)~~
~~we could offer same deal to another party by borrowing & paying off w/ this Annuity schedule~~

If we are allowed to borrow $L > V_0$ on this payment schedule, I should borrow this pocket $L - V_0$ and purchase Annuity ~~at V_0. Then~~ With the same repayment schedule @ V_0.

Thus I have mad a risk free profit Since this is not allowed