Stochastic Integration Introduction

Abstract

Stochastic integration summary.

1 Integration by Taylor theorem and Reimann sums

Review Taylor's theorem in calculus (non-stochastic case). Let f be defined on $\mathbb{R} \times \mathbb{T}$ with time set $\mathbb{T} = [0, T]$. We create a discrete Riemann sum, for function $t \to x(t)$ for $t \in \mathbb{T}$ then

$$f(T, x(T)) = f(0, x(0)) + \sum_{k=1}^{N} f\left(t_k, x(t_k)\right) - f\left(t_{k-1}, x(t_{k-1})\right)$$

where we take (for example) $\Delta t = \frac{1}{N}$ and $t_k = k/N$. We also write $x_k = x(t_k)$. Let f have one derivative in space and one derivative in time, then

$$\begin{aligned} f\left(t_k, x(t_k)\right) - f\left(t_{k-1}, x(t_{k-1})\right) &= f_t\left(t_{k-1}, x_{k-1}\right)\left(t_k - t_{k-1}\right) + f_x\left(t_{k-1}, x_{k-1}\right)\left(x_k - x_{k-1}\right) \\ &\quad + o(t_k - t_{k-1}) + o(x_k - x_{k-1}) \end{aligned}$$

Approximation Here the notation o indicates a term vanishing with respect to the argument, for example $y_\epsilon = o(\epsilon)$ if say $|y_\epsilon| < C\epsilon^\alpha$ for some $\alpha > 1$ and $C < \infty$. Moreover, $x_k - x_{k-1} = x'_{k-1}(t_k - t_{k-1}) + o(t_k - t_{k-1})$. The important point is $\sum_k (t_k - t_{k-1})^\alpha \to 0$, as $N \to 0$.

From this we have that

$$\begin{aligned} f(T, x(T)) &= f(0, x(0)) + \sum_{k=1}^{NT} f_t\left(t_{k-1}, x_{k-1}\right)\left(t_k - t_{k-1}\right) + f_x\left(t_{k-1}, x_{k-1}\right)\left(x_k - x_{k-1}\right) \qquad (1.1) \\ &= f(0, x(0)) + \int_0^T \left[f_t\left(s, x(s)\right) + f_x\left(s, x(s)\right) x'(s)\right] \mathbf{d}s \end{aligned}$$

2 Stochastic integration by similar approach of Taylor's theorem and Riemann sums

We attempt to reproduce this calculation for functions $f(t, W_t)$ for Wiener process W_t.

In the deterministic case, fluctuations of $x(t)$ are proportional to x' in time Δt. However, we know that brownian motion W_t fluctuations are of order $\sqrt{\Delta t}$ in a time step Δt. Therefore, if we consider only one spatial derivative we come across the following issue

$$\begin{aligned} u(t + \Delta t, W_{t+\Delta t}) &= u(t, W_t) + \dot{u}(t, W_t)[\Delta t] + u'(t, W_t)[W_{t+\Delta t} - W_t] \\ &\quad + o(\Delta t) + o(W_{t+\Delta t} - W_t) \end{aligned}$$

the second term of little o remainders cannot be counted upon to sum up to something finite.

Ito's formula for functions of Brownian processes We therefore take 2 derivatives, and using Taylor's theorem we have

$$\begin{aligned} u(t+\Delta t, W_{t+\Delta t}) &= u(t, W_t) + \dot{u}(t, W_t)[\Delta t] + u'(t, W_t)[W_{t+\Delta t} - W_t] \\ &\quad + \tfrac{1}{2}u''(t, W_t)[W_{t+\Delta t} - W_t]^2 + o(\Delta t) + o([W_{t+\Delta t} - W_t]^2) \end{aligned} \tag{2.1}$$

putting these terms into a summation like (1.1) we have,

$$\begin{aligned} u(T, W_T) &= u(0, W_0) + \sum_{i=0}^{NT-1} \dot{u}(t_i, W_{t_i})[t_{i+1} - t_i] \\ &\quad + \sum_{i=0}^{NT-1} u'(t_i, W_{t_i})[W_{t_{i+1}} - W_{t_i}] + \sum_{i=0}^{NT-1} \tfrac{1}{2}u''(t_i, W_{t_0})[W_{t_{i+1}} - W_{t_i}]^2 \end{aligned} \tag{2.2}$$

we have dropped the little o terms for convenience. It is reasonable to expect that $[W_{t_{i+1}} - W_{t_i}]^2 \sim (t_{i+1} - t_i)$ as $t_{i+1} - t_i \to 0$ by the 1/2-space time scaling. This is in fact true, although it is not a trivial point to show, but we will assume it here for convenience. Therefore, we have Ito's formula,

$$u(T, W_T) = u(0, W_0) + \int_0^T \dot{u}(t, W_t)\mathbf{d}t + \tfrac{1}{2}\int_0^T u''(t, W_t)\mathbf{d}t + \int_0^T u'(t, W_t)\mathbf{d}W_t \tag{2.3}$$

this formula is equivalent to the differential form

$$\mathbf{d}u(t, W_t) = \left[\dot{u}(t, W_t) + \tfrac{1}{2}u''(t, W_t)\right]\mathbf{d}t + u'(t, W_t)\mathbf{d}W_t \tag{2.4}$$

Ito's formula for functions of Stochastic processes measurable with respect to Brownian motion Now that we understand the equation

$$\mathbf{d}Z_t = X_t\mathbf{d}t + Y_t\mathbf{d}W_t \tag{2.5}$$

we can reproduce (2.3), replacing $W_t \to Z_t$. To see how this plays out, return to (2.1)

$$\begin{aligned} u(t+\Delta t, Z_{t+\Delta t}) &= u(t, Z_t) + \dot{u}(t, Z_t)[\Delta t] + u'(t, Z_t)[Z_{t+\Delta t} - Z_t] \\ &\quad + \tfrac{1}{2}u''(t, Z_t)[Z_{t+\Delta t} - Z_t]^2 + o(\Delta t) + o([Z_{t+\Delta t} - Z_t]^2) \end{aligned} \tag{2.6}$$

the differences in (2.6) are a discrete version of (2.5) i.e.

$$[Z_{t+\Delta t} - Z_t] \sim X_t[\Delta t] + Y_t[W_{t+\Delta t} - W_t]$$

and

$$[Z_{t+\Delta t} - Z_t]^2 \sim X_t^2[\Delta t]^2 + 2X_tY_t[(\Delta t)(W_{t+\Delta t} - W_t)] + Y_t^2[W_{t+\Delta t} - W_t]^2$$

as before, $[W_{t+\Delta t} - W_t]^2 \sim \Delta t$ but the other terms vanish. Therefore, carrying out (2.2) we have the following version of Ito's formula,

$$u(T, Z_T) = u(0, Z_0) + \int_0^T \dot{u}(t, Z_t)\mathbf{d}t + \tfrac{1}{2}\int_0^T u''(t, Z_t)Y_t^2\mathbf{d}t + \int_0^T u'(t, Z_t)\mathbf{d}Z_t \tag{2.7}$$

which is equivalent to the differential form

$$\mathbf{d}u(t, Z_t) = \left[\dot{u}(t, Z_t) + \tfrac{1}{2}u''(t, Z_t)Y_t^2\right]\mathbf{d}t + u'(t, Z_t)\mathbf{d}Z_t \tag{2.8}$$

In both (2.7) and (2.8) the term $\mathbf{d}Z_t$ can be replaced by (2.5).

3 Product rule

To derive product rule, let's remind ourselves again about the product rule in the deterministic case.

$$\Delta[f(t)g(t)] = f(t+\Delta t)g(t+\Delta t) - f(t)g(t)$$

Adding zero to this equarion gives us,

$$\begin{aligned}
\Delta[f(t)g(t)] &= f(t+\Delta t)g(t+\Delta t) - f(t+\Delta t)g(t) + f(t+\Delta t)g(t) - f(t)g(t) \qquad (3.1)\\
&= [f(t+\Delta t)g(t+\Delta t) - f(t)g(t+\Delta t) - f(t+\Delta t)g(t) + f(t)g(t)] \\
&\quad + [f(t+\Delta t)g(t) - f(t)g(t)] + [f(t)g(t+\Delta) - f(t)g(t)] \\
&= [f(t+\Delta t) - f(t)] \cdot [g(t+\Delta t) - g(t)] \\
&\quad + [f(t+\Delta t) - f(t)]\, g(t) + [g(t+\Delta) - g(t)]\, f(t)
\end{aligned}$$

This obtains for small Δt,

$$\Delta[f(t)g(t)] = [f'(t)g'(t)](\Delta t)^2 + [f'(t)g(t) + f(t)g'(t)](\Delta t)$$

Thus the $[\Delta t]^2$ term vanishes in the ratio

$$[f(t)g(t)]' = \frac{\Delta[f(t)g(t)]}{[\Delta t]} = f'(t)g(t) + f(t)g'(t)$$

Of course this is easily extended to $f_1(t, x(t)) = f(t)$.

Now do the same for stochastic processes for $i = 1, 2$,

$$\mathbf{d}Z_t^{(i)} = X_t^{(i)}\mathbf{d}t + Y_t^{(i)}\mathbf{d}W_t$$

Reverting again to the discrete version,

$$\begin{aligned}
\Delta[Z_t^{(1)}Z_t^{(2)}] &= \left[Z_{t+\Delta t}^{(1)} - Z_t^{(1)}\right] \cdot \left[Z_{t+\Delta t}^{(2)} - Z_t^{(2)}\right] \qquad (3.2)\\
&\quad + \left[Z_{t+\Delta t}^{(1)} - Z_t^{(1)}\right] Z_t^{(2)} + \left[Z_{t+\Delta t}^{(2)} - Z_{t+\Delta t}^{(2)}\right] Z_t^{(1)}
\end{aligned}$$

Again we determine which are the higher order terms which vanish, the differences on the second line in (3.2) become our $\mathbf{d}$ terms and on the first line,

$$\begin{aligned}
\left[Z_{t+\Delta t}^{(1)} - Z_t^{(1)}\right] \cdot \left[Z_{t+\Delta t}^{(2)} - Z_t^{(2)}\right] &= X_t^{(1)}X_t^{(2)}[\Delta t]^2 + [X_t^{(1)}Y_t^{(2)} + Y_t^{(2)}X_t^{(1)}](\Delta t)[W_{t+\Delta t} - W_t] \\
&\quad + Y_t^{(1)}Y_t^{(2)}[W_{t+\Delta t} - W_t]^2
\end{aligned}$$

and the only term which survives is the final term. There fore the product rule is

$$\mathbf{d}[Z_t^{(1)}Z_t^{(2)}] = Z_t^{(1)}\mathbf{d}Z_t^{(2)} + Z_t^{(2)}\mathbf{d}Z_t^{(1)} + Y_t^{(1)}Y_t^{(2)}\mathbf{d}t$$

4 Examples

Let's find the derivative of some terms.

4.1 A stock price with log Brownian forcing

Consider $u(t,x) = u_0 e^{at+bx}$. The derivatives are

$$\dot{u} = au_0 e^{at+bx} = au,\ u' = bu_0 e^{at+bx} = bu \text{ and } u'' = b^2 u_0 e^{at+bx} = b^2 u.$$

Therefore, for $Z_t = u(t, W_t)$, and applying (2.4) on u we have,

$$\mathbf{d}Z_t = \mathbf{d}u(t, W_t) = (a + \tfrac{1}{2}b^2)u\mathbf{d}t + bu\mathbf{d}W_t = (a + \tfrac{1}{2}b^2)Z_t\mathbf{d}t + bZ_t\mathbf{d}W_t$$

Notice if we set $b = 0$ we have a deterministic process increasing exponentially at rate a. On the other hand setting $a = 0$ gives Brownian forcing proportional to the value of the process. This forms the stochastic model of a stock with forcing independent on disjoint intervals.

Thus, if we expect the value of the stock to drift at a rate of μ at a given time in proportion to its current value and fluctuate at intensity σ relative to its current value the stochastic differential equation of the stock is

$$\mathbf{d}S_t = \mu S_t\mathbf{d}t + \sigma S_t\mathbf{d}W_t$$

which has solution

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

This corresponds to the process we derived for scaling the N step binomial model

$$\log\frac{S_t}{S_0} = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t$$

Stock with continuous dividends Suppose the stock pays $\delta\frac{1}{N}S(t)$ on each time step. Then in the continuous model we have

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t$$

So that

$$S_t = S(0)e^{(r-\delta-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

5 Portfolios

At time t, holding is $x(t), y(t)$ portfolio has value,

$$V(t, S(t)) = x(t)S(t) + y(t)A(t)$$

For $S(t)$ value of stock and $A(t) = e^{rt}$ the value of the bond.

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad dA(t) = re^{rt}dt$$

European option/replicating portfolio Let V replicate a European option.

Recall in discrete version: $S^{\pm}(t) = S(t)(1 + r\frac{1}{N} \pm \sigma\frac{1}{\sqrt{N}})$

$$x(t) = \frac{V(t + 1/N, S^+(t)) - V(t + 1/N, S^-(t))}{S^+(t) - S^-(t)}$$

that is $x(t)$ is derivative of V with respect to S value,

$$x(t) = V'(t, S(t)).$$

On the other hand, in the discrete version,

$$y(t) = \frac{S^+(t)V(S^-(t)) - S^-(t)V(S^+(t))}{A(t+1/N)(S^+(t) - S^-(t))}$$

then 'adding zero'

$$y(t) = \frac{S^+(t)V(S^-(t)) - S^+(t)V(S^+(t)) + S^+(t)V(S^+(t)) - S^-(t)V(S^+(t))}{A(t+1/N)(S^+(t) - S^-(t))}$$

Thus, taking the limit we have

$$y(t) = \frac{z(t)}{A(t)} = \frac{V(t, S(t)) - S(t)V'(t, S(t))}{A(t)}$$

Stochastic formula From Ito formula,

$$dx(t) = dV'(t, S(t)) = [\dot{V}' + \frac{1}{2}\sigma^2 S^2(t)V''']dt + V''dS(t)$$

or

$$dx(t) = dV'(t, S(t)) = [\dot{V}' + rS(t)V'' + \frac{1}{2}\sigma^2 S^2(t)V''']dt + \sigma S(t)V''dW(t)$$

Differential Change in value, use Ito formula,

$$dV(t) = x(t)dS(t) + S(t)dx(t) + \sigma^2 S^2(t)V''dt + y(t)dA(t) + A(t)dy(t).$$

Assume self financing:

$$0 = S(t)dx(t) + \sigma^2 S^2(t)V''dt + A(t)dy(t)$$

Then

$$dV_t = x_t dS_t + y_t dA_t = r(x_t S_t + y_t A_t)dt + x_t \sigma S_t dW_t = rV_t dt + \sigma V_t' S_t dW_t$$

Thus

$$d(e^{-rt}V(t)) = (-re^{-rt}V(t) + re^{-rt}V(t))dt + \sigma e^{-rt}S(t)V'dW(t) = \sigma e^{-rt}S(t)V'dW(t)$$

It follows that $\widetilde{V}(t) = e^{-rt}V(t)$ is a martingale so that

$$\widetilde{V}(0) = \mathbb{E}(\widetilde{V}(t))$$

for any t.