Ito Calc & SDE ...

Initial example of stock price -

$$S_t = S_0 e^{(r - \frac{1}{2}\sigma^2) t + \sigma W_t}$$

where $W_t \equiv$ Brownian motion.

* MODEL for stock w/ ~~independent~~ forcing independent on disjoint intervals.

SDE:

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t .$$

Eg - suppose the stock pays $\frac{\delta}{m} S_t$

ie - a dividend @ a rate of its present value

m times per year for large m ~ say $\underline{m \geq 12}$

then the SDE is

$$dS_t = r S_t dt - \delta S_t dt + \sigma S_t dW_t$$

$$\cancel{S_t = S_0 e^{(r-\delta)t}} \quad S_t = S_0 e^{(r - \delta - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

Solving SDE –

First, recall solving ODEs – find $x(t)$ for the ode.

$$\frac{dx}{dt} = p(t)\, x(t) + q(t) \;;\; x(0) = x_0$$

Integrating factor $g(t) = e^{-\int_0^t p(s)\, ds}$

then find – (write $p_t = p(t)$ etc)

$$\frac{d}{dt}(xg) = \left(\frac{d}{dt}x\right) g + \left(\frac{d}{dt} g\right) x$$

$$= (p_t x_t + q_t)\, g_t + x_t g_t (-p_t) = q_t g_t$$

$\therefore$ Fundamental theorem of Calc $\Rightarrow$

$$x_t g_t = x_0 g_0 + \int_0^t g(s)\, q_s\, ds$$

$$= x_0 + \int_0^t q_s\, e^{-\int_0^s p(u)\, du}\, ds$$

$$\Rightarrow \quad x_t = x_0 e^{\int_0^t p_u\, du} + \int_0^t q_s\, e^{\int_s^t p_u\, du}\, ds$$

Now instead of deterministic purturbation
- let us add noisy pertorbation
- additively.

$$dx = p_t x_t \, dt + q_t \, dW_t .$$

Same integrating factor $g_t = e^{-\int_0^t p\,ds}$

$$d(x_t g_t) = (dx_t) g_t + x_t \, dg_t$$

$$= g_t (p_t x_t \, dt + q_t \, dW_t) + x_t g_t (-p_t) \, dt$$

$$= g_t q_t \, dW_t$$

$$\Leftrightarrow \quad X_t g_t = X_0 + \int_0^t q_s g_s \, dW_t$$

$$X_t = X_0 e^{\int_0^t p\,ds} + e^{\int_0^t p\,ds} \int_0^t q_s g_s \, dW_t$$

Note $\mathbb{E} \int_0^t dW_t = 0 \rightarrow \mathbb{E} \int_0^t q_s g_s \, dW_t = 0$

$\therefore \quad \mathbb{E} X_t = X_0 e^{\int_0^t p\,ds}$

Eg Ornstein Uhlenbeck

~Model for a noisy ~~[illegible]~~ Security having value μ ~ or time dependent variance

$$dX_t = \cancel{\rho}k(\theta - X_t)\,dt + \sigma\,dW_t$$

$k, \theta,$ ~~[illegible]~~, $\sigma > 0$

try integrating factor e^{kt}

$$d(X_t e^{kt}) = e^{kt}\,dX_t + k e^{kt} X_t\,dt$$

$$= (k\theta\,dt + \sigma\,dW_t)e^{kt}$$

$$\Rightarrow \cancel{X_t e^{\rho t} = X_0 + \int_0^t \rho\mu\,dt}$$

$$X_t e^{kt} = X_0 + k\mu \int_0^t e^{ks}\,ds + \sigma \int_0^t e^{ks}\,dW_s$$

$$X_t = X_0 e^{-kt} + \mu\{e^{kt} - 1\}e^{-kt} + \sigma \int_0^t e^{-k(t-s)}\,dW_s$$

$$= \underbrace{(X_0 - 1)e^{-kt}}_{\substack{\text{initial}\\ \text{position}\\ \text{decays}}} + \underbrace{\theta}_{\substack{\text{attraction}\\ \text{point}}} + \underbrace{\sigma \int_0^t e^{-k(t-s)}\,dW_s}_{\substack{\text{as } t \gg s\\ \text{the noise}\\ \text{at timesegments}\\ s \text{ ie } dW_s\\ \text{decays exponentially}}}$$

~problem for Finance = value X_t may be < 0 !

HESTON MODEL ~

MODEL for stochastic variance

$V_t \equiv$ variance @ time t

the stock price is

$$dS_t = r S_t\, dt + \sqrt{V_t}\, S_t\, dW_t^{(1)}$$

suppose V_t satisfies

$$dV_t = k(\theta - V_t)\, dt + \sigma\sqrt{V_t}\, dW_t^{(2)}$$

$W^{(1)}, W^{(2)}$ are correlated B.M.

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}\left(W_t^{(2)} \cdot W_t^{(1)}\right) = \rho \quad \equiv \text{correlation}$$

→ this is also written as

$$\mathbb{E}\left(dW^{(1)}\, dW^{(2)}\right) = \rho\, dt.$$

* Variance is always positive.

* $\rho \equiv$ correlation → shocks to volatility are correlated to shocks in stock prices
  → sharp increase in stock price corresponds to sharp increase in volatility.

* No closed form soln but we can write down soln in terms of P.D.E.