We have defined risk neutral measure so that
<u>all</u> securities in the market have the property:

discrete time:

$$S(t) = \tilde{E}\left(\frac{1}{1+r} S(t+1) \,\middle|\, \mathcal{F}_t\right)$$

continuous time:

$$S(\tau) = \tilde{E}\left(e^{-(t-\tau)r} S(t) \,\middle|\, \mathcal{F}_\tau\right)$$

note that $\tilde{E}\left(S(t+s)e^{-sr} - S(t) \,\middle|\, \mathcal{F}_t\right) = 0$

$\forall s > 0$, so we may write

$$\tilde{E}\left(d\tilde{S}_t\right) = 0$$

where $\tilde{S}_t = e^{-tr} S_t$.

$\therefore$ In the continuous case $\tilde{S}_t = e^{-tr} S_t$, discrete case $\tilde{S}_t = \frac{1}{(1+r)^t} S_t$

are respectively called <u>martingales</u>

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* Note S_t may be value of security or combination of securities.

How does it translate for stocks?

$$S_t = S_0 \, e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

$\therefore$

$$d\tilde{S}_t = \left(d(e^{-tr})\right) S_t + e^{-tr} dS_t$$

$$= \underline{-re^{-tr}S_t dt + e^{-tr} rS_t dt} + e^{-tr}\sigma S_t dW_t$$

$$= \sigma e^{-tr} S_t \, dW_t$$

$\therefore$ only change left over is from $dW_t \rightsquigarrow$ which has fluctuations adding up to 0

ie for any $f(t)$

$$\mathbb{E} \int_0^t f(s)\, dW_s = 0$$

$\therefore$

$$\mathbb{E}\, \tilde{S}_t = \mathbb{E} \int_0^t \sigma e^{-sr} S_s \, dW_s = 0$$

Suppose V_t is a portfolio of the Security + Bond...

$$V_t = X_t S_t + Y_t A_t$$

X_t, Y_t is determined by $\cancel{\{\mathcal{F}_s\}_{s<t}}$ $\mathcal{F}_t$.

* predictable processes.
* Self financing...

~~Let~~

$$dV_t = (dX_t)S_t + (dY_t)A_t + X_t\,dS_t + Y_t\,dA_t$$

$$= X_t\,dS_t + Y_t\,dA_t$$

ie

$$(dX_t)S_t + (dY_t)A_t = 0$$

ie dV_t = change in value due to change in stock + bond.

Let

$$M_\tau = \tilde{E}\left(V_T e^{-rT} \mid \mathcal{F}_\tau\right)$$

$$= e^{-r\tau} \tilde{E}\left(V_T e^{-r(T-\tau)} \mid \mathcal{F}_\tau\right)$$

$$= e^{-r\tau} V(\tau, T)$$

M_τ is a martingale $u < \tau$

$$\tilde{E}[M_\tau \mid \mathcal{F}_u] = \tilde{E}\left(\tilde{E}\left(V_T e^{-rT} \mid \mathcal{F}_\tau\right) \mid \mathcal{F}_u\right)$$

$$= \tilde{E}\left(V_T e^{-rT} \mid \mathcal{F}_u\right) = M_u$$

Let us assume the value of the portfolio

V_T depends only on S_T ... ie

portfolio V_T replicates a European option

w/ payoff $g(S_T) \rightsquigarrow V_T = g(S_T)$.

Then $\rightarrow$

$$M_\tau = e^{-r\tau} V(\tau, T, x)$$

find "d" of M_t

* Recall Ito's formula:

$u(t, Z_t)$ st $dZ_t = X_t dt + Y_t dW_t$

$$du_t = \{\dot{u}_t + u'_t X_t + \tfrac{1}{2} u''_t Y_t^2\} dt + u'_t Y_t dW_t$$

* And $dS_t = r S_t dt + \sigma S_t dW_t$

$\therefore$

$$dM_t = \{-rV + \dot{V} + V' r S_t + \tfrac{1}{2}\sigma^2 S_t^2 V''\} e^{-rt} dt$$

$$+ \; e^{-rt} \sigma S_t V' dW_t$$

+ deterministic part of M_t is zero...

$\therefore$ V satisfies

$$rV = \dot{V} + V' r x + \tfrac{1}{2}\sigma^2 x^2 V''$$

w/ bdry condition $V(t, 0) = g(0)$

$V(T, x) = g(x)$.

$V(t,x) \equiv$ value of option @ time t w/ value $g(S_T)$ @ time T. if $S_t = x$.

Black-Scholes equation ~

$\frac{\partial}{\partial t} V = \dot{V}$; $\frac{\partial}{\partial x} V \equiv V'$

$$rV = \dot{V} + V' r x + \frac{1}{2}\sigma^2 x^2 V''$$

w/ bdry condition $\begin{cases} V(t,0) = g(0) \\ V(T,x) = g(x) \end{cases}$

On the other hand ~

$$M_t = e^{-rt} \tilde{E}\left(g(S_T) e^{-r(T-t)} \middle| S_t = x\right) = f(t, W_t)$$

$$= e^{-rt} V(t,x)$$

Ito formula:

$$df = \left(f_t + \frac{1}{2} f_{xx}\right) dt + f_x \, dW_t .$$

$$\therefore f_t + \frac{1}{2} f_{xx} = 0$$

w/ $f(T,x) = e^{-rT} g(x)$

$f(t,0) = e^{-r(T-t)} g(0)$

Let $\phi(t,x) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$

Then
$$\left(\frac{\partial}{\partial t}\right)\phi = k\left(\frac{\partial}{\partial x}\right)^2 \phi$$

let us write
$$u(t,x) = \int_{\mathbb{R}} \phi(x-y, t)\, g(y)\, dy$$

Then u solves
$$\mathring{u} = k\, u'' \quad \text{on } (x,t) \in \mathbb{R} \times (0,\infty)$$

w/ bdry conditions $\left\{ u(x,0) = g(x) \right.$.

Let us discuss the replicating portfolio of the value of the portfolio V.

We will find the replicating portfolio by considering the discrete version & taking the limit.

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let $N \equiv$ # of steps per unit of time

Binomial model over TN steps...

$S(t) \equiv$ value of stock @ time t.

$$\left.\begin{aligned} S^+(t) &= S(t)\left(1 + m_u^N\right) \\ S^-(t) &= S(t)\left(1 + m_d^N\right) \end{aligned}\right\} \text{value of stock at time } (t + 1/N)$$

$x(t) \equiv$ holding of stock

$y(t)$ holding in bond

$$V_t = x_t S_t + y_t A_t .$$

from the discrete model we have

Stock holding:

$$x_t = \frac{V(t+1/N,\, S_t^+) - V(t+1/N,\, S^-(t))}{S_t^+ - S_t^-}$$

as $N \to \infty$ limit approaches

$$x_t \approx \frac{V(t, S_t^+) - V(t, S_t^-)}{S_t^+ - S_t^-} \xrightarrow[\lim_{N\to\infty}]{} V'(t, S_t)$$

$$\therefore \quad X_t = V'(t, S_t)$$

bond holding

$$y_t = \frac{S_t^+ V(S_t^-) - S_t^- V(S_t^+)}{A(t+1/N)\left(S_t^+ - S_t^-\right)}$$

$$= \frac{\{S_t^+ V(S_t^-) - S_t^+ V(S_t^+)\}}{A_{t+1/N}\{S_t^+ - S_t^-\}} + \frac{\{S_t^+ V(S_t^+) - S_t^- V(S_t^+)\}}{A_{t+1/N}\{S_t^+ - S_t^-\}}$$

$$\xrightarrow[\lim_{N\to\infty}]{} \frac{-S_t^+ V'(S_t)}{A_t} + \frac{V(S_t)}{A_t} \equiv Y_t .$$