European Option.

Let H be an option w/ payoff $g(S_T)$ @ time T.

$V(t,x) \equiv$ value of option @ time t if $S_t = x$ & if value at time $T \equiv$ expiry is $g(S_T)$

$V(t,x)$ solves Black Scholes Equation –

$$rV = \dot{V} + rxV' + \frac{1}{2}\sigma^2 x^2 V''$$

w/ bdry condition $\begin{cases} V(t,0) = g(0) \\ V(T,x) = g(x) \end{cases}$

Hedging:
$$V_t = (V')_t S_t + \left(\frac{V_t - S_t V'_t}{A_t} \right) A_t$$

$$dV_t = (V')_t \, dS_t + r(V_t - S_t V'_t)\,dt$$

$$= rV_t\,dt + \sigma S_t (V')_t \, dW_t.$$

Value of the European Call:

$$C_E(0) = \tilde{\mathbb{E}}\{e^{-rT}(S(T) - X)^+\}$$

$$= \tilde{\mathbb{E}}\left\{e^{-rT}\left(S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma W_T} - X\right)^+\right\}$$

$W_T \sim N(0,T)$ ie $W_T = \sqrt{T}\,W$ for $W \sim N(0,1)$.

$$\therefore C_E(0) = \int_{-\infty}^{\infty} e^{-rT}\left(S_0\, e^{(r-\frac{1}{2}\sigma^2)T + \sigma w\sqrt{T}} - X\right)^+ \frac{e^{-w^2/2}\,dw}{\sqrt{2\pi}}$$

find lower bound of integrand:

$$S_0\, e^{(r-\frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\,d} - X = 0$$

$$\hookrightarrow d = \frac{\ln\frac{S_0}{X} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$C_E(0) = \int_{-d}^{\infty} e^{-rT}\left(S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma W\sqrt{T}} - X\right) \frac{e^{-W^2/2}\,dW}{\sqrt{2\pi}}$$

$$= S_0 e^{-\frac{1}{2}\sigma^2 T}\int_{-d}^{\infty} e^{\sigma\sqrt{T}W - W^2/2}\frac{dW}{\sqrt{2\pi}}$$

$$- e^{-rT} X \int_{-d}^{\infty} e^{-W^2/2}\frac{dW}{\sqrt{2\pi}}$$

Note:
$$\sigma\sqrt{T}W - W^2/2 = \frac{-1}{2}(W - \sigma\sqrt{T})^2 + \frac{1}{2}\sigma^2 T$$

$$W = -d \Rightarrow W - \sigma\sqrt{T} = -(d + \sigma\sqrt{T}) = -\left(\frac{\ln\frac{S_0}{X} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$\tilde{d} = d + \sigma\sqrt{T}$$

$$C_E(0) = S_0\int_{-\tilde{d}}^{\infty} e^{-W^2/2}\frac{dW}{\sqrt{2\pi}} - e^{-rT}X\int_{-d}^{\infty} e^{-W^2/2}\frac{dW}{\sqrt{2\pi}}$$

$$= S_0\,\mathbb{P}(Z > -\tilde{d}) - e^{-rT}X\,\mathbb{P}(Z > -d)$$

where $Z = N(0,1)$.

$\star$ One can check that this solution solves Black Scholes PDE.

SIMILAR EQUATION for time $0 \le t < T$ . .

$$C_E(t) = S_t \,\mathbb{P}(Z > -\tilde{D}_t) - e^{-r(T-t)} X \,\mathbb{P}(Z > -D_t)$$

$$D_t = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

$$\tilde{D}_t = \frac{\ln\left(\frac{S_t}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}}$$

P-C parity:

$$C_E - P_E = S_t - X e^{-r(T-t)} \rightarrow P_E = C_E - S_t + X e^{-r(T-t)}$$

$\therefore$

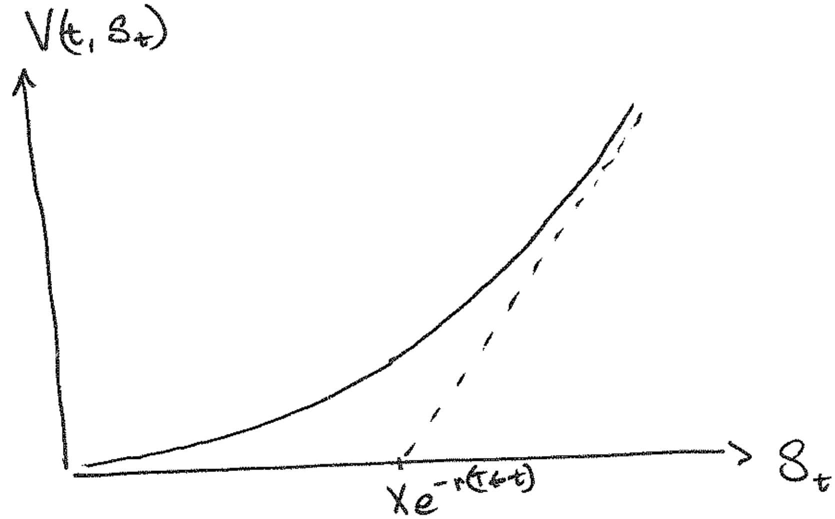
$$P_E(t) = -S_t \,\mathbb{P}(Z < -\tilde{D}_t) + e^{-r(T-t)} X \,\mathbb{P}(Z < -D_t)$$

GREEKS: $V_t = V(t, S_t) \equiv$ value of option @ time t.

Δ Delta

$$\Delta \equiv V' = \frac{\partial}{\partial S} V(t, S)$$

Recall: value of Call wrt S_t (fixed t)



as $S_t \gg Xe^{-r(T-t)} \rightsquigarrow V' \to 1$.

$S_t \ll Xe^{-r(T-t)} \rightsquigarrow V' \to 0$

$\therefore$ * when deep in money $S_t - Xe^{-r(T-t)} \gg 0$

Call option acts like stock

* when out of money

Call option acts like one own a nothing

~ {option value falls @ as $t \to T$ for constant S}

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Θ ~ Theta $\Theta \equiv \frac{\partial V(t,S)}{\partial t} = \dot{V}$

θ is sensitivity to ~~interest rate~~ time.

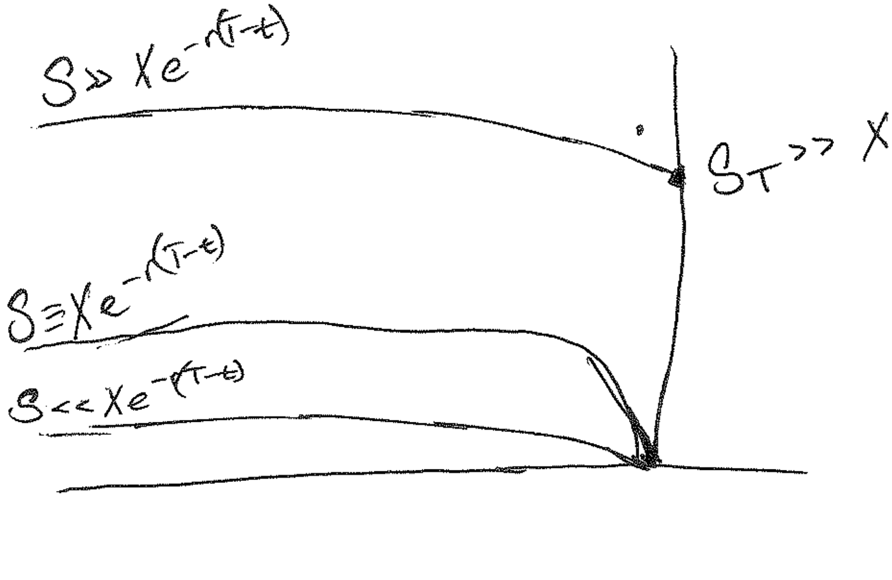
~~value always decreases w/ time.~~

For Euro option, $\Theta < 0$,
the value always decreases w/ time.

* For American Call (= Euro Call) theta is negative

* For American Put, Θ may be positive or negative.

The largest values of $|\Theta|$ are @ the money
& close to expiration.



Vega $\quad \nu = \dfrac{\partial V}{\partial \sigma}$

Sensitivity to volatility –

from formula we may observe that ν is high at large times and decreases w/ time.

Indeed large volatility may put you in the money if enough time remains but will not if time is limited

Eg $\quad S_0 = 100$