Pricing Perpetual American Put (I)
Martingales + Optimal Sampling.

Recall: Discounted stock value is a martingale (mg)

$$\tilde{S}_t = \frac{1}{(1+r)^t} S_t .$$

i.e. $\tilde{E}(\tilde{S}_{t+\tau} \mid \mathcal{F}_t) = \tilde{S}_t$ (continuous or discrete).

Thus all portfolios created by stocks + bonds are mg.

$$\tilde{V}_t = \frac{1}{(1+r)^t}\left(X_t S_t + Y_t A_t \right)$$

$$= X_t \tilde{S}_t + Y_t A_0$$

$\therefore \tilde{V}_t$ is a mg.

For any mg, with fixed time T:

Let M_t be a mg then for fixed time T:

$$E(M_T) = M_0 . \quad *$$

What if T is a random variable?

Does $(*)$ hold?

Example where (*) does not (always) hold (for random T)

Game: flip coins in sequence

$$\Omega = \{\omega : \omega_i = A \text{ or } B\} \; ; \; X_i(\omega) = \begin{cases} 1 & \text{if } \omega_i = A \\ -1 & \text{if } \omega_i = B \end{cases}.$$

$X_i \equiv$ outcome of betting on ith flip.

$Y_i \equiv$ # of bets purchased on ith flip.

Then the value @ Nth flip is:

$$V_N = Y_1 X_1 + \cdots + Y_N X_N$$

notice that

$$\mathbb{E}(V_{N+1} \mid \mathcal{F}_N) = Y_1 X_1 + \cdots + Y_N X_N + \underbrace{\mathbb{E}(Y_{N+1} X_{N+1})}$$

But $\mathbb{E}\big(Y_{N+1}\mathbb{E}(X_{N+1} \mid Y_{N+1})\big) = \mathbb{E}(Y_{N+1} \cdot 0) = 0$.

$\therefore \; \mathbb{E}(V_{N+1} \mid \mathcal{F}_N) = V_N$ ✓

V_N is a <u>mg</u>

If $Y_i \equiv 1$ ~~then~~ + T is a random variable independent of (X_i) it is not hard to show:

$$\mathbb{E}(V_T) = \sum_{i=1}^{T} \mathbb{E}(X_i) = 0.$$

$$\mathbb{E}(V_T) = \sum_{k=1}^{\infty} \mathbb{E}\left(V_k \, 1_{(T=k)}\right) = \sum_{k=1}^{\infty} P(T=k) \sum_{i=1}^{k} \mathbb{E}(X_i) = 0.$$

"Martingale Betting Strategy"

~~☆~~ ☆ These are pathological mg, we will need to ~~☆~~ remove these mg from consideration.

Strategy: 1^{st} flip $\rightarrow$ Buy 1 bet.

(i) If you win on bet $k \geq 1$ stop betting

(ii) If you lose on bet $k \geq 1$ buy twice as many bets on step $k+1$

$\therefore$ we stop betting on step τ where $\tau \equiv 1^{st}$ bet won.

At k flips, * if $k < \tau$: all bets $1, \ldots, k$ are lost: $X_1 = \cdots = X_k = -1$.

$$V_k = -Y_1 - Y_2 - \cdots - Y_k$$

But $Y_1 = 1$; $Y_{i+1} = 2Y_i \Rightarrow Y_{i+1} = 2^i$

$$V_k = -1 - 2 - \cdots - 2^{k-1} = -2^k + 1.$$

* if $k = \tau$: $X_1 = \cdots = X_{k-1} = -1$; $X_k = 1$.

$$V_\tau = V_k = -1 - 2 - \cdots - 2^{k-2} + 2^{k-1} = 1 - 2^{k-1} + 2^{k-1} = 1.$$

$\therefore$ at all outcomes we have

$$V_\tau = 1.$$

Although V is a mg $\mathbb{E}(V_{k+1} | \mathcal{F}_k) = V_k$.

we have

$$0 = V_0 \neq \mathbb{E}(V_\tau) = 1.$$

Def$^{\underline{n}}$ A random variable $\tau : \Omega \to \mathbb{T}$ is a <u>stopping time</u> if $\{\tau = t\} \in \mathcal{F}_t$.

That is, we only need to know information up to time t to determine ~~whether~~ whether to stop @ time t.
In other words we cannot "look into the future" to determine if we should stop.

<u>Eg</u> ~~Mg~~ For a coin flipping game

$$\tau = \min\{i : \omega_i = A\}$$

τ is a stopping time.

<u>Eg</u> Walk on a line.

for $Y_i \equiv 1$

~~$\tau = \min\{i : V_t\}$~~

$$\tau = \min\{i : V_i \in \{-N, M\}\}$$

that is stopping time is the first time V_t goes below $-N$ or above M

This is typically known as the Gambler's Ruin.

Optional Sampling Theorem.

Let $\mathbb{T}$ index the time values ie $\mathbb{T} = [0, \infty)$ or $\mathbb{T} = \{0, 1, 2, \dots\}$

If M_t is a Martingale and τ is a stopping time (wrt $\mathcal{F}_t$) so that $\mathbb{P}(\tau < \infty) = 1$.

and

(i) $\mathbb{E}(|M_\tau|) < \infty$

(ii) $\lim_{t \to \infty} \mathbb{E}\{|M_t| \mathbb{1}_{\{\tau > t\}}\} = 0$

then $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$

in particular, if M_0 is a constant:

$\mathbb{E}(M_\tau) = M_0$.

Eg Walk on line

let $0<S_0<N$ be initial position of a random walk.

$$S_k = S_0 + X_1 + \cdots + X_k \ ; \quad S_{k+1} = S_k + X_{k+1}.$$

Let $\tau = \min\{k : S_k = 0 \text{ or } N\}$

$\tau \equiv$ the first time S_t hits 0 or N.

Apply O.S.T.

* $\mathbb{P}(\tau > k) \to 0$ as $k \to \infty$.
* $\mathbb{E}(|S_\tau|) \le N \because S_\tau = 0$ or N
* $\mathbb{E}\{|S_t| \mathbb{1}_{\tau>t}\} \le N\,\mathbb{P}(\tau>k) \to 0$ as $k \to \infty$

Optional Sampling $\Rightarrow$

$$S_0 = \mathbb{E}(S_\tau) = 0 \cdot \mathbb{P}(S_\tau = 0) + N \cdot \mathbb{P}(S_\tau = N)$$

$$\hookrightarrow \mathbb{P}(S_\tau = N) = S_0/N$$

$$\mathbb{P}(S_\tau = 0) = 1 - \mathbb{P}(S_\tau = N) = \frac{N-S_0}{N}.$$

Eg Martingale Betting.

$$|V_k| = 2^{k}-1 \quad \text{if } \tau > k$$

$$+ \ \mathbb{P}(\tau > k) = (1/2)^k$$

$$\therefore \mathbb{E}\left(|V_k|\, \mathbb{1}_{\{\tau>k\}}\right) = \left(\tfrac{1}{2}\right)^k (2^k - 1) \sim 1$$

$\therefore$ Optimal stopping does not apply which is why we contradict the conclusion of O.S.T.