Pricing perpetual American Put. (II)

Consider American option w/ intrinsic value $H(S_t)$

ie intrinsic value only depens on value of security @ excerize.

Stock Value: $dS_\tau = rS_t dt + \sigma S_t dW_t$

Let the stopping time $\tau \equiv$ time we ~~excer~~ exercise option.

(Deciding on stopping time is deciding on strategy for exercizeng option.)

Value of option w/ intrinsic value H + stopping time τ:

$$C_t^{(\tau)} = \mathbb{E}\left(e^{-r(\tau-t)} H(S_\tau) \,/\, \mathcal{F}_t\right)$$

If option has expiry date T then we are constrained by $\tau \leq T$.

If option does not expire we may choose any ~~option~~ τ, $\therefore$ current time t is irrelevent to price

S_t has "Markov Property"

* W_t is time homogeneous
* $W_{t_i} - W_{s_i}$ are independent on non intersecting (s_i, t_i).

$\therefore$ shift time to zero

$$G^{(\tau)}_{S_t = x} = G^{(\tau)}_x = \mathbb{E}\left(e^{-r\tau} H(S_\tau) \mid S_0 = x\right).$$

Letting $U_t = e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}$ we have

$$G^{(\tau)}_x = \mathbb{E}\left(e^{-r\tau} H(x U_\tau)\right)$$

~~$\therefore$ decision to stop process does not depend on current time or position $\rightarrow$~~

~~There exists $A \subset [0,\infty)$ so that~~

~~τ~~

The value of the contract is given by the stopping time that performs the best ie.

$$V_x = \sup_\tau G^{(\tau)}_x .$$

or $\quad V(x,t) = \sup_\tau G^{\tau}_{x,t}$ if expiry time is finite.

In the No expiry case,

Optimal Stopping time does not depend on current value or time.

Thus, for τ satisfying:

$$V_x = G_x^{\tau} = \sup_{\hat{\tau}} G_x^{(\hat{\tau})}$$

The stopping time is of the form:

$$\tau = \min\{t : S_t \in A\} \text{ for some } A \subset [0,\infty)$$

Clearly, for the put, the stopping time is of the form,

$$\tau := \inf\{t : S_t \leq L\}$$

Thus we can set,

$$V(S_0) = f_L(S_0) = \mathbb{E}\{e^{-r\tau}(X - S_\tau)^+ \mid S_0\}$$

$$= (X - L)\,\mathbb{E}\{e^{-r\tau} \mid S_0\}$$

* It is true that L is independent of S_0 ... this follows from Harmonic theory.

Let $U_t = e^{(r-\frac{1}{2}\sigma^2)t + \sigma W_t}$

then $e^{-rt}U_t$ is a martingale.

Indeed, $d(e^{-rt}U_t) = (e^{-rt}U_t)\sigma\, dW_t$.

$\therefore\ \tilde{S}_t = e^{-rt}S_t$ is a mg.

Let us consider dialating σ by λ:

$$U_t^\lambda = e^{\lambda(r-\frac{1}{2}\sigma^2)t + \sigma\lambda W_t}$$

then

$$d(U_t^\lambda) = \left\{\lambda(r-\tfrac{1}{2}\sigma^2) + \frac{(\sigma\lambda)^2}{2}\right\}U_t^\lambda\, dt + \sigma\lambda U_t^\lambda\, dW_t.$$

Cancel drift to make this a mg.

$$\text{Let } Z_t = e^{-\left\{\lambda(r-\frac{1}{2}\sigma^2) + \frac{(\sigma\lambda)^2}{2}\right\}t}\, U_t^\lambda$$

$$= e^{-\frac{1}{2}\sigma^2\lambda^2 t + \sigma\lambda W_t}$$

then Z_t is a mg.

Observe, we can use OST to see that

$$Z_0 = \mathbb{E}(Z_\tau)$$

as Z_t is a mg.

But

$$\mathbb{E}(Z_\tau) = \mathbb{E}\left\{ e^{-\left\{ \lambda(r - \frac{1}{2}\sigma^2) + \frac{(\sigma\lambda)^2}{2} \right\}\tau} U_\tau^\lambda \right\}$$

and $$U_\tau^\lambda = \left(\frac{S_\tau}{S_0}\right)^\lambda = \left(\frac{L}{S_0}\right)^\lambda$$

so

$$1 = Z_0 = \left(\frac{L}{S_0}\right)^\lambda \mathbb{E}\left(e^{-\left\{ \lambda(r - \frac{1}{2}\sigma^2) + \frac{(\sigma\lambda)^2}{2} \right\}\tau} \right)$$

But we need $\mathbb{E}\, e^{-r\tau}$

$\therefore$ let

$$r = \left\{ \lambda(r - \tfrac{1}{2}\sigma^2) + \frac{(\sigma\lambda)^2}{2} \right\}$$

the solution is $\lambda_+ = 1$; $\lambda_- = -2r/\sigma^2$

$$\therefore \quad 1 = \left(\frac{L}{S_0}\right)^{-\frac{2r}{\sigma^2}} \mathbb{E}\left(e^{-r\tau}\right)$$

$$\mathbb{E}\left(e^{-r\tau}\right) = \left(\frac{L}{S_0}\right)^{\frac{2r}{\sigma^2}}$$

Finally, we have:

$$f_L(S_0) = (X-L)\left(\frac{L}{S_0}\right)^{\frac{2r}{\sigma^2}}$$

let us find the max over L: $f = \frac{(X-L)L^b}{S^b}$

$$\frac{\partial}{\partial L} f_L(S_0) = -\frac{L^b}{S^b} + \frac{(X-L)\,b\,L^{b-1}}{S^b} = 0$$

$$\therefore \quad L = (X-L)\,b$$

$$\Rightarrow \quad L = \frac{b}{b+1} X$$

thus stopping price is

$$L^* = \frac{\frac{2r}{\sigma^2}}{\frac{2r}{\sigma^2}+1} X = \frac{2r}{2r+\sigma^2} X$$

Thus the value of the perpetual american Put is:

$$V(S) = \begin{cases} (X-L^*)\left(\frac{L^*}{S_0}\right)^{\frac{2r}{\sigma^2}} = X\,\frac{\sigma^2}{2r+\sigma^2}\left(\frac{2rX}{(2r+\sigma^2)S_0}\right)^{\frac{2r}{\sigma^2}} & ; S_0 > L^* \\ X-S & ; S \le L^* . \end{cases}$$

Black Sholes "PDE" ← actually no time derivative.

$$rV = \dot{V} + \frac{1}{2}\sigma^2 x^2 V'' + r x V'$$

$$\dot{V} = 0$$

$\Rightarrow$

$$rV = \frac{1}{2}\sigma^2 x^2 V'' + r x V' \qquad (*)$$

Boundary condition:

$$V(t, L) = (X - L)$$

Thus $V(x) = V(t, x)$ is an ODE w/ eq $(*)$

+ boundary $V(L) = (X - L)$

$V(\infty) = 0$