Continuous Compound Interest.

Define Continuous Compound interest w/ rate r

as Compound interest with rate r

where frequency m goes to infinity

Recall Compound interest formula,

for $t = \frac{1}{m}, 2\frac{1}{m}, \ldots$

$$V_t = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$\left\{ \begin{array}{l} P \equiv \text{principle} \\ r \equiv \text{interest rate} \\ m \equiv \text{frequency of compounding.} \end{array} \right.$$

Recall from calculus that

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e \cong 2.71828\ldots$$

At continuous interest

$$\therefore \quad V_t = \lim_{m \to \infty} P\left(1 + \frac{r}{m}\right)^{mt} \underset{x = m/r}{=} \lim_{x \to \infty} P\left(1 + \frac{1}{x}\right)^{xrt}$$

$$= P\left\{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x\right\}^{rt} = Pe^{rt}.$$

Last lecture, we saw comparison of

Compounding interest at different freq:

<u>$k < m$</u>

$$V_k(t) = P\left(1+\frac{r}{k}\right)^{kt} < P\left(1+\frac{r}{m}\right)^{mt}$$

equivalently
(limit $\lim_{x\to\infty}$ $\left(1+\frac{1}{x}\right)^x$ is <u>increasing</u> in x)

Thus continuous compound interest accrues return faster than compound interest for fixed interest rate:

ie fixed r:

$$V_m \doteq P\left(1+\frac{r}{m}\right)^{mt} < Pe^{rt}.$$

Suppose a loan charges 6% interest, continuously compounded.

At what time ~~t~~ does the value double?

$V_t = Pe^{rt}$ is the value of the contract with initial loan P and interest rate r,

$$\therefore \quad V_t = 2P \Rightarrow \quad 2 = e^{(.06)t}$$

$$\log 2 = (.06)t$$

$$t = \frac{\log 2}{.06}.$$

In general, the loan increases by a factor of M at time

$$t = \frac{\log M}{r}$$

It is usefull to consider the ~~ratio change of values~~ logrithmic ~~chan~~ return instead of the arithmetic return: $s<t$

$$k(s,t) = \log \frac{V(t)}{V(s)}$$

The logrithmic return is additive: for $s<u<t$,

$$k(s,t) = \log \frac{V(t)}{V(s)} = \log \frac{V(t)}{V(u)} + \log \frac{V(u)}{V(s)} = k(s,u) + k(u,t)$$

For contracts at continuous interest we have:

$$k(s,t) = \log \frac{Pe^{tr}}{Pe^{sr}} = \log e^{(t-s)r} = (t-s)r.$$

Interest of bonds accrue effectively an exponential rate,

Suppose ~~a~~ contract A_s pays \$120 at $t=1+s$ and can be purchased at $t=s$ for \$100.

Let us write this as $A_s(t) = \begin{cases} 100 & \text{for } s \le t < 1+s \\ 120 & \text{for } t \ge 1+s \end{cases}$

Let $V(t)$ be the "proper" value at time t

Consider 2 options for $V(t)$.

Option 1

* Simple interest

$$V_0^{(1)}(t) = 100(1 + t(.2))$$

or for a bond issued @ time s

$$V_s^{(1)}(t) = 100\left(1 + (t-s)(.2)\right)$$

Option 2

* Continuously compounded interest

$$V_0^{(2)}(t) = 100(1.2)^t$$

$$V_s^{(2)}(t) = 100(1.2)^{(t-s)}$$

Suppose you find someone who values contract @ $V^{(k)}$ ...

Let us assume Contract A can be purchased any time, ie it can be purchased at time 0 or $t = 1/2$.

Thus $V_0^{(1)}(0) = 100 = V_0^{(2)}(0)$

$V_0^{(1)}(1/2) = 110$ $\qquad$ $V_0^{(2)}(1/2) = 109.5$

$V_0^{(1)}(1) = 120$ $\qquad$ $V_0^{(2)}(1) = 120$

$V_{1/2}^{(1)}(1/2) = 100 = V_{1/2}^{(2)}(1/2)$

$V_{1/2}^{(1)}(1) = 110$ $\qquad$ $V_{1/2}^{(2)}(1) = 109.5.$

We will ask our counterparty to buy A(1/2) @ time $t = 1/2$ & time $t = 1$.

Let us compare $V^{(1)}_{(1/2)}$ ~~to~~ $V^{(2)}_{(1/2)}$

$$V^{(2)}_{(1/2)} = 100 \times 1.095 = \$109.5$$

ie $\quad V^{(2)}(1/2) < V^{(1)}(1/2)$.

Suppose you believe value is $V^{(2)}$ and someone else believes value is $V^{(1)}$, how can you use this to your advantage?

Ask the other person to buy these contracts @ 6 mo maturity for $110

Thus,
- Borrow \$1000 (10×100) in loan, promising to pay $10 \times 120 = 1200$ @ $t=1$
- Buy 10 contracts for \$1000 @ time 0, $t=0$

$t = 1/2 = 6$ mo,
- Sell 10 contracts for $10 \times \$110 = \$1100$
- Buy 11 contracts

$t = 1$ year,
- Sell 11 contracts for $\$11 \times 110 = \$1210$
- Pay off original loan @ \$1200.

We pocket \$10 !

Similarly if some one thinks the value is low, say \$109 can you make a similar arguement?

hint: Ask the other person to buy contracts at time $t=0$ & $t = 1/2$ which you will buy off them @ $t = 1/2$ & $t=1$ resp.

We define the effective rate to be

$$\left(1+\frac{r}{m}\right)^m = 1 + r_e$$

the rate @ frequence $m=1$.

Continuous freq:

$$e^r = 1 + r_e.$$

It is reasonable now to make ext.
$V(t) = (1+r_e)^t$

As demo'd in the above example given any loan w/ effective rate r_e we have

$$V(t) = P(1+r_e)^t$$

or if it has freq m: $V(t) = P\left(1+\frac{r}{m}\right)^{mt}$

—— continuous freq: $V(t) = Pe^{rt}$

If one loan has freq m_1 w/ rate r_1 and a second has freq m_2 w/ rate r_2

and $\bar{r}_1$ and $\bar{r}_2$ are the effective rates we say loan 1 is prefferable to loan 2 only if $\bar{r}_1 > \bar{r}_2$.

∴ loan w/ monthly compounding
w/ rate 5% what is effective rate?

$$V(1) = \left(1+\frac{.05}{12}\right)^{12} = 1.05116 = 1+r_e$$

$$r_e \cong .0512$$

Continuous loan w/ rate 6% what is effective rate?

$$e^{.06} = 1.06184 = 1+r_e.$$

$$r_e = .0618$$

Notice loans continuous @ rate r is pref to compound @ rate r