(Zero) Coupon Bond

Recall: Bonds pay risk free face value F at maturity date T. Some Bonds pay Coupons of value C at regular dates $\tau, 2\tau, \dots, n\tau$ until maturity $n\tau = T$.

Zero Coupon

Suppose a Coupon with face value F has maturity date T. The effective interest is given implicitly by:

$$\cancel{V(0) = (1+r_e)^{-T} F}$$

$$(1+r_e)^T V(0) = F. \quad *$$

equivalently:

$$r_e = \left(\frac{F}{V_0}\right)^{1/T} - 1.$$

The discount factor is given by

$$V(0) = (1+r_e)^{-T} F = \beta_T F$$

$$\therefore \beta_T = (1+r_e)^{-T}$$

The discount ~~val~~ factor from maturity time T to time $t>0$ (Assuming constant interest)

$$\beta_{T-t} = (1+r_e)^{-(T-t)}$$

In general, when interest is not constant,

$B(t,T) \equiv$ discount factor from maturity date T to time t.

Coupon Bonds

We find value of Coupon Bond by summing up discounted value of Coupons and discounted Face value - similar to calc value of Annuities:

Bond paying C @ T/n, $2\frac{T}{n}$, ..., $\frac{n}{n}T$ (let $\tau = T/n$) + F @ time T.

$$V(0) = C\beta_{\tau} + C\beta_{2\tau} + \dots + C\beta_{n\tau} + \beta_T F.$$

By assm. $\beta_{n\tau} = \beta_T$.

and $\beta_{\tau}^{n} = \beta_{n\tau}$.

$\therefore$

$$V(0) = C\beta_{\tau} \cancel{\beta} \left(\frac{1-\beta_{\tau}^{n}}{1-\beta_{\tau}} \right) + \beta_{\tau}^{n} F$$

$$= C \, \frac{1-(1+r_e)^{-n\tau}}{(1+r_e)^{\tau}-1} + (1+r_e)^{-n\tau} F.$$

Moreover $\tau > \underline{\underline{t > 0}}$

$$V(t) = \left(C \, \frac{1-(1+r_e)^{-n\tau}}{(1+r_e)^{\tau}-1} + (1+r_e)^{-n\tau} F \right) \frac{1}{\beta_t}$$

Eg. Suppose Coupon bond pays \$10 coupons each year for 5 years. Maturity is 5 years & it pays \$100. Effective interest is $r_e = .05$.

Find Value over time:

At time $t=0$

$$V(0) = 10 \frac{1}{.05}\left(1-(1.05)^{-5}\right) + (1.05)^{-5} 100 \cong 121.64.$$

At time $t=k$ ~ After k^{th} Coupon is paid:

$$V(k) = 10\left(\beta_\tau + \cdots + \beta_\tau^{5-k}\right) + 100\,\beta_\tau^{5-k}$$

$$\therefore \quad V(1) = 10\frac{1}{.05}\left(1-(1.05)^{-4}\right) + (1.05)^{-4}\,100 \cong 117.73$$

$V(2) = 113.61$

$V(3) = 109.29$

$V(4) = 104.76$

$V(5) = 0$.

For t so that $k < t < k+1$

$$\cancel{V(t) = 10\,(\beta_{t-k}} \quad V(t) = V(k)\frac{1}{\beta_{t-k}}$$

~~∴ Just before payments:~~

On the other hand

for $k < t < k+1 < 5$

$$V(t) = V(k+1)\,\beta_{(k+1)-t} + C\,\beta_{(k+1)-t}$$

for $4 < t < 5$

$$V(t) = (F + C)\,\beta_{5-t}$$

$\therefore$

$V(1-) = 127.73 \equiv$ Just befor a payment.

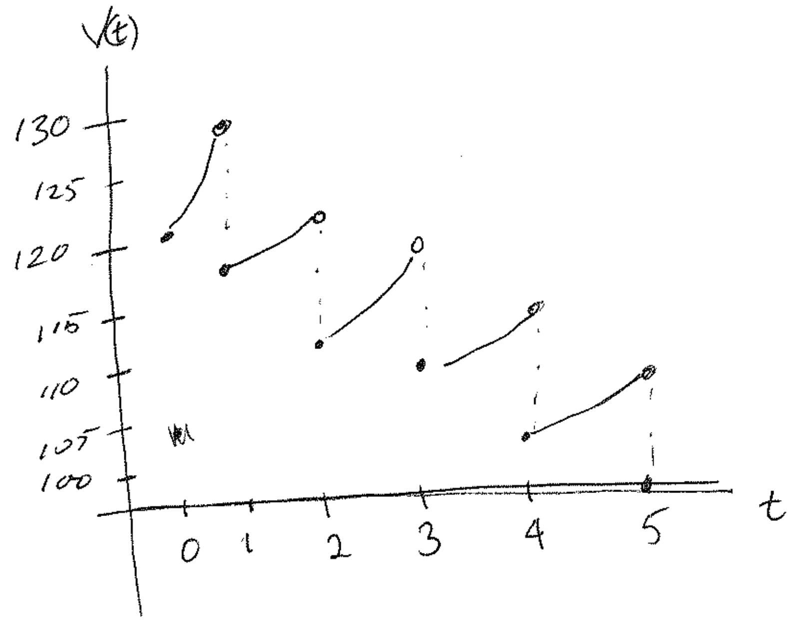
$V(2-) = 123.61$

$V(3-) = 119.29$

$V(4-) = 114.76$

$V(5-) = 110 \approx \underline{\underline{F + C}}$

GRAPH OF $V(t)$



Bonds at Par:

In a special case for Coupon Bonds, we have

$$V(0) = F.$$

here $F = ?$, $C = ?$

$\tau = 1$ year , $T = n$ years.

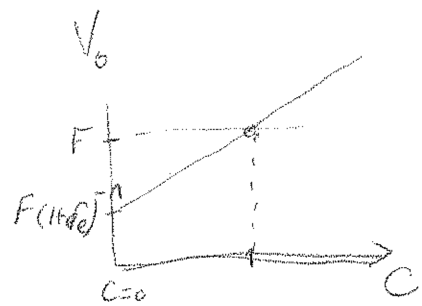
Notice the formula for the value is

$$V(0) = C \frac{1}{r_e}\left(1 - (1+r_e)^{-n}\right) + F(1+r_e)^{-n}$$

* the first term is increasing and second term is constant as C increases.

$$\therefore \quad V(0) = F(1+r_e)^{-n} < F \quad @ \ C = 0$$

$$\frac{d}{dC} V(0) = \frac{1}{r_e}\left(1 - (1+r_e)^{-n}\right) = PA(r_e, n)$$



Let us find C s.t. $V(0) = F$.

$$C \frac{1}{r_e} \left(1 - (1+r_e)^{-n} \right) + F (1+r_e)^{-n} = F$$

$$C \frac{1}{r_e} \left\{ 1 - (1+r_e)^{-n} \right\} = F \left\{ 1 - (1+r_e)^{-n} \right\}$$

$$\Leftrightarrow \quad C = r_e F \checkmark.$$

Notice the formula of C does not depend on n,

$$V(0) = (r_e F)\, PA(r_e, n) + F (1+r_e)^{-n}$$

for $n > m > 0$

$$V(m) = (r_e F)\, PA(r_e, n-m) + F (1+r_e)^{-(n-m)}$$

$$\therefore \ \forall \ i = 0, 1, \dots, n-1$$

$$V(i) = F. \ ; \ \lim_{t \nearrow i} V(t) = (1+r_e) F$$

$V(t)$