Several Securities, $n>2$.

In the 2 security case, we saw attainable set forms a hyperbola in the (σ, μ) plane.

For $n>2$ we see the problem reduces to 2 security market.

$$S_i(t) \equiv \text{value of } i^{th} \text{ security of time } t \quad \begin{cases} t = 0, 1 \\ i = 1, \dots, n \end{cases}$$

Let $x_i \equiv$ amt of i^{th} stock purchased @ time 0.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Value of portfolio with holdings x

$$V_x = x_1 S_1 + \dots + x_n S_n = x^T S$$

Weights

$$w_i = \frac{x_i S_i(0)}{V_x(0)} = \text{weight of } i^{th} \text{ security.}$$

Attainable portfolio,

$$W_n = \{ w \in \mathbb{R}^n : w^T \mathbb{1}_n = 1 \}$$

$$\text{for } \mathbb{1}_n = \left.\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right\} n$$

Return variables,

$$K_i = \frac{S_i(1) - S_i(0)}{S_i(0)} \quad ; \quad \mu_i = \mathbb{E}(K_i) \quad ; \quad \mu_K = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$$

let $\quad c_{ij} = \mathrm{Cov}(K_i \cdot K_j)$

$$\Sigma_K \equiv \text{covariance matrix.} \quad (\Sigma_K)_{ij} = c_{ij}$$

Assume $\det \Sigma \neq 0 \Rightarrow$ all eigenvalues are positive.

$$K_w = w^T K = w_1 K_1 + \cdots + w_n K_n .$$

Expected Return

$$\mu_w = w^T \mu_K$$

$(\text{Risk})^2$

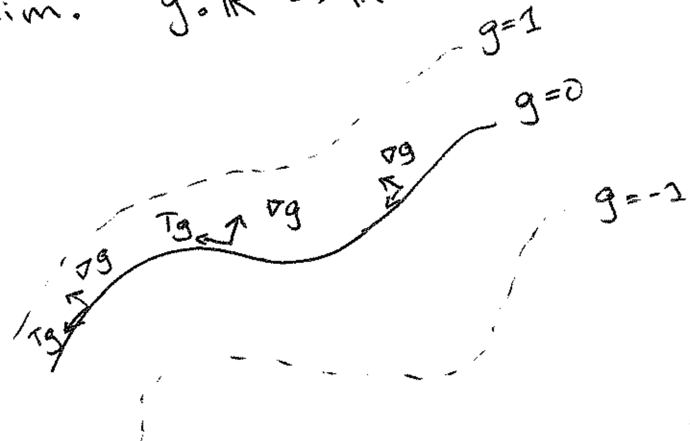
$$\sigma_w^2 = \mathrm{var}(K_w) = w^T \Sigma_K w$$

Find minimum of σ_w^2 with restriction $w^T \mathbb{1} = 1$.

Recall Lagrange Multiplier Method,

1 dim. $g: \mathbb{R}^2 \to \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}$, maximize f along $g=0$.



∇g is always $\perp$ to tangent of curve $g=0$, $Tg \equiv$ tangent ~~lines~~ vectors

if $\nabla f \nparallel \nabla g$ at some point $x \in \{x : g=0\}$

move alonge curve in direction Tg ($Tg \cdot \nabla f$).

this will increase f as you move along curve.

$\therefore$ any point on $g=0$ st f is maximized obeys $\underline{\underline{\nabla f = \lambda \nabla g}}$ for some $\lambda \in \mathbb{R}$.

The same extends to case $g: \mathbb{R}^n \to \mathbb{R}$.

~~Re~~

In the case of Multi dim constraint.

$g : \mathbb{R}^n \to \mathbb{R}^m$, ie $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, m$

and we wish to maximize $f : \mathbb{R}^n \to \mathbb{R}$

with constraint $g_i = 0 \quad \forall i = 1, \dots, m$.

The proof follows from ~~1 dim constraint~~, idea similar to 1 dim.

define $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$

Lagrange function,

$$L(x, \lambda) = f(x) - \lambda^T g$$

Maximal of L is at $\nabla L = 0$.

But $\nabla L = \begin{pmatrix} \nabla_x L \\ \nabla_\lambda L \end{pmatrix} = \begin{pmatrix} \nabla f - (\nabla g_1 \cdots, \nabla g_m) \lambda \\ g \end{pmatrix}$

~~But $g = 0$ along constraint~~ so.

~~we require~~

$\therefore$ max of L is at $g = 0$

and

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \dots + \lambda_m \nabla g_m$$

Lagrange multiplier to minimize σ_w^2

Lagrange function,

$$L(w,\lambda) = \sigma_w^2 - \lambda(w^T \mathbb{1} - 1) \quad ; \quad \sigma_w^2 = w^T \Sigma w$$

$$0 = \nabla L = \begin{pmatrix} \nabla_w L \\ \frac{\partial}{\partial \lambda} L \end{pmatrix}$$

$$0 = \nabla_w L = \nabla_w \left(w^T \Sigma w - \lambda(w^T \mathbb{1} - 1) \right)$$

$$(\dagger) \qquad = 2\Sigma w - \lambda \mathbb{1}$$

Solve for λ.

$$\lambda \Sigma^{-1} \mathbb{1} = 2w$$

multiply $\mathbb{1}^T$ on the left

$$\lambda \mathbb{1}^T \Sigma^{-1} \mathbb{1} = 2 \mathbb{1}^T w = 2$$

$$\lambda = \frac{2}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}$$

Replace λ ~~in~~to $(\dagger)$

$$0 = 2\Sigma w - \frac{2}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}} \mathbb{1}$$

min risk portfolio: $\quad w_m = \dfrac{\Sigma^{-1} \mathbb{1}}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}$

Thus the risk + return of minimal risk portfolio,

$$W_m = \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$$

Expected return,

$$K_m = W_m^T K$$

$$\mu_m = \mathbb{E}(K_m) = w_m^T \mu_K$$

$$= \frac{\mathbb{1}^T\Sigma^{-1}\mu_K}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$$

$(\text{Risk})^2$

$$\sigma_m^2 = w_m^T \Sigma_K w_m = \frac{\mathbb{1}^T\Sigma^{-1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}} \Sigma \frac{\Sigma^{-1}\mathbb{1}}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$$

$$\hookrightarrow \quad \sigma_m^2 = \frac{1}{\mathbb{1}^T\Sigma^{-1}\mathbb{1}}$$

Definition ~

Given portfolios $w^{(1)}$ + $w^{(2)} \in W$

We say $w^{(2)}$ is preferable to $w^{(1)}$

* If the expected return of $w^{(2)}$ is greater than the expected return of $w^{(1)}$ (w/ equal risk).

– or –

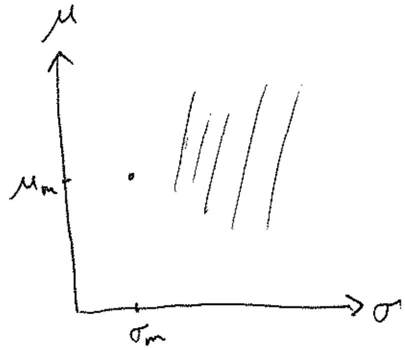
* If the risk of $w^{(2)}$ is less than the risk of $w^{(1)}$ (w/ equal respected return)

In other words,

$w^{(2)}$ is preferred to $w^{(1)}$ if

$$\begin{cases} \mu_{w^{(2)}} \geq \mu_{w^{(1)}} \\ \sigma_{w^{(2)}} \leq \sigma_{w^{(1)}} \end{cases}$$

We know that there is a unique minimal risk portfolio



Let us find the minimum risk for each given return μ.

Lagrange: minimize σ_w^2

w/ restriction $\begin{cases} \mu_w = \mu \\ w^T \mathbb{1} = 1. \end{cases}$

Lagrange Function,

$$L(w,\lambda) = \sigma_w^2 - \lambda_1 (w^T \mu_K - \mu) - \lambda_2 (w^T \mathbb{1} - 1).$$

$$0 = L(w,\lambda) = \begin{pmatrix} \nabla_w L \\ \nabla_\lambda L \end{pmatrix}$$

$$(+) \quad 0 = \nabla_w L = 2 \Sigma w - \lambda_1 \mu_K - \lambda_2 \mathbb{1}.$$

$$0 = \frac{\partial}{\partial \lambda_1} L = w^T \mu_K - \mu$$

$$0 = \frac{\partial}{\partial \lambda_2} L = w^T \mathbb{1} - 1.$$

Solve (†) for w:

$$(\ddagger) \quad w = \frac{\lambda_1}{2} \Sigma^{-1} \mu_K + \frac{\lambda_2}{2} \Sigma^{-1} \mathbb{1}.$$

multiply (‡) on left by μ_K^T,

$$\mu = \frac{\lambda_1}{2} \mu_K^T \Sigma^{-1} \mu_K + \frac{\lambda_2}{2} \mu_K^T \Sigma^{-1} \mathbb{1}.$$

multiply (‡) on left by $\mathbb{1}^T$.

$$1 = \frac{\lambda_1}{2} \mathbb{1}^T \Sigma^{-1} \mu_K + \frac{\lambda_2}{2} \mathbb{1}^T \Sigma^{-1} \mathbb{1}$$

$\therefore$

$$\begin{pmatrix} \mu \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

for $a = \frac{1}{2} \mu_K^T \Sigma^{-1} \mu_K$

$b = \frac{1}{2} \mu_K^T \Sigma^{-1} \mathbb{1}.$

$d = \frac{1}{2} \mathbb{1}^T \Sigma^{-1} \mathbb{1}.$

$$A = \begin{pmatrix} \frac{1}{2}\mu_K \Sigma^{-1} \mu_K & \frac{1}{2}\mu_K \Sigma^{-1} \mathbb{1} \\ \frac{1}{2}\mu_K \Sigma^{-1} \mathbb{1} & \frac{1}{2} \mathbb{1}^T \Sigma^{-1} \mathbb{1} \end{pmatrix}$$

$$\begin{pmatrix} \mu \\ 1 \end{pmatrix} = A \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = A^{-1} \begin{pmatrix} \mu \\ 1 \end{pmatrix}$$ <u>linear in μ</u>

min Risk given μ:

$$w_\mu = \frac{\lambda_1}{2} \Sigma^{-1} \mu_K + \frac{\lambda_2}{2} \Sigma^{-1} \mathbb{1}$$

min Risk w/ $\mu = 0$ $\Rightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} -b \\ a \end{pmatrix}$

$$w_0 = \frac{-b}{2(ad-bc)} \Sigma^{-1} \mu_K + \frac{a}{2(ad-bc)} \Sigma^{-1} \mathbb{1}$$

Min Risk for all μ:

$$w_m = \frac{\Sigma^{-1} \mathbb{1}}{\mathbb{1}^T \Sigma^{-1} \mathbb{1}}.$$

Go back to original model:

Return Portfolios are given by

$$K_w = w^T K \quad \text{for } w \in W = \{w \in \mathbb{R}^n : w^T \mathbb{1} = 1\}.$$

w_μ is linear in μ. ie ~~it~~ is a line

But

$$\bar{w}_s = (1-s) w_0 + s w_m \quad \text{is linear in } s$$

$\therefore$ line $\{\bar{w}_s : s \in \mathbb{R}\}$ is the same as line $\{w_\mu : \mu \in \mathbb{R}\}$.

this is called min variance line.

Consider market formed by

Return variables

$$K_{w_0} + K_{w_m}$$

This is the minimal risk market for each given μ.

$$K_s = s K_{w_m} + (1-s) K_{w_0}$$

Therefore, we have, the market formed by

K_{w_m}, K_{w_0} is a 2 security market w/ attainable set a hyperbola, all other portfolios lie to right of line.

Natural question, given μ can minimal risk be attained w/o short selling?

Consider example w/ 3 securities,

~~$W = \{w^T 1\!\!1$~~ $W = \{ w : w^T 1\!\!1 = 1 \}$ ← Porfolio set.

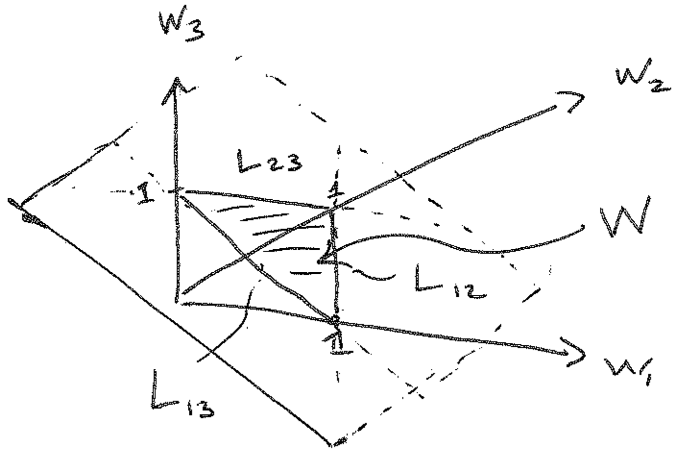
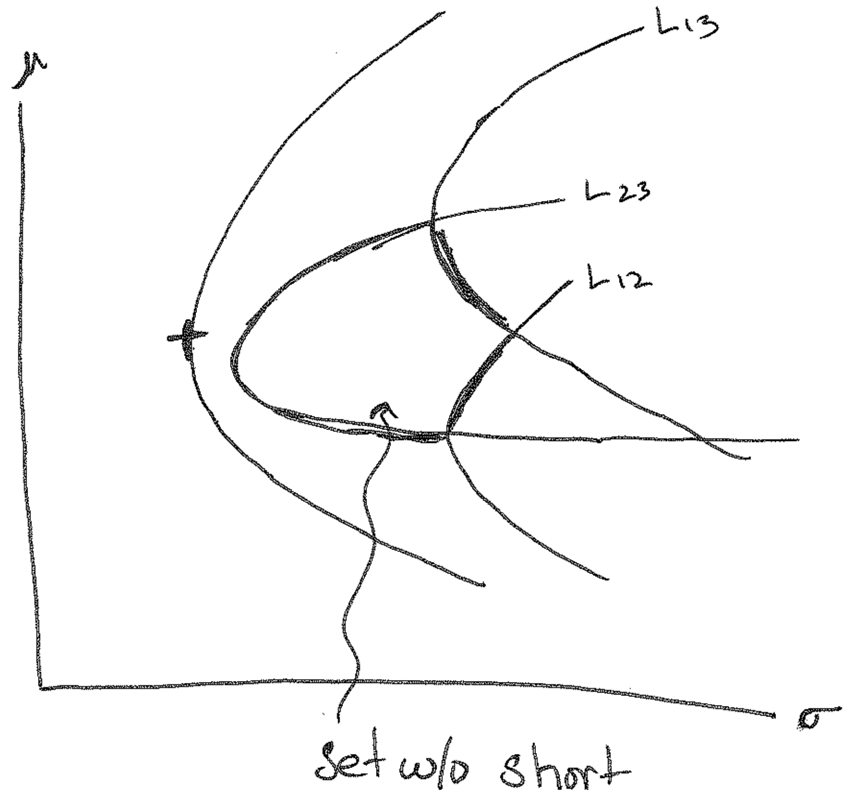


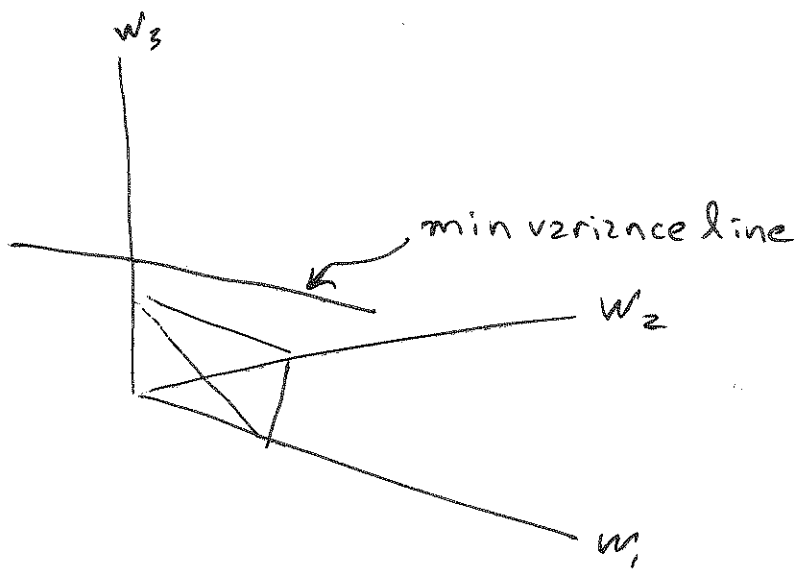
Image of L_{12}, L_{13}, L_{23} in (σ, μ) plane are hyperbola as they are 2 security markets.

EG MIN VARIANCE LINE DOES NOT

PASS THROUGH

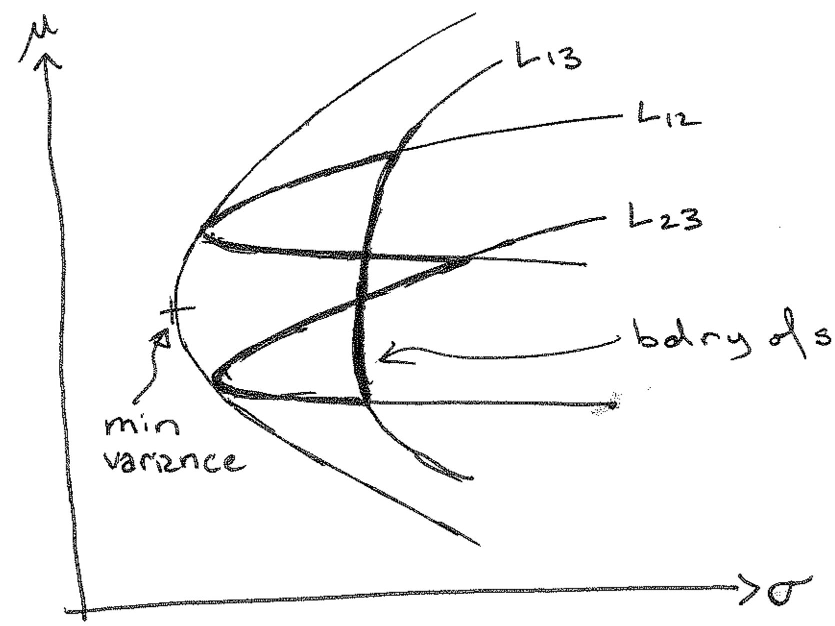
set w/o short.

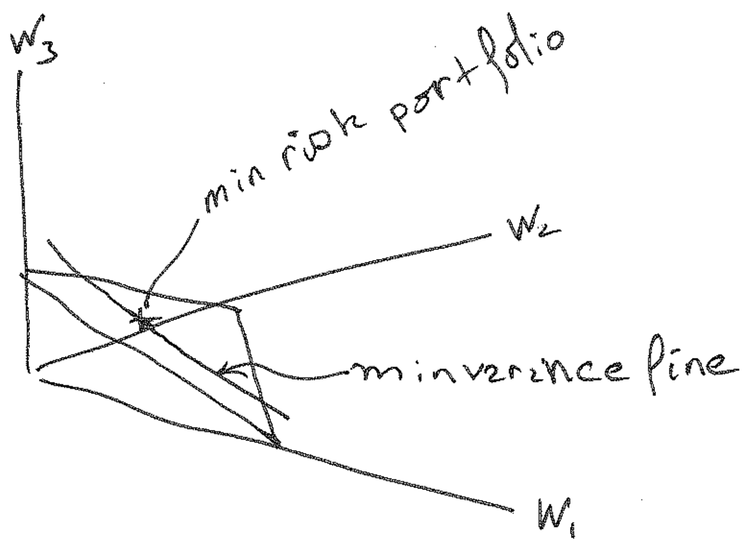




Eg If min variance line goes through interior of set w/o short selling,

Min risk portfolio inside set w/o short





Eg Min variance line going through set w/o short

min risk portfolio ~~~~ requires short selling.