CapM, Several risky securities + BOND

$S_1, \ldots, S_n \leftarrow$ Risky securities values.

$x_i \equiv$ holding in i^{th} security

$S_i(0)$ given, $S_i(1)$ random

$K_i = \dfrac{S_i(1) - S_i(0)}{S_i(0)}$ Return of ith security

$V_x = x_1 S_1 + \cdots + x_n S_n =$ value of portfolio x.

$$K_x = \frac{V_x(1) - V_x(0)}{V_x(0)}$$

for $w_i = \dfrac{x_i S_i(0)}{V_x(0)} \equiv$ weight of ith investement.

$$K_w = w^T K = w_1 K_1 + \cdots + w_n K_n$$

$$K_w = K_x$$

Set of Portfolio weights:

$$W_n = \{ w \in \mathbb{R}^n : w^T \mathbb{1}_n = 1 \}.$$

$$\mathbb{1}_n = \left.\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\right\} n$$

~~Expected Return of portfolio $\mu_w = w \mu_K$~~

Expected return portfolio

$$\mu_i = \mathbb{E} K_i \quad , \quad \mu_K = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$$

$$c_{ij} = \operatorname{cov}(K_i, K_j)$$

$$(\Sigma_K)_{ij} = c_{ij}$$

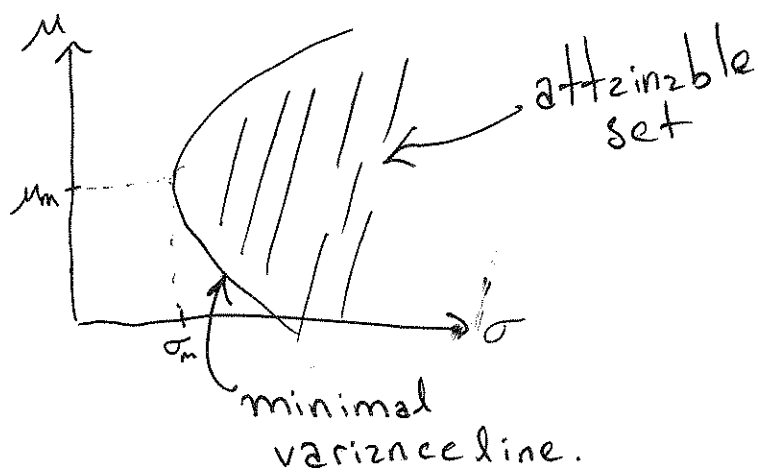
Expected return:

$$\mu_W = W^T \mu_K$$

$(\text{Risk})^2$

$$\sigma_W^2 = W^T \Sigma_K W .$$

Risk + Return of portfolios:



(μ_m, σ_m) risk + return of unique min risk portfolio.

Add to system a bond.

Eg: $A(0) = 1$; $A(1) = 1+r$; $K_B = r = \dfrac{A(1) - A(0)}{A(0)}$

portfolio: $(x_1, \dots, x_n, y)$

y is holding of bonds.

$$V_p = yA + x_1 S_1 + \dots + x_n S_n = V_B + V_x$$

$$s = \frac{V_B(0)}{V_p(0)} \; ; \quad w_i = \frac{x_i S_i(0)}{V_x(0)} \; ; \quad (1-s) w_i = \frac{x_i S_i(0)}{V_x(0)} \frac{V_x(0)}{V_p(0)} .$$

Portfolio weights:

$$w_p = \left((1-s) w_1, \dots, (1-s) w_n, s \right)$$

$$w_p^T \mathbb{1}_{n+1} = 1.$$

$$W_{n+1} = \{ w \in \mathbb{R}^{n+1} : w^T \mathbb{1}_{n+1} = 1 \}$$

weight set for n risky securities + 1 bond.

$w \in \mathbb{R}^n$, $s \equiv$ weight invested in bond.

Return:

$$K_p = \frac{V_p(1) - V_p(0)}{V_p(0)} = w_1(1-s)K_1 + \cdots + w_n(1-s)K_n + s K_B$$

$$= (1-s)\, w^T K + s r$$

Risk + Return:

$$\mu_p = \mathbb{E} K_p = (1-s)\, w^T \mu_K + s r$$
$$= (1-s)\mu_w + s r$$

$$\sigma_p^2 = \text{Var } K_p = \text{var}\left((1-s)\, w^T K + \underset{\uparrow \text{ const.}}{s r} \right)$$

$$= \text{var}\left((1-s)\, w^T K\right)$$

$$= (1-s)^2 \text{ var}\left(w^T K\right)$$

$$= (1-s)^2\, w^T \Sigma_K w$$

$$\sigma_p = (1-s)\sqrt{w^T \Sigma_K w} = (1-s)\,\sigma_w$$

Both risk + return are linear in s.

$\therefore$ given $w \in W_n$ the portfolio formed by the securities in risky market,

find risk + return formed by combining this portfolio w/ bond.

Let s be weight of investment in bond.

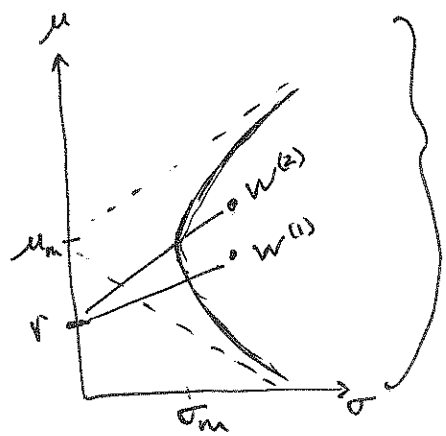
Risk and Return of total portfolio:

$$(\mu, \sigma) = ((1-s)\mu_w + sr, (1-s)\sigma_w) = (\mu_w + s(r-\mu_w), (1-s)\sigma_w)$$

$$\text{Slope: } \frac{\mu_w - r}{\sigma_w} \equiv \frac{\Delta \text{ return}}{\Delta \text{ risk}}$$

Notice greater $\frac{\Delta \text{ return}}{\Delta \text{ risk}}$ is preferrable.

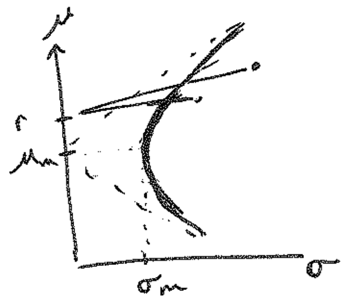
Given $w^{(1)}, w^{(2)} \in W$,



Line formed by $w^{(2)}$ is preferable to line formed by $w^{(1)}$

Line formed by $w^{(2)}$ has greater slope $\frac{\Delta \text{ return}}{\Delta \text{ risk}}$.

If $r \geq \mu_m$



No optimal line since slope increases to be (=) to slope of asymptote.

If $r < \mu_m$ let us find optimal line.

the line segment is from $(0, r)$ to $(\sigma_w, \mu_w) = \left(\sqrt{w^T \Sigma w}, w^T \mu_K\right)$

$$\text{slope} = \frac{w^T \mu_K - r}{\sqrt{w^T \Sigma w}}$$

$$(\text{slope})^2 = \frac{(w^T \mu_K - r)^2}{w^T \Sigma w}$$

We would like to maximize slope (equivalently $(\text{slope})^2$) wrt constraint $w^T \mathbb{1} = 1$.

Lagrange multiplier problem,

Lagrange function:

$$L(w,\lambda) = \frac{(w^T\mu_K - r)^2}{w^T\Sigma w} - \lambda(w^T\mathbb{1} - 1)$$

maximizing L occurs at point,

$$0 = \nabla L = \begin{pmatrix} \nabla_w L \\ \frac{\partial}{\partial\lambda} L \end{pmatrix}$$

$$(\dagger) \quad 0 = \frac{\partial}{\partial\lambda} L = w^T\mathbb{1} - 1$$

$$(\ddagger) \quad 0 = \nabla_w L = 2\,\frac{(w^T\mu_K - r)}{w^T\Sigma w}\,\mu_K - 2\,\frac{(w^T\mu_K - r)^2}{(w^T\Sigma w)^2}\,\Sigma w - \lambda\mathbb{1}.$$

Multiply (‡) on left by w^T

$$\lambda = 2\,\frac{(w^T\mu_K - r)}{w^T\Sigma w}\, w^T\mu_K - 2\,\frac{(w^T\mu_K - r)^2}{w^T\Sigma w}$$

$$\Leftrightarrow \quad \lambda = \frac{2r\,(w^T\mu_K - r)}{w^T\Sigma w}$$

Replace λ into (‡)

$$(‡) \quad \left(\frac{w^T\mu_K - r}{w^T\Sigma w}\right)(\mu_K - r\mathbb{1}) = \frac{(w^T\mu_K - r)^2}{(w^T\Sigma w)^2}\,\Sigma w$$

Multiply on left by Σ^{-1}

$$\frac{(w^T\Sigma w)}{(w^T\mu_K - r)}\,\Sigma^{-1}(\mu_K - r\mathbb{1}) = w$$

Multiply on left by $\mathbb{1}^T$

$$\left(\frac{w^T \Sigma w}{w^T \mu_K - r}\right)\left[\mathbb{1}^T \Sigma^{-1} (\mu_K - r\mathbb{1})\right] = 1$$

– or –

$$\frac{w^T \Sigma w}{w^T \mu_K - r} = \frac{1}{\mathbb{1}^T \Sigma^{-1} (\mu_K - r\mathbb{1})}$$

Replace into (‡)

$$\mu_K - r\mathbb{1} = \left[\frac{1}{\mathbb{1}^T \Sigma^{-1} (\mu_K - r\mathbb{1})}\right] \Sigma w$$

Solving for w

$$w_{mp} = \frac{1}{\mathbb{1}^T \Sigma^{-1} (\mu_K - r\mathbb{1})} \Sigma^{-1} (\mu_K - r\mathbb{1})$$

w_{mp} is known as the market portfolio.

Risk + Return of market portfolio,

$$\mu_{mp} = \frac{\mu_K \Sigma^{-1}(\mu_K - r\mathbb{1})}{\mathbb{1}^T \Sigma^{-1}(\mu_K - r\mathbb{1})}$$

$$\sigma^2_{mp} = \frac{(\mu_K - r\mathbb{1})^T \Sigma^{-1}(\mu_K - r\mathbb{1})}{\left[\mathbb{1}^T \Sigma^{-1}(\mu_K - r\mathbb{1})\right]^2}$$

Slope of market portfolio w/ Bond

$$\text{slope} = \frac{\mu_{mp} - r}{\sigma_{mp}} = \frac{\left(\frac{(\mu_K - r\mathbb{1})^T \Sigma^{-1}(\mu_K - r\mathbb{1})}{\mathbb{1}\Sigma^{-1}(\mu_K - r\mathbb{1})}\right)}{\sqrt{\frac{(\mu_K - r\mathbb{1})^T \Sigma^{-1}(\mu_K - r\mathbb{1})}{\left[\mathbb{1}^T \Sigma^{-1}(\mu_K - r\mathbb{1})\right]^2}}}$$

$\therefore$

$$\text{slope} = \sqrt{(\mu_K - r\mathbb{1})^T \Sigma^{-1}(\mu_K - r\mathbb{1})}$$

* Recall Σ is positive definite $\Leftrightarrow$ Σ^{-1} is positive definite

$$w^T \Sigma^{-1} w > 0 \quad \forall \ w \in \mathbb{R}^n.$$

Now suppose $w \in W$ is "index portfolio" (or standin for market portfolio.)

Let K_w be return of index portfolio.

Let us evaluate an alternative investment opportunity, with return K_u

Form 2 security market of w + u.

We want to evaluate the new investment wrt an established investment.

let $C = \operatorname{cov}(K_w, K_u) = u \Sigma w$

Form 2 security market of u + w

$$K_s = s K_u + (1-s) K_w.$$

$$\sigma_s^2 = \operatorname{var} K_s = s^2 \sigma_u^2 + (1-s)^2 \sigma_w^2 + 2s(1-s) C$$

$$\mu_s = s \mu_u + (1-s) \mu_w$$

For small amount invested into new opportunity

$$\text{find } \frac{\Delta \text{ return}}{\Delta \text{ risk}} \approx \frac{\frac{d}{ds}\mu_s\big|_{s=0}}{\frac{d}{ds}\sigma_s\big|_{s=0}}$$

$$\frac{d}{ds}\mu_s = \mu_u - \mu_w$$

$$\frac{d}{ds}\sigma_s^2 = 2s\sigma_u^2 - 2(1-s)\sigma_w^2 + 2(1-s)C - 2sC$$

$$\frac{d}{ds}(\sigma_s^2)^{1/2} = \frac{1}{2}\frac{1}{(\sigma_s^2)^{1/2}}\frac{d}{ds}\sigma_s^2$$

$$\frac{d}{ds}\sigma_s\Big|_{s=0} = \cancel{\frac{1}{2\sigma_s}}\ \frac{-\sigma_w^2 + C}{\sigma_w}$$

$$\therefore \quad \frac{\Delta \text{return}}{\Delta \text{risk}} = \frac{\mu_u - \mu_w}{\left(\frac{C - \sigma_w^2}{\sigma_w}\right)}$$

Assume, the security u obeys:

$$\frac{\mu_u - \mu_w}{\left(\frac{c - \sigma_w^2}{\sigma_w}\right)} = \frac{\mu_w - r}{\sigma_w} = \frac{\Delta \text{ return}}{\Delta \text{ risk}} \text{ of index porfolio vs bond.}$$

Solve for μ_u

$$\mu_u - \mu_w = \frac{1}{\sigma_w^2} (c - \sigma_w^2)(\mu_w - r)$$

$$= \frac{c}{\sigma_w^2}(\mu_w - r) - (\mu_w - r)$$

$$\mu_u = r + \frac{c}{\sigma_w^2}(\mu_w - r)$$

Define $\beta = \frac{c}{\sigma_w^2} = \frac{\operatorname{cov}(K_u, K_w)}{\sigma_w^2}$

* $\beta > 1$ risk increase quickly wrt return.
* $0 < \beta < 1$ low risk wrt index
* $0 > \beta > -1$, low risk, varies inverse to market
* $-1 > \beta$ larger risk, inverse to market.

Application of CAPM.

Suppose we write contract to pay value H @ time 1.

put aside money @ time $t=0$ to pay it off.

ie Put $V(0)$ into portfolio $\begin{cases} \text{amt: } w_B V(0) \text{ in Bonds} \\ \text{amt: } w_M V(0) \text{ in the Market} \end{cases}$

Value at time 1 is $V(1) = V(0)\left(w_M(1+K_M) + w_B(1+R)\right)$

Ideally $V(1) = H(1)$ to clear obligation.

But @ time 1 we have to correct the error:

$$\varepsilon = H(1) - V(1).$$

We try to minimize the error.

ie 1st set expectation to zero:

$$\mathbb{E}(\varepsilon) = 0 = \mathbb{E}(H(1)) - V(0)\left(W_M(1+\cancel{K_M}) + W_B(1+R)\right)$$

☆ Let us write $H(1) = V(0)(1+K_H)$

then $\mu_H = \mathbb{E}K_H$

$$\Longrightarrow \quad \mu_H = W_M \mu_M + W_B R.$$

$$\mathrm{Var}(\varepsilon) = \mathrm{Var}(H(1) - V(1)) = \mathrm{Var}\left(V(0)\{\cancel{1} + K_H\} - V_0[W_M(1+\mu_M) + W_B(1+R)]\right)$$

$$= V(0)^2 \; \mathrm{Var}(K_H - \{W_M K_M + W_B R\})$$

$$= V(0)^2 \left\{ \mathrm{var}\,K_H + \mathrm{var}(W_M K_M + W_B R) - 2\mathrm{cov}(K_H, W_M K_M + W_B R) \right\}$$

$$= V(0)^2 \left\{ \mathrm{var}\,K_H + W_M^2 \,\mathrm{var}\,K_M - 2 W_M \,\mathrm{cov}(K_H, K_M) \right\}$$

$$\therefore \quad \mathrm{Var}(\varepsilon) = V^2(0) \left\{ \mathrm{var}(K_H) + w_M^2 \, \mathrm{var}\, K_M - 2 w_M \, \mathrm{cov}(K_H \cdot K_M) \right\}.$$

$$0 = \frac{d}{d w_M} \mathrm{var}(\varepsilon) = V^2(0) \left\{ 2 w_M \, \mathrm{var}\, K_M - 2 \, \mathrm{cov}(K_H \cdot K_M) \right\}$$

$$\hookrightarrow \quad w_M = \frac{\mathrm{cov}(K_H \cdot K_M)}{\mathrm{var}\, K_M} = \beta_H.$$

$\therefore$ Minimum Risk:

$$\sigma_{min}^2 = V^2(0) \left\{ \sigma_H^2 + \underbrace{\beta_H^2 \sigma_M^2 - 2 \beta_H c_{MH}}_{\frac{\mathrm{cov}^2}{\sigma_M^2} - 2 \frac{\mathrm{cov}^2}{\sigma_M^2}} \right\}$$

$$\frac{\sigma_{min}^2}{V^2(0)} = \sigma_H^2 \left\{ 1 - \frac{\mathrm{cov}^2}{\sigma_H^2 \sigma_M^2} \right\}$$

$$= \sigma_H^2 \left\{ 1 - \rho^2 \right\}$$

$$= \sigma_H^2 \left\{ 1 - \frac{\beta_H^2}{\sigma_H^2} \right\}.$$

β_H measures risk of portfolio relative to risk of market.

$\beta_H \lesssim 1$ low risk relative to market.

$\beta_H > 1$ high risk relative to market.