Computational pricing of options

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Organization

- Anderson localization in more than one particle systems
  - What is Anderson Localization?
  - One particle localization - illustrate by fractional moment
  - multiparticle/many body localization
- Introduction to Holstein model - distinguished particle with many Bosons
  - Intermediate multiparticle to many body model

Scaling Binomial model

- Assume initial security price S_0 with volatility σ.
- Assume risk-free bond with yearly effective interest rate r_e
- Divide year into N steps
- The step-wise interest becomes:

$$r\delta = (1 + r_e)^{1/N} - 1 \approx r_e/N$$

  The value of a bond is: $A_{\frac{k}{N}} = (1 + r\delta)^{\frac{k}{N}}$
- The stepwise fluctuations are

$$\sigma\Delta = \sigma/\sqrt{N}$$

- Ie $\Delta^2 \approx \delta$

- The upwards and downwards fluctuations are,

$$m_{\pm} = r\delta \pm \sigma\Delta$$

- The values of the stock price are

$$S_0, S_{1/N}, S_{2/N}, \dots$$

so that

$$\mathbb{P}(S_{\frac{k+1}{N}} = (1 + m_+)S_{\frac{k}{N}}) = \mathbb{P}(S_{\frac{k+1}{N}} = (1 + m_-)S_{\frac{k}{N}}) = \frac{1}{2}$$

Cox - Rubinstein - Ross

- Consider a European call with payoff $H = (S_T - X)^+$ at time of expiry T, in discrete steps the expiry time is step $n = \lfloor NT \rfloor$
- Lower integration bound $S_T > X$:

$$k_0 = \left\lfloor \frac{\ln \frac{X}{(1+m_-)^n S_0}}{\ln \frac{1+m_+}{1+m_-}} \right\rfloor$$

- Value:

$$V_0 = \sum_{k=k_0}^{n} \frac{1}{2^n} \binom{n}{k} \left[S_0 \left[\frac{1+m_+}{1+\Delta_r} \right]^k \left[\frac{1+m_-}{1+\Delta_r} \right]^{n-k} - \frac{X}{(1+\Delta_r)^n} \right]$$

Gaussian approximation of the CRR formula

- Shifted probability: $q_{\pm} = \frac{1}{2}\frac{1+m_{\pm}}{1+r\delta} = \frac{1}{2}\frac{1+r\delta\pm\sigma\Delta}{1+r\delta}$
- Value:

$$V_0 = S_0\mathcal{N}_+\left(\frac{k_0 - q_+ n}{(q_+ q_- n)^{1/2}}\right) - \frac{X}{(1+r_e)^T}\mathcal{N}_+\left(\frac{k_0 - n\frac{1}{2}}{(\frac{1}{4}n)^{1/2}}\right)$$

- Where $\mathcal{N}_+(w) = \mathbb{P}(Z \geq w)$ for a standard normal random variable Z.

Approximating the approximation

Terms: as $N \to \infty$

$$k_0 \approx \frac{\ln \frac{X}{S_0} - n \ln(1+m_-)}{\ln \frac{1+m_+}{1+m_-}} \approx \frac{\ln \frac{X}{S_0} - NT(r\delta - \sigma\Delta - \frac{1}{2}\sigma^2\Delta^2)}{2\sigma\Delta}$$

$$q_+ n = \left(\frac{1 + r\delta + \sigma\Delta}{1 + r\delta}\right)\frac{NT}{2}$$

$$(q_- q_+ n)^{1/2} = \frac{\left((1+r\delta)^2 - \sigma^2\Delta^2\right)^{1/2}}{1 + r\delta}\frac{N^{1/2}T^{1/2}}{2}$$

$$\frac{k_0 - q_+ n}{(q_- q_+ n)^{1/2}} \to \frac{\ln \frac{X}{S_0} - T(r + \frac{1}{2}\sigma^2)}{\sigma T^{1/2}} =: D_0^+$$

$$\frac{2k_0 - n}{n^{1/2}} \to \frac{\ln \frac{X}{S_0} - T(r - \frac{1}{2}\sigma^2)}{\sigma T^{1/2}} =: D_0^-$$

Stochastic model

- Stock price model:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

solution:

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$$

- European option, payoff of $H_T = g(S_T)$ a function of the security value at expiry.
- Value of option at time t if present value of security is x,

$$V(t,x) = e^{-r(T-t)}\mathbb{E}(g(S_T)|S_t = x)$$

Black Scholes PDE for European Option

- V satisfies the PDE:

$$rV = \dot{V} + rxV' + \frac{1}{2}\sigma^2 x^2 V''$$

in $x > 0$ and $0 < t < T$ with boundary condition,

$$V(T, x) = g(x) \qquad\qquad V(t, 0) = g(0)$$

Black Scholes value of European Call

- Value of Euro call at expiry: $H_T = (S_T - X)^+$.
- Solution to the PDE:

$$V(t, S_t) = S_t \mathcal{N}_+(-D^+_{t,S_t}) - Xe^{-r(T-t)} \mathcal{N}_+(-D^-_{t,S_t})$$

where

$$D^{\pm}_{t,S} = \frac{\ln \frac{S}{X} + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

- Comparison to CRR solution, as $N \to \infty$:

$$V_0 = S_0 \mathcal{N}_+ \left(-D^+_0\right) - Xe^{-rT} \mathcal{N}_+ \left(-D^-_0\right)$$

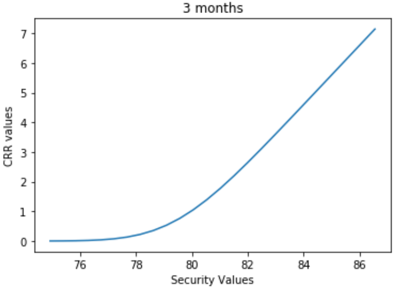
- That is, the values converge as the step size goes to 0.

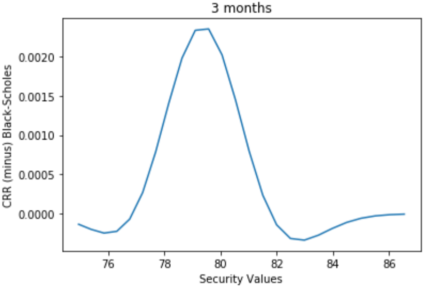
Example - European call

- Security: $S_0 = 80$; $\sigma = .03$. Bond rate $r = 1.5\%$
- Option $T = 9$ months, strike price $X = 80$.
- Time steps per year: $N = 100$.
- Value at time of issue (t = 0):

$$V_0 = \$1.3497$$

Example - European call - Value at 3 months





Computation - layout of program

- Input parameters
  - Stock initial value S_0 and volatility $= \sigma$
  - Risk free interest rate $= r$
  - Option properties: Strike $= X$, and maturity $= T$
  - Step size/ total number of steps $= n$
- Model parameters:
  - T/n amount of time per step
  - $(1 + rT/N)$ Discount factor over 1 step
  - stock price fluctuation up / down per step
    $u = (1 + rT/N + \sigma\sqrt{T/N})$
    $d = (1 + rT/N - \sigma\sqrt{T/N})$
  - Probability fluctuation up = Probability fluctuation down = 1/2.

- Introduce $n+1 \times n+1$ matrix S for the values of the Stock.
- $S[0,0] = S_0$
  for $i = 1$ to $n+1$: $S[i,0] = S[i-1,0](1+u)$;
  for $j = 1$ to i: $S[i,j] = S[i-1,j-1](1+d)$

  $\therefore S[i,j] \equiv$ security price at timestep i with $i-j$ steps up and j steps down.
- Note according to the model, for $t = iT/n$

$$\mathbb{P}(S_t = S[i,j]) = \binom{i}{j}\frac{1}{2^i}$$

European Valuation

- Introduce $n + 1 \times n + 1$ matrix EC for the values of the (European Call) Option.
- Find the values of the option at expiry:
  For $i = 0$ to n: $EC[n, i] = g(S[n, i]) = (S[n, i] - X)^+$
- Evolve the values from expiry back to time 0:
  For $i = n - 1$ to 0:
  For $j = 0$ to i:
  $EC[i, j] = (1 + rT/n)^{-1} \left(\frac{1}{2}EC[i + 1, j] + \frac{1}{2}EC[i + 1, j + 1]\right)$

American Valuation

Extra step in the American (Put) Valuation, testing for early exercise.

- Introduce $n+1 \times n+1$ matrix *Am* and *H* for the values of the Option and intrisic value.
- Find Intrisic values of option
  For $i = 0$ to n:
  For $j = 0$ to i:
  $H[i,j] = (X - S[i,j])^+$
- Set values of *Am* at expiry equal to intrinsic value.
- Evolve the values from expiry back to time 0:
  For $i = n - 1$ to 0:
  For $j = 0$ to i:
  $G = (1 + rT/n)^{-1} \left(\frac{1}{2}Am[i+1,j] + \frac{1}{2}Am[i+1,j+1]\right)$
  $Am[i,j] = max(G, H[i,j])$

Greeks

The Greeks quantify the sensitivity of Options to variables in the model - ie, these are partial derivatives.

- Delta: $\Delta \equiv \frac{\partial V}{\partial S}$. Sensitivity with respect to security price.
- Gamma: $\Gamma = \frac{\partial^2 V}{\partial S^2}$. Second order sensitivity with respect to security price.
- Theta: $\Theta \equiv \frac{\partial V}{\partial t}$. Sensitivity with respect to time.
- Vega: $\nu \equiv \frac{\partial V}{\partial \sigma}$. Sensitivity with respect to security's implied volatility.

Delta

We can compute the Greeks directly from the results of the program:

- Delta: $\Delta \equiv \frac{\partial V}{\partial S}$.
- Introduce $n+1 \times n+1$ matrix *Delta*

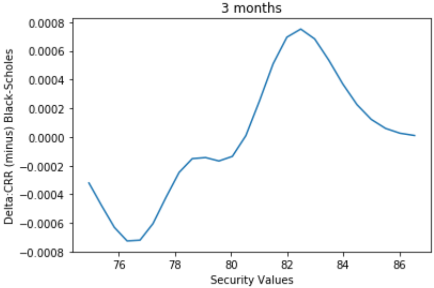
$$\Delta[i,j] = \frac{EC[i+1,j] - EC[i+1,j+1]}{S[i+1,j] - S[i+1,j+1]}$$

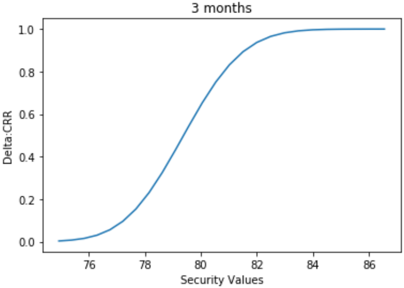
- According to Black Scholes model:

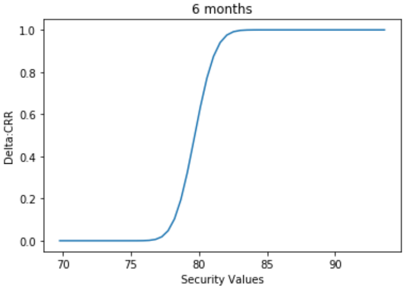
$$\Delta_{t,S} = \mathcal{N}_{+}(-D^{+}_{t,S}) = \mathcal{N}(D^{+}_{t,S})$$

- This is the proportion of the replicating portfolio invested in the stock.

Example - Delta







Delta Hedging

- Fix time step η for each hedging event – we hedge times $0, \eta, 2\eta, .., n\eta = T$
- Calculate current value of portfolio V_0 and define initial replicating portfolio

$$V_0 = X_0 S_0 + Y_0 A_0$$

  setting $X_0 = \Delta[0, S_0]$; $A_0 = 1$;
  $Y_0 = V_0 - X_0 S_0$
- At time step $(k+1)\eta$ let $X_{(k+1)\eta} = \Delta[(k+1)\eta, S_{(k+1)\eta}]$ solve for $Y_{(k+1)\eta}$:

$$Y_{(k+1)\eta} = (X_{k\eta} - X_{(k+1)\eta})S_{(k+1)\eta} + Y_{k\eta}$$

- As the stock price varies continuously some errors occur from the discritization. Find

$$V_T - X_{(n-1)\eta} S_T - Y_{(n-1)\eta} A_T$$

Gamma

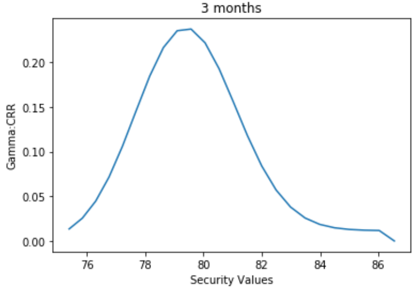
- Gamma: $\Gamma \equiv \frac{\partial^2 V}{\partial S^2}$
- Introduce $n+1 \times n+1$ matrix *Gamma* (restrict to index $[2:n-1, 2:n-1]$)

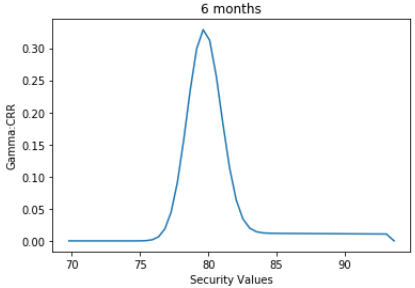
$$\Gamma[i,j] = \frac{EC[i,j+1] - 2 * EC[i,j] + EC[i,j-1]}{(S[i,j] - S[i,j-1])^2}$$

- Formula due to Black-Scholes:

$$\Gamma_{t,S} = \frac{\phi(D^{+}_{t,S})}{S\sigma\sqrt{T-t}}$$

Example - Gamma





Theta

- Theta: $\Theta \equiv \frac{\partial V}{\partial t}$
- Introduce $n+1 \times n+1$ matrix *Gamma* (restrict to index $[0:n-2,:]$)

$$\Theta[i,j] = \frac{EC[i+2,j+1] - EC[i,j]}{2\delta}$$

$\delta \equiv$ time step

- Formula due to Black-Scholes:

$$\Theta_{t,S} = -\frac{S\sigma\phi(D^{+}_{t,S})}{2\sqrt{T-t}} - rXe^{-r(T-t)}\mathcal{N}(D^{-}_{t,S})$$

Asian Option

- Asian options allow exchanges of security based on average price.
- Let

$$A_T = \frac{1}{T}\int_0^T S_t dt$$

- The Asian Call with fixed strike price has payoff at expiry

$$H_T = (A_T - X)^+$$

- The Asian Call with floating strike has payoff at expiry

$$H_T = (S_T - A_T)^+$$

- A mathematically simpler version of this option replaces the arithmetic average with a geometric average, that is replace A_T with

$$G_T = e^{\frac{1}{T}\int_0^T \ln S_t dt}$$

Monte Carlo Valuation of Asian option

Note the Asian option is path dependent. Thus, for an n step discrete model the total number of terms/paths to average over is 2^n.

This is too large to effectively implement.

Soln: Conduct trials indexed by j. For each j, simulate stock value over n steps:

$$A_T^{(j)} = \frac{1}{n}\sum_{i=1}^{n} S_i^{(j)}.$$

Then approximate the value of the call by the average of results over M trials:

$$H_T \approx \sum_{j=1}^{M}(A_T^{(j)} - S_T)^+$$

Example - Fixed strike Asian Call

$$H_T = (A_T - X)^+$$

- Security: $S_0 = 80$; $\sigma = .03$. Bond rate $r = 1.5\%$
- Option $T = 9$ months, strike price $X = 80$.
- Time steps per year: $N = 100$. Total number of trials: $M = 1000$
- Value at time of issue (t = 0):

$$V_0 = \$0.7225$$

- Value for geometric version: \$0.7188.

Example - Fixed strike Asian Call - Distribution of values

- $\mathbb{P}(H_T < .01) = \frac{362}{1000}$
- Value frequencies above 1¢