Homework 8

1. Let $u(t,x) = t^n x^m$. Let Z_t be the stochastic process $Z_t = u(t, W_t)$. Find dZ_t.
2. Let $u(t,x) = t^2 \sin(x)$. Let Z_t be the stochastic process $Z_t = u(t, W_t)$. Find dZ_t.
3. We wish to calculate $\int_0^t W_s dW_s$ ... it is not exactly clear how to do this. Consider $u(t,x) = \frac{1}{2}(x^2 - t)$ and $u(t, W_t)$ take the differential 'd' to find the value of the integral.
4. Let $a(s)$ and $b(s)$ be 'nice' let $Z_t = e^{\int_0^t a(s)ds + \int_0^t b(s)dW_s}$, find dZ_t.

   First show for $Y_t = e^{-\int_0^t \frac{1}{2} b_s^2 ds + \int_0^t b_s dW_s}$ ... Ito's formula implies:

$$dY_t = b_t Y_t dW_t$$

5. Product rule: Suppose Z_t^i for $i = 1, 2$ has differential $dZ_t^i = X^i dt + Y^i dW_t$. Let's derive $d(Z_t^1 Z_t^2)$.

   First derive the product rule from calculus: let $\Delta f(t) = f(t+\delta) - f(t)$. Derive:

$$\Delta(f(t)g(t)) = g(t)\Delta f(t) + f(t)\Delta g(t) + \Delta f(t)\Delta g(t)$$

   argue that $\Delta f(t) \sim f'(t)\delta$ so the last term vanishes, the product rule $(fg)' = f'g + g'f$. As the same difference equation holds for the stochastic functions

$$\Delta(Z_t^1 Z_t^2) = Z_t^1 \Delta Z_t^2 + Z_t^2 \Delta Z_t^1 + \Delta Z_t^1 \Delta Z_t^2$$

   the last term now is of the form

$$\Delta Z_t^1 \Delta Z_t^2 = (X_t^1 \delta + Y_t^1 \delta^{1/2})(X_t^2 \delta + Y_t^2 \delta^{1/2})$$

   thus the product rule includes a $Y_t^1 Y_t^2 dt$ term.

6. In this example, we find the solution of the general linear equation,

$$dX_t = (p_t X_t + g_t)dt + (q_t X_t + f_t)dW_t$$

   let us define the multiplier

$$\mu = e^{-\int_0^t (p_s - \frac{1}{2} q_s^2) ds - \int_0^t q_s dW_s}.$$

   Calculate $d(\mu X)$... As $d\mu$ has a fluctuation term dW_t – don't forget to use the proper Ito product rule. Note the terms with X_t cancel. Thus we may write

$$\mu_t X_t = \mu_0 X_0 + \int_0^t d(\mu_t X_t)$$

   and simplify for X_t.