# Homework 1

1. $$\mathcal{F}_2 = \{\emptyset, \Omega, A_H, A_T, A_{HH}, A_{HT}, A_{TH}, A_{TT},$$
$$A_{HH} \cup A_{TH}, A_{TT} \cup A_{HT},$$
$$A_{HH}^c, A_{HT}^c, A_{TH}^c, A_{TT}^c\}$$

Notice $(A_{HH} \cup A_{TH})^c = A_{TT} \cup A_{HT}$

$$A_H^c = A_T$$
$$A_{HH}^c = A_{HT} \cup A_{TH} \cup A_{TT} = A_{HT} \cup A_T$$
$$\vdots$$

2. ~~Suppose $A \in \mathcal{F}_i$ and $B^c$~~

Suppose $A, B \in \mathcal{F}$ then $A \in \mathcal{F}_i$ and $B \in \mathcal{F}_j$.

For $m > i, j$, $A, B \in \mathcal{F}_m$ so $A \cup B \in \mathcal{F}_m$ and $A^c \in \mathcal{F}_m$

$\therefore$ $A \cup B$ and $A^c$ are in $\mathcal{F}$.

Finally $\emptyset$ and $\Omega$ are in $\mathcal{F}_1$ so $\emptyset, \Omega \in \mathcal{F}$.

$\underline{3.}$

(a) $\int_a^b f_X(x)\,dx = 1.$

$$\int_a^b cx\,dx = \frac{cx^2}{2}\Big|_a^b = c\frac{(b^2-a^2)}{2}$$

$$\therefore\ c = \frac{2}{b^2-a^2}$$

(b) $$\mathbb{E}X = \int_a^b cx^2\,dx = c\,\frac{b^3-a^3}{3} = \frac{2}{3}\,\frac{b^3-a^3}{b^2-a^2}$$

$$= \frac{2}{3}\,\frac{b^2+ab+b^2}{a+b}$$

$$\mathbb{E}(X^2) = \int_a^b cx^3\,dx = c\,\frac{b^4-a^4}{4} = \frac{1}{2}\,\frac{b^4-a^4}{\cancel{b^2-a^2}} = \frac{1}{2}\,(b^2+a^2)$$

$$\mathrm{var}(X) = \frac{1}{2}\,(b^2+a^2) - \frac{4}{9}\left(\frac{b^2+ab+b^2}{a+b}\right)^2$$

4. $f_{XY}(x,y) = \begin{cases} cxy & - \; x \geq 0 \,;\; y \geq 0 \,;\; x+2y \leq 2 \\ 0 & - \; \text{ow}. \end{cases}$

(a) $\iint f_{XY}(x,y)\,dx\,dy = c\frac{1}{6} = 1 \Rightarrow c = 6.$

(b) $f_X(x) = cx\int_0^{1-\frac{1}{2}x} y\,dy = \frac{c}{8}(4x - 4x^2 + x^3)$

$f_Y(y) = cy\int_0^{2(1-y)} x\,dx = 2c\,(y - 2y^2 + y^3)$

(c) $\mathbb{E}(X) = \int_0^2 \frac{c}{8}(4x^2 - 4x^3 + x^4)\,dx = \frac{c}{8}\left(\frac{4}{3}x^3 - x^4 + \frac{x^5}{5}\right)\Big|_0^2$

$= 4/5$

$\mathbb{E}(Y) = \int_0^1 y\, f_Y(y)\,dy = 2c\int_0^1 (y^2 - 2y^3 + y^4)\,dy$

$= 2/5$

$\mathbb{E}(X^2) = 4/5 \quad , \quad \mathbb{E}(Y^2) = 1/5$

$Var(X) = \frac{4}{5} - \frac{16}{25} = \frac{4}{25} \quad ; \; Var(Y) = \frac{1}{5} - \frac{4}{25} = \frac{1}{25}.$

d $\quad \mathbb{E}(XY) = \int_0^1 \int_0^{2(1-y)} 6x^2y^2 \, dx\, dy$

$= 4/15.$

$\mathrm{Cov}(X,Y) = \frac{4}{15} - \frac{8}{25} = \frac{-4}{75}$

e $\quad f_Y(y|X) = \begin{cases} \dfrac{y}{\frac{1}{8}(4-4X+X^2)} & 0<y<1-\frac{1}{2}x \\ 0 & \text{o/w.} \end{cases}$

$f_X(x|Y) = \begin{cases} \dfrac{x}{2(1-2Y+Y^2)} & 0<x<2(1-Y) \\ 0 & \text{o/w} \end{cases}$

(f) $\quad \mathbb{E}(X|Y) = \int_0^2 \frac{x^2}{2(1-2Y+Y^2)} \, dx = \frac{8}{6(1-Y)^2} = \frac{4}{3} \frac{1}{(1-Y)^2}$

$\mathbb{E}(Y|X) = \int_0^1 8 \frac{y^2}{(2-X)^2} \, dy = \frac{8}{3} \frac{1}{(2-X)^2}$

(g) $f_X(x \mid W=\omega) = \dfrac{f_{XY}\left(x, \frac{\omega-x}{2}\right)}{f_W(\omega)}$

$$f_W(\omega) = \int_0^{\omega} c\,x\left(\frac{w-x}{2}\right) dx$$

$$= \frac{c}{2}\int_0^{w} x\,(w-x)\,dx = \frac{c}{2}\,\frac{\omega^3}{6} = \omega^3/2.$$

$$f_X(x|W) = \begin{cases} \dfrac{2c\,x\left(\frac{w-x}{2}\right)}{W^3} & 0 \le x \le W \\ 0 & \text{o.w.} \end{cases}$$