Homework 8 Solutions

1. $u = t^n x^m$; then $\dot{u} = n\, t^{n-1} x^m$

$$u' = m\, t^n\, x^{m-1}$$

$$u'' = m(m-1)\, t^n x^{m-2}$$

$$Z_t = u(t, W_t)$$

$$dZ_t = \left(\dot{u} + \tfrac{1}{2}u''\right) dt + u'\, dW_t$$

$$= \left(n W_t^2 + \frac{m(m-1)}{2}\, t\right) t^{n-1} W_t^{m-2}\, dt$$

$$+ m\, t^n\, W_t^{m-1}\, dW_t .$$

2. $u = t^2 \sin x$; $\left.\begin{aligned} \dot{u} &= 2t\sin x \\ u' &= t^2 \cos x \\ u'' &= -t^2 \sin x \end{aligned}\right\} Z_t = u(t, W_t)$

$$dZ_t = \left(\dot{u} + \tfrac{1}{2}u''\right) dt + u'\, dW_t$$

$$= \left(2 - \frac{t}{2}\right) t \sin W_t\, dt + t^2 \cos W_t\, dW_t .$$

3. $\int_0^t W_s \, dW_s = ?$

$u(t,x) = \frac{1}{2}(x^2 - t)$; $\dot{u} = -\frac{1}{2}$; $u' = x$; $u'' = 1$

$Z_t = u(t, W_t)$

$$dZ_t = (\dot{u} + \tfrac{1}{2}u'')dt + u' dW_t$$
$$= (-\tfrac{1}{2} + \tfrac{1}{2})dt + W_t dW_t = W_t dW_t.$$

$$Z_t - Z_0 = \int_0^t W_s \, dW_s.$$

$$\tfrac{1}{2}(W_t^2 - t) = \int_0^t W_s \, dW_s.$$

4.

$$Y_t = e^{-\int_0^t \frac{1}{2} b_s^2 ds + \int_0^t b_s dW_s}$$

for given t write $t = T+\tau$.

$$Y_t = \left(e^{-\int_0^T \frac{1}{2} b_s^2 d_s + \int_0^T b_s dW_s}\right)\left(e^{-\int_T^{T+\tau} \frac{1}{2} b_s^2 d_s + \int_T^{T+\tau} b_s dW_s}\right)$$

$$= Y_T \, Y_{T,\tau}$$

$\therefore \; dY_t = Y_T \, \underline{\underline{dY_{T,\tau}}}$ $\therefore$ we can think of dY_t @ just $t=0$.

~~[illegible]~~ for $0<\tau<\delta$ we have $b_s \sim b_0$ $\quad 0<s<\delta$

$$dY_\tau \approx d\left(e^{-\frac{1}{2} b_0^2 \tau + b_0 W_\tau}\right)$$

$$= \left(-\frac{1}{2} b_0^2 Y_\tau + \frac{1}{2} b_0^2 Y_\tau\right) Y_\tau dt + b_0 Y_\tau dW_\tau .$$

$$= b_0 Y_\tau dW_\tau \; ; \; \therefore dY_t = b_t Y_t dW_t .$$

$$Z_t = \left(e^{\int_0^t (a_s + \frac{1}{2} b_s^2) ds}\right)\left(e^{\int_0^t -\frac{1}{2} b_s^2 ds + \int_0^t b_s dW_s}\right)$$

$$\Rightarrow \; dZ_t = \left(a_t + \frac{1}{2} b_t^2\right) Z_t \, dt + b_t Z_t \, dW_t .$$

5.

$$\Delta\, f(t)\, g(t)$$

$$= f(t+\delta)\, g(t+\delta) - f(t)\, g(t)$$

$$=$$

$$f(t+\delta)\, g(t+\delta)$$
$$- f(t)\, g(t+\delta) \qquad + f(t)\, g(t+\delta)$$
$$- f(t+\delta)\, g(t) \qquad\qquad + f(t+\delta)\, g(t)$$
$$+ g(t)\, f(t) \qquad - g(t)\, f(t) \qquad - g(t)\, f(t)$$

$$=$$

$$\big(f(t+\delta) - f(t)\big)\big(g(t+\delta) - g(t)\big)$$
$$+ \big(f(t+\delta) - f(t)\big)\, g(t) + f(t)\big(g(t+\delta) - g(t)\big)$$
$$= (\Delta f)(\Delta g) + (\Delta f) g + f(\Delta g)$$

$(\Delta f)(\Delta g) \approx \delta^2$ so $\frac{\Delta f g}{\delta} \to \frac{\Delta f}{\delta} g + f \frac{\Delta g}{\delta} \Rightarrow f' g + f g'$

Stoch process: $Z^i_{t+\delta} \approx Z^i_t + X^i_t \delta + Y^i_t W_\delta$

But $\frac{1}{\delta} W_\delta^2 \to 1$ as $\delta \to 0$ $\therefore$ $Z^i_{t+\delta} \approx Z^i_t + X^i_t \delta + Y^i_t \delta^{1/2}$

$$\Delta Z^1_t Z^2_t = (\Delta Z^1_t)(\Delta Z^2_t) + Z^1_t (\Delta Z^2_t) + (\Delta Z^1_t)\, Z^2_t$$

$$= (X^1_t \delta + Y^1_t \delta^{1/2})(X^2_t \delta + Y^2_t \delta^{1/2}) + \cancel{Z_t (X^i_t \delta + Y^i_t \delta^{1/2})}\; Z^1 (\Delta Z^2) + (\Delta Z^1) Z^2$$

$$\approx (Y^1_t Y^2_t \delta + O\delta^{3/2}) + Z^1 \Delta Z^2 + (\Delta Z^1) Z^2$$

$$\therefore \quad d Z^1_t Z^2_t = Y^1_t Y^2_t\, dt + Z^1_t\, dZ^2_t + Z^2_t\, dZ^1_t$$

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Let μ be defined as -

$d\mu_t = \mu A_t dt + \mu D_t dW_t$

$dX_t = (p_t X_t + g_t)dt + (q_t X_t + f_t) dW_t$.

$= X_t(p_t dt + q_t dW_t) + g_t dt + f_t dW_t$.

$d(\mu_t X_t) = \mu X_t(D_t q_t)dt + \mu_t dX_t + X_t d\mu_t$

$= \mu X_t(D_t q_t + p_t + A_t)dt + \mu X_t(q_t + D_t) dW_t$

$+ \mu_t(D_t f_t + g_t) dt + \mu_t f_t dW_t$

choose μ s.t. X terms are $\underline{\underline{*0}}$

$\therefore D_t q_t + p_t + A_t = 0$

$q_t + D_t = 0$

$\hookrightarrow D_t = -q_t$

$\hookrightarrow -q_t^2 + p_t + A_t = 0$

$\hookrightarrow A_t = q_t^2 - p_t$

$$d\mu_t = \mu\,(q_t^2 - p_t)\,dt + \mu\,(-q_t)\,dW_t$$

$$\mu_t = e^{\int_0^t q_s^2 - p_s\,ds - \int_0^t \frac{1}{2} q_s^2\,ds - \int_0^t q_s\,dW_s}$$

$$= e^{-\int_0^t (p - \frac{1}{2} q_s^2)\,ds - \int_0^t q_s\,dW_s}.$$

$\mu_0 = 1$.

$\Leftrightarrow$

$$\mu_t X_t - X_0 = \int_0^t \mu_s (D_s f_s + g_s)\,ds + \int_0^t \mu_s f_s\,dW_s.$$

$$X_t = \mu_t^{-1} X_0 + \mu_t^{-1} \int_0^t \mu_s (D_s f_s + g_s)\,ds + \mu_t^{-1} \int_0^t \mu_s f_s\,dW_s.$$