Quiz 1 Name: Key

In all problems, you may use symmetry where appropriate and calculations where neccessary.

1. Let X be a real random variable given by a PDF (for some a)

$$f_X(x) = \begin{cases} a(1-x^2), & \text{for } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the proper value of a that makes this a probability density function.

$$1 = \int f_X(a)dx = a(x - \frac{1}{3}x^3)|_{-1}^{1} = a(4/3)$$

Thus $a = 3/4$.

(ii) Find $\mathbb{E}(X)$

f_X is symmetric around 0 ie $f_X(0+s) = f_X(0-s)$ so $\mathbb{E}(X) = 0$.

Let $Y = X^2$

(iii) Find the PDF of Y

Two functions for x^{-1}, these are $x_\pm(y) = \pm\sqrt{y}$. Find, $|\frac{d}{dy}(\pm\sqrt{y})| = \frac{1}{2\sqrt{y}}$

$$f_Y(y) = f_X(x_+(y))\left|\frac{d}{dy}x_\pm(y)\right| + f_X(x_-(y))\left|\frac{d}{dy}x_\pm(y)\right| = \frac{3}{4}(y^{-1/2} - y^{1/2})$$

2. Let $A = \{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}$, Let the pair (X,Y) be uniformly distributed on A.

(i) Find the joint density $f_{X,Y}$ of X and Y

We must have $f_{X,Y}(x,y) = c$ for $(x,y) \in A$ and 0 otherwise.

Integrate over the density,

$$1 = \int\int f_{X,Y}(x,y)dxdy = \left(\int\int\right)_A c\ dxdy$$

where the right hand side indicates the integral over the triangle A. The area of the triangle is $1/2$ so $c = 2$.

(ii) Find the marginal distributions of X and Y

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)dy = \mathbf{1}_{\{0<x<1\}} \int_0^x 2dy = 2x\mathbf{1}_{\{0<x<1\}}$$

Here $\mathbf{1}_{\{0<x<1\}} = 1$ if $0 < x < 1$ and $\mathbf{1}_{\{0<x<1\}} = 0$ otherwise.

$$f_Y(y) = \int_{\mathbb{R}} f_{X,Y}(x,y)dx = \mathbf{1}_{0<y<1} \int_y^1 2dy = 2(1-y)\mathbf{1}_{\{0<y<1\}}$$

(iii) Find the Expectation of X and Y

$$\mathbb{E}X = \int_{\mathbb{R}} x f_X(x)\ dx = \int_0^1 2x^2\ dx = 2/3$$

$$\mathbb{E}Y = \int_{\mathbb{R}} y f_Y(y)\ dy = \int_0^1 2(1-y)y\ dx = 1/3$$

(iv) Find the variance of X and Y

$$\mathbb{E}(X^2) = \int_{\mathbb{R}} x^2 f_X(x)\ dx = \int_0^1 2x^3\ dx = 1/2$$

Then $var(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = 1/2 - (2/3)^2 = 1/18$

The marginals distributions of X and Y are symmetric around $1/2$:

$f_X(1/2 + s) = f_Y(1/2 - s)$

Thus $var(X) = var(Y)$.

(v) Find the covariance of X and Y, $cov(X,Y)$, write the convariance matrix.

$$\mathbb{E}(XY) = \int\int xy f_{X,Y}(x,y)\ dxdy = \int_0^1 \int_0^x 2xydydx = \int_0^1 x^3 dx = 1/4$$

So $cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 1/36$

$$\Sigma = \frac{1}{36}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(vi) Find the conditional probability of X with respect to Y for any $Y = y$, ie find $f_X(x|Y=y)$

$$f_X(x|Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{1-y}\mathbf{1}_{\{0<y<x<1\}}$$

(vi) Find $E(X|Y)$

$$
\begin{aligned}
E(X|Y=y) &= \int x f_X(x|Y=y)dx \\
&= \mathbf{1}_{\{0<y<1\}} \int_y^1 \frac{x}{1-y} dx \\
&= \mathbf{1}_{\{0<y<1\}} \frac{1-y^2}{1-y} \\
&= \mathbf{1}_{\{0<y<1\}} (1+y)
\end{aligned}
$$

Thus, taking Y as a random variable:

$$E(X|Y) = (1+Y)$$