1.7 Problems

An intuitive idea of limits (tables and graphs)

Question 1. Consider the table of values below of $f(x)$, a strictly decreasing function. Find the largest value δ so that $|x-2| < \delta$ guarantees that $|f(x)-5| < 1$.

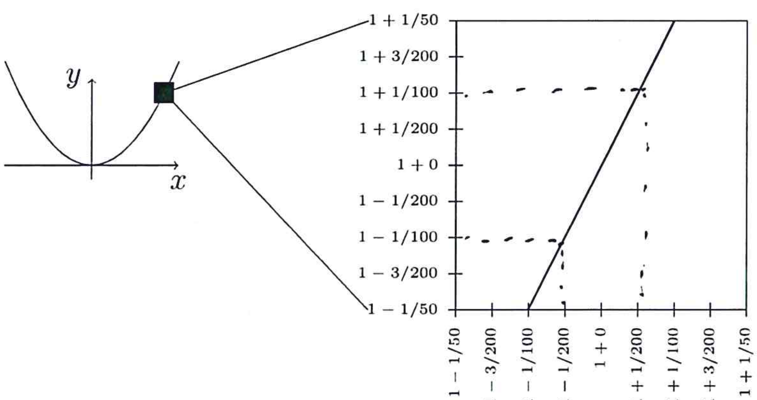
x	1.5	1.6	1.7	1.8	1.9	2	2.1	2.2	2.3	2.4	2.5
$f(x)$	6.5	6.4	5.9	5.8	5.3	5	4.3	4.1	3.9	3.1	1.7

$|1.7-2| = .3$

$|2.2-2| = .2$

$\therefore \delta = .2$

Question 2. Use a graph to find a number $\delta > 0$ such that if $|x-1| < \delta$, then $|x^2-1| < \dfrac{1}{100}$.



guess $\delta = \frac{1}{200}$

Question 3. By definition, "$\lim_{x \to 0} x^2 = 0$" means that by taking x very close to zero, we can make $|x^2 - 0|$ smaller than any small number you can name.

(a) Find a number $\delta > 0$ such that if $-\delta < x < \delta$, then $|x^2 - 0| < \frac{1}{100}$.

$\delta = \frac{1}{10} \rightarrow$ then

$|x| < \frac{1}{10}$

$|x^2| < \frac{1}{100}$

$|x^2 - 0| < \frac{1}{100}$

(b) Use algebra to find a number $\delta > 0$ such that if $-\delta < x - 1 < \delta$, then $|x^2 - 1| < \frac{1}{100}$.

$-\frac{1}{100} < x^2 - 1 < \frac{1}{100}$

$\Updownarrow$

$-\frac{1}{100} < (x-1)(x+1) < \frac{1}{100}$

$(x+1) < 3$

$-\frac{1}{300} < x - 1 < \frac{1}{300} \therefore \delta = \frac{1}{300}$ suffices.

(c) When constructing this problem, $\frac{1}{100}$ was used as an arbitrary, smallish number. Could you have done the previous problems if we replaced $\frac{1}{100}$ by $\frac{1}{10,000}$? How about $\frac{1}{1,000,000}$?

in part (a) given ε, let $\delta = \sqrt{\varepsilon}$

(b) given ε let $\delta = \frac{1}{3}\varepsilon$.

Question 4. Determine where the fallacy lies below.

Given $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$. Here prove that $\lim_{x \to 0} f(x) = 0$.

Solution: Take $\epsilon > 0$ and define $\delta = \epsilon$ and suppose $|x - 0| < \delta$ then we have:

$$|x - 0| < \delta$$
$$|x - 0| < \epsilon$$
$$|f(x) - 0| < \epsilon \quad \leftarrow \quad f(x) \neq x \text{ for } x \in (-\epsilon, 0). \quad \text{(since } f(x) = x\text{)}$$

Therefore we have shown $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$ so by definition $\lim_{x \to 0} f(x) = 0$.

Being careful with inequalities

Question 5. Perform the operations below

(a) Square both sides of $1 < 4$.

$1 < 16$

(b) Square both sides of $-4 < 1$.

$16 > 1$

(c) Square all sides of $1 < 3 < 4$.

$1 < 9 < 16$

(d) Take the square root of $4 < 9$.

$2 < 3$

(e) Take the square root of $-4 < 9$.

$\emptyset$

Question 6. Is the statement below true? Why or why not?

Suppose $x - 1 < 5$ then we know that $\sqrt{x-1} < \sqrt{5}$.

No, we can write

$|x-1| < 5 \Rightarrow \sqrt{|x-1|} < \sqrt{5}$.

Solving limit problems given equations

Question 7. How close to -3 do we have to take x so that $\dfrac{1}{(x+3)^4} > 10,000$?

$$\frac{1}{y^4} > 10{,}000$$

if $y < \frac{1}{10}$

$\therefore \quad |x+3| < \frac{1}{10}$

$|x-(-3)| < \frac{1}{10}$ ✓.

Question 8. Prove that $\lim_{x\to 1} \dfrac{2+4x}{3} = 2$ using the precise definition of the limit.

given $\varepsilon > 0$ let $\delta = \frac{3}{4}\varepsilon$

$|x-1| < \delta = \frac{3}{4}\varepsilon$

$\left|\frac{4}{3}x - \frac{4}{3}\right| < \varepsilon$

$\left|\frac{2}{3} + \frac{4}{3} - 2\right| < \varepsilon$ ✓

Question 9. Prove that $\lim_{x \to 2} [x^2 - 4x + 5] = 1$ using the precise definition of the limit.

notice

$$x^2 - 4x + 5 = (x-2)^2 + 1.$$

$$|(x-2)^2| < \varepsilon$$

$$|x-2| < \sqrt{\varepsilon}.$$

proof. given $\varepsilon > 0$ let $\delta = \sqrt{\varepsilon}$.

$$|x-2| < \delta = \sqrt{\varepsilon}$$

$$|(x-2)^2| < \varepsilon$$

$$|(x-2)^2 + 1 - 1| < \varepsilon$$

$$|[x^2 - 4x + 5] - 1| < \varepsilon.$$