2.3 Problems

Standard Problems

Example 1. Differentiate the functions:

(a) $f(x) = \pi^2$ f constant $\Rightarrow$ f' is zero

$$f'(x) = 0$$

(b) $g(x) \quad (x \quad 2)(2x + 3)$ product rule.

$$g'(x) = (x-2)'(2x+3) + (x-2)(2x+3)'$$
$$= (2x+3) + (x-2)(2)$$
$$= 4x - 1$$

(c) $h(x) = \dfrac{\sqrt{x} + x}{x^2}$ quotient rule.

$$h'(x) = \frac{x^2(\sqrt{x} + x)' - (\sqrt{x} + x)\,2x}{(x^2)^2} = \frac{\frac{1}{2}x^{3/2} + x^2 - x^{1/2} - 2x^2}{x^4}$$

$$= \frac{-x^2 + \frac{1}{2}x^{3/2} - x^{1/2}}{x^4}$$

(d) $k(x) = \dfrac{x}{x + \frac{c}{x}}$

$$k'(x) = \frac{(x + \frac{c}{x})(x)' - x\,(x + \frac{c}{x})'}{(x + \frac{c}{x})^2} = \frac{x + \frac{c}{x} - x(1 - \frac{c}{x^2})}{(x + \frac{c}{x})^2}$$

$$= \frac{\frac{c}{x} + \frac{c}{x^2}}{(x + \frac{c}{x})^2} - \frac{c(x+1)}{(x^2+c)^2}$$

Example 2. Find the equation of the tangent line of the curve $y \quad \frac{3x+1}{x^2+1}$ through the point $(1,2)$.

$$y' = \frac{(x^2+1)(3x+1)' - (3x+1)(x^2+1)'}{(x^2+1)^2} = \frac{3(x^2+1) - 2x(3x+1)}{(x^2+1)^2}$$

$$= \frac{-3x^2 - 2x + 3}{(x^2+1)^2}$$

$$\frac{\hat{y} - 2}{\hat{x} - 1} = y'(1) = \frac{-3-2+3}{(1+1)^2} = -1$$

Example 3. Find the points of the curve $y \quad 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

" tangent is horizontal" ≡ " $y' = 0$ "

$$y' = 6x^2 + 6x - 12 = 6(x^2 + x - 2)$$

$$y' = 6(x+2)(x-1)$$

$$0 = y' = 6(x+2)(x-1) \Rightarrow x = 1 \text{ or } x = -2$$

Example 4. Let $f(x) \quad \begin{cases} ax^2 & \text{if } x \le 1 \\ 3x + b & \text{if } x > 1 \end{cases}$. Find the values of a and b that make f differentiable everywhere.

f is differentiable $\Rightarrow$ f is continuous.

$$a = a(1)^2 = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = 3(1) + b \therefore a = 3 + b$$

f is differentiable @ $\underline{1}$ if f' is continuous @ 1.

$$2a = \lim_{x \to 1^-} f'(x) = \lim_{x \to 1^+} f'(x) = 3$$

$$\hookrightarrow a = \frac{3}{2} \ \& \ b = -\frac{3}{2}.$$

Non-Standard (Fun) Problems

Example 5. Suppose an Environmental scientist measures the depth of a lake and the concentration of Phosphorus. They calculate that the lake is 1.2×10^9 m^3 and has 1.5×10^{-8} mg/m^3 phosphorus concentration. A year later they repeat the measurement and find the volume of the lake is 1.4×10^9m^3 and has 1.4×10^{-8} mg/m^3 phosphorus concentration. How has the total phosphorus level changed/what is the average total phosphorus change per year?

(Total phosphorus) = (concentration $\frac{mg}{m^3}$) (volume m^3).

@ $t=0$ $\quad V(0) = 1.2 \times 10^9 \, m^3$; $p(0) = 1.5 \times 10^{-8} \, mg/m^3$

$Q \equiv$ total ph. $\quad Q(0) = 1.2 \times 1.5 \times 10$ mg.

@ $t=1$ $\quad V(1) = 1.4 \times 10^9 \, m^3$; $p(1) = 1.4 \times 10^{-8} \, \frac{mg}{m^3}$

$Q(1) = (1.4)^2 \times 10$ mg.

Avg change per year: $\{(1.4)^2 - (1.2)(1.5)\} \times 10$ mg/year

$= 1.6$ mg/year.

Suppose another scientist took measurements over the year and found the volume of the lake to follow the curve $V(t) = \sqrt{t/2 + 1.5} \times 10^9$ m^3 and the concentration of phosphorus follows the curve $p(t) = 1.4 + (1-t)^2/10$ mg/m^3. Fin $Q \equiv$ total quantity of phosphorus, what is the instantaneous rate of change of the quantity of phosphorus?

$p_t = \left[1.4 + \frac{(1-t)^2}{10}\right] \times 10^{-8} \frac{mg}{m^3}$

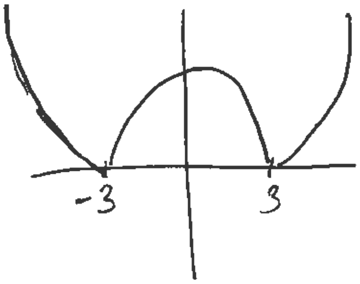
$Q(t) \equiv$ total phosphorus @ time $= V(t)\, p(t)$

$$Q(t) = \sqrt{t/2 + 1.5}\left(1.4 + \frac{(1-t)^2}{10}\right) \times 10 \text{ mg}$$

$$Q'(t) = \left\{ \frac{1}{4}\left(\frac{t}{2}+1.5\right)^{-\frac{1}{2}}\left(1.4 + \frac{(1-t)^2}{10}\right) \times 10 + \left(\frac{t}{2}+1.5\right)^{\frac{1}{2}}\left(-(1-t)\,2\right) \right\} \text{mg}$$

Example 6.

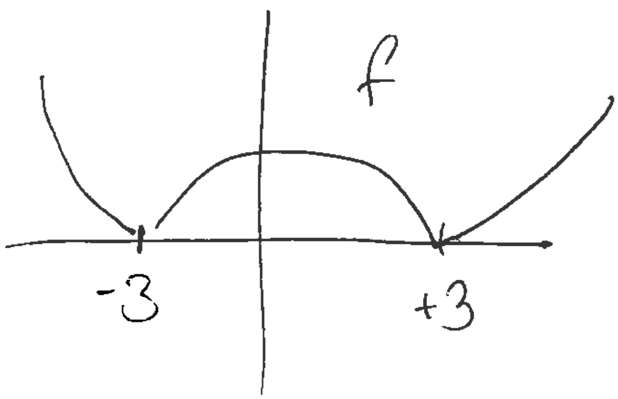
(a) For what values of x is the function $f(x) = |x^2 \quad 9|$ differentiable? Find a formula for f'

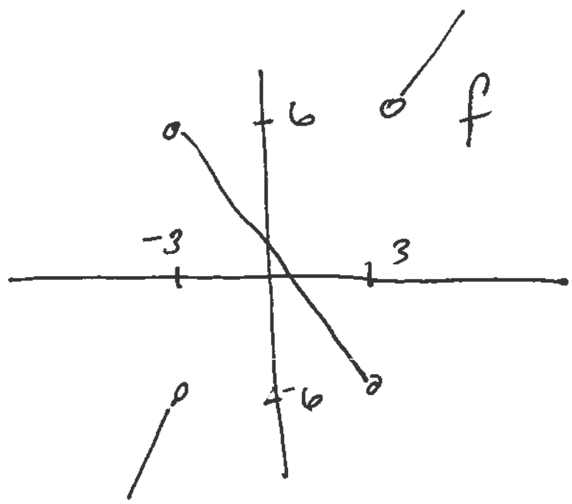


$$f = \begin{cases} x^2 - 9, & x^2 > 9 \\ 0, & x^2 = 9 \\ 9 - x^2, & x^2 < 9 \end{cases}$$

$$f' = \begin{cases} 2x, & |x| > 3 \\ -2x, & |x| < 3. \end{cases}$$

(b) Sketch the graphs of f and f'





Example 7 Evaluate $\lim_{x \to 1} \frac{x^{1000} - 1}{x - 1}$ $= \lim_{h \to 0} \frac{(1+h)^{1000} - 1}{(1+h) - 1} = 1000$

$(1+h)^{1000} = 1 + (1000)h + \binom{1000}{2} h^2 + \cdots$

since $f(x) = x^{1000}$

$f'(1) = \lim_{h \to 0} \frac{(1+h)^{1000} - 1}{h}$

+ $f'(x) = 1000x^{999}$

$f'(1) = 1000.$