2.4a Problems

Fun Trig Limits

Example 1. Find the limits:

(a) $\lim\limits_{x\to0} \dfrac{\sin 4x}{\sin 7x}$

Hint: Recall $\lim_{x\to0}\frac{\sin x}{x}=1$*, use division of limits rules.*

$$\lim_{4x\to0}\frac{\sin 4x}{4x}=\lim_{y\to0}\frac{\sin y}{y}=1$$

$$\therefore \lim_{x\to0}\frac{\sin 4x}{x}=4.$$

$$\lim_{x\to0}\frac{\sin 4x}{\sin 7x}=\lim_{x\to0}\underbrace{\frac{\sin 4x}{x}}_{4}\ \underbrace{\frac{x}{\sin 7x}}_{\frac{1}{7}}=\frac{4}{7}.$$

(b) $\lim\limits_{t\to0} \dfrac{\sin 4t}{\cos(-3t)}$

$$\lim_{t\to0}\cos(-3t)=1.\ ;\ \lim_{t\to0}\sin 4t=0$$

Case: $\frac{0}{N}$

$$\therefore \lim_{t\to0}\frac{\sin 4t}{\cos(-3t)}=0.$$

(c) $\lim\limits_{\theta\to0} \dfrac{\sin\theta}{\theta+\tan\theta}$

Hint: Find $\lim_{\theta\to0}\frac{\theta+\tan\theta}{\theta}$ 1*, use division of limits rules.*

$$\lim_{\theta\to0}\frac{\theta+\tan\theta}{\theta}=\lim_{\theta\to0}\left(1+\underbrace{\frac{\sin\theta}{\theta}}_{\to1}\ \underbrace{\frac{1}{\cos\theta}}_{\to1}\right)=2$$

$$\lim_{\theta\to0}\frac{\sin\theta}{\theta+\tan\theta}=\lim_{\theta\to0}\underbrace{\frac{\sin\theta}{\theta}}_{1}\ \underbrace{\frac{\theta}{\theta+\tan\theta}}_{1/2}=\frac{1}{2}.$$

Example 2. Find the limits:

(a) $\lim_{t\to\pi/4} \dfrac{1-\tan t}{\sin t - \cos t}$ $= \lim_{t\to\frac{\pi}{4}} \dfrac{1-\tan t}{\sin t - \cos t}\,\dfrac{\cos t}{\cos t}$

$$= \lim_{t\to\frac{\pi}{4}} \underbrace{\frac{\cos t - \sin t}{\sin t - \cos t}}_{-1}\,\frac{1}{\cos t} = \frac{1}{\sqrt{2}/2} = \sqrt{2}$$

(b) $\lim_{x\to 0} \dfrac{\sec x}{1-\sin x}$, $\sec x = \dfrac{1}{\cos x}$; $\lim_{x\to 0}\dfrac{1}{\cos x} = 1$

$$\lim_{x\to 0}\frac{1}{1-\sin x} = 1$$

$$\therefore \lim_{x\to 0}\frac{\sec x}{1-\sin x} = 1.$$

(c) $\lim_{t\to 0} \dfrac{t^3}{\tan^3 2t}$ $= \lim_{t\to 0}\left(\dfrac{t^3}{\sin^3 2t}\,\dfrac{\cos^3 2t}{1}\right)$

$$= \lim_{t\to 0}\left(\frac{t}{\sin 2t}\right)^3 \lim_{t\to 0}\frac{(\cos 2t)^3}{1.}$$

$$= \left(\underbrace{\lim_{t\to 0}\frac{t}{\sin 2t}}_{1/2}\right)^3 = \frac{1}{8}$$

Derivatives and Such

Example 3. Find the equation of the tangent line to the curve

(a) $y = (1 + x)\cos x$ through the point $(0, 1)$. $(\hat{x}, \hat{y}) \equiv$ tangent line. $(x_0\ y_0) = 0\ 1$

Recall eq: $\dfrac{\hat{y} - y_0}{\hat{x} - x_0} = y'(0)$ Tangent line:

$$y = \cos x + (1+x)(-\sin x)$$

$$y'(0) = 1 + 0 = 1.$$

$$\frac{\hat{y} - 1}{\hat{x}} = 1$$

$$\hat{y} = 1 + \hat{x}\ .$$

(b) y $\cos x$ $\sin x$ through the point $(\pi\ -1)$.

$$y' = -\sin x - \cos x$$

$$y'(\pi) = 0 - (-1) = 1$$

$$\frac{\hat{y} - (-1)}{\hat{x} - \pi} = 1.$$

$$\hat{y} = \hat{x} - \pi - 1$$

(c) y $2x \sin x$ through the point $(\pi/2, \pi)$.

$$y' = 2\sin x + 2x(\cos x)$$

$$y'\left(\frac{\pi}{2}\right) = 2 \cdot 1 + 2\left(\frac{\pi}{2}\right) \cdot 0 = 2$$

$$\frac{\hat{y} - \pi}{\hat{x} - \frac{\pi}{2}} = 2$$

$$\hat{y} = \pi - \pi + 2\hat{x}$$

$$\therefore \hat{y} = 2\hat{x}\ .$$

Example 4. Suppose that $f(\pi/3) = 4$ and $f'(\pi/3) = -2$, and let $g(x) = f(x)\sin x$ and $h(x) = \dfrac{\cos x}{f(x)}$. Find

(a) $g'(\pi/3)$

(b) $h'(\pi/3)$

$$g' = f'\sin x + f(\sin x)'$$

$$g'\left(\tfrac{\pi}{3}\right) = (-2)\sin\left(\tfrac{\pi}{3}\right) + 4\cos\left(\tfrac{\pi}{3}\right)$$

$$= (-2)\left(\tfrac{1}{2}\right) + 4\left(\tfrac{\sqrt{3}}{2}\right) = -1 + 2\sqrt{3}$$

$$h = \frac{f(\cos x)' - (\cos x)f'}{f^2}$$

$$h'\left(\tfrac{\pi}{3}\right) = \frac{4\left(-\sin\frac{\pi}{3}\right) - \left(\cos\frac{\pi}{3}\right)(-2)}{4^2} = \frac{-1/2}{4} + \frac{\sqrt{3}}{4^2}$$

$$= \frac{1}{4^2}\left(-2 + \sqrt{3}\right).$$

Example 5. Find the points on the curve $y = \dfrac{\cos x}{2 + \sin x}$ at which the tangent is horizontal.

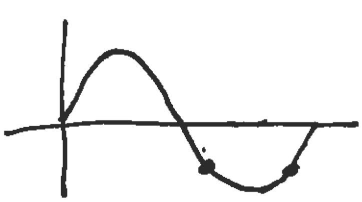
"tangent is horizontal" ≡ "$y' = 0$"

$$y' = \frac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2+\sin x)^2}$$

{denominator is always > 0}

∴ set numerator = 0

$$0 = -2\sin x - \sin^2 x - \cos^2 x = -2\sin x - 1$$



$\frac{4}{3}\pi$ $\frac{5}{3}\pi$

$$\frac{1}{-2} = \sin x$$

$x = \frac{4}{3}\pi + k2\pi$

-or-

$x = \frac{5}{3}\pi + k2\pi$

$k = \ldots, -1, 0, 1, 2 \ldots$