4.1 Problems

Table and Graph Problems

Example 1. Estimate the area under the curve $f(x) = \dfrac{1-x}{\sqrt{-x}}$ on $[-4,-1]$ using 4 rectangles and right sums.

4 rectangles for interval of length 3 ⇒ each interval is length $\frac{3}{4}$.

$f(-4+\frac{3}{4}) = \frac{4.25}{\sqrt{3.25}}$; $f(-4+\frac{6}{4}) = \frac{3.5}{\sqrt{2.5}}$; $f(-4+\frac{9}{4}) = \frac{2.75}{\sqrt{1.75}}$

$f(-4+\frac{12}{4}) = \frac{2}{\sqrt{1}} = 2$

$$\text{Area} \approx \frac{3}{4}\cdot\frac{4.25}{\sqrt{3.25}} + \frac{3}{4}\,\frac{3.5}{\sqrt{2.5}} + \frac{3}{4}\,\frac{2.75}{\sqrt{1.75}} + \frac{3}{4}\cdot 2.$$

Unseen Difficulties

Example 2. Consider the function $f(x) = x(x-4)^2$ on the interval $[0,6]$.

(a) Estimate the area under f using left-hand end points and

(i) $n = 3$

(ii) $\Delta x = 1$

x	0	1	2	3	4	5
f	0	9	8	3	0	5

(i) Area $\approx f(0)\Delta x + f(2)\Delta x + f(4)\Delta x = 0\cdot 2 + 8\cdot 2 + 0\cdot 2$
$= 16$

(ii) Area $\approx f(0)\Delta x + f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x$

$= 0\cdot 1 + 9\cdot 1 + 8\cdot 1 + 3\cdot 1 + 0\cdot 1 + 5\cdot 1$

$= 25$

(b) Estimate the area under f using right-hand end points and

(i) $\Delta x = 2$

(ii) $n = 6$

$f(6) = 24$

(i) Area $\approx f(2)\Delta x + f(4)\Delta x + f(6)\Delta x$

$= 8\cdot 2 + 0\cdot 2 + 24\cdot 2 = 64.$

(ii) Area $\approx f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x + f(6)\Delta x$

$= 9\cdot 1 + 8\cdot 1 + 3\cdot 1 + 0\cdot 1 + 5\cdot 1 + 24\cdot 1 = 49$

(c) Use your curve sketching abilities and 4.1 video notes to explain why none of the previous estimates are technically upper sums.

$f = x^3 - 8x^2 + 16x$; $f' = 3x^2 - 16x + 16$

$f' = 0 \iff x = \frac{4}{3}, 4 \therefore f \begin{cases} > 0 \text{ for } (-\infty, \frac{4}{3}) \cup (4, \infty) \\ < 0 \text{ for } (\frac{4}{3}, 4). \end{cases}$

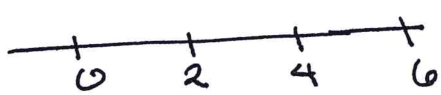
$\therefore$ max sum must take on value $f(\frac{4}{3})$

for interval containing $\frac{4}{3}$.

(d) Find an upper sum of f using 3 rectangles.

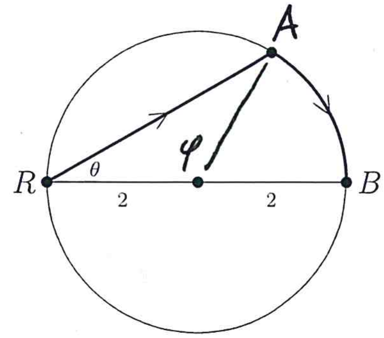
Area $\leq 2\cdot f(\frac{4}{3}) + 2\, f(2) + 2\, f(6)$

$= 2\left(\frac{4}{3}\cdot\left(\frac{8}{3}\right)^2\right) + 2\cdot 8 + 2\cdot 24.$



Example 3.

Ryan is at point R on the shore of a circular lake with radius 2 mi and all of a sudden really has to use the bathroom which is at point B diametrically opposite R (see the picture to the right). He can run at the rate of 8 mi/h and row a boat at 4 mi/h. How should he proceed?



$$|RA| = \left(2\sin\frac{\varphi}{2}\right)2$$

$$\text{arc}(AB) = 2\cdot(\pi - \varphi)$$

find minimum of

$$T = (1/4)\,|RA| + (1/8)\,\text{arc}(AB)$$

$$= \sin\left(\frac{\varphi}{2}\right) + \frac{\pi - \varphi}{4}$$

$$T' = \frac{1}{2}\cos\frac{\varphi}{2} - \frac{1}{4} = 0 \Leftrightarrow \cos\frac{\varphi}{2} = \frac{1}{2}$$

$$\frac{\varphi}{2} = \frac{\pi}{3}$$

$$\varphi = \frac{2}{3}\pi.$$

$$T'' = -\frac{1}{4}\sin\frac{\varphi}{2} \quad \therefore\ \varphi = \frac{2}{3}\pi \text{ is local } \underline{\text{max}}$$

$\varphi = 0$

$$T = \pi 2 \frac{1}{8} = .78 \text{ hour}$$

$\varphi = \pi$

$$T = 4\,\frac{1}{4} = 1 \text{ hour}$$

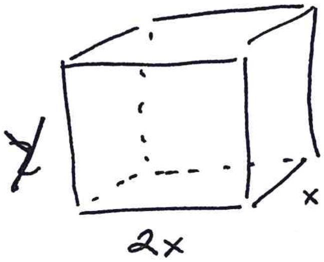
$\varphi = \frac{2}{3}\pi$

$$T = \frac{\sqrt{3}}{2} + \frac{\pi - \frac{2}{3}\pi}{4}$$

$$= 1.12 \text{ hour}.$$

$\therefore$ min: Run around pond.

Example 4. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of the materials for the cheapest such container.



$$V = 2x^2 \cdot y = 10$$

Base cost: $2x^2 \, \$10$

Side cost: $\cancel{2} \times (2xy + xy) \times \$6$

$$\text{Cost} = 20x^2 + 32xy$$

$$y = \frac{5}{x^2}$$

$$\text{Cost} = 20x^2 + 32 \cdot \frac{5}{x}$$

$$(\text{Cost})' = 40x - \frac{160}{x^2} = 0 \iff x^3 = 4$$

$$x = 4^{1/3}.$$

$(\text{Cost})'' = 40 + \frac{320}{x^3}$ $\therefore$ concave up $x = 4^{1/3}$ is local min.

Cheapest cost:

$$\text{Cost} = 20 \cdot 4^{2/3} + \frac{160}{4^{1/3}}.$$