SYMPLECTIC RESOLUTION OF ISOLATED ALGEBRAIC SINGULARITIES

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In [McW] the authors described a method of gluing symplectic manifolds along a class of hypersurfaces called *ω-compatible hypersurfaces.* They then applied this gluing to "symplectically resolve" the isolated orbifold singularities in symplectic 4-manifolds. Subsequently, Y. Eliashberg pointed out to us that, using contact hypersurfaces, it should be possible to "symplectically resolve" isolated algebraic singularities on a symplectic 4-manifold. The purpose of this note is to state and prove this result.

It is unfortunately the case that, at present, there is no definition of an isolated symplectic singularity, let alone a construction of its resolution. Certainly, any reasonable definition must include the algebraic singularities. Accordingly we regard this paper as a first step in a program towards the definition and resolution of this class of singularities. The resolution we give here relies on the resolution of the singularity in the algebraic category. This feature limits the technique. A more illuminating proof would involve only symplectic techniques as, for example, in the resolution of orbifold singularities [McW]. Such a procedure is not currently available.

Let X be a topological space and $p \in X$ such that $X \setminus \{p\}$ is a symplectic manifold with symplectic form ω. Such a point p is called an *isolated singularity.*

Definition 1. *We say p is an isolated algebraic singularity in the symplectic manifold $(X \setminus \{p\}, \omega)$ if there are:*

(1) a neighborhood $N(p)$ of p in X.

(2) an algebraic variety $V \subset \mathbb{C}^N$.

(3) an isolated singularity $q \in V$.

(4) a neighborhood $U(q) \subset V$ of q such that $U(q) \setminus \{q\}$ has symplectic form η induced from $\mathbb{C}^N$.

(5) a symplectic diffeomorphism

$$\psi : (N(p) \setminus \{p\}, \omega) \to (U(q) \setminus \{q\}, \eta).$$

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Recall the notion of symplectic resolution from [McW]: Suppose that p is an isolated singularity of X. Let U be a neighborhood of p in X.

Definition 2. *We say a symplectic manifold* $(\tilde{X}, \tilde{\omega})$ *is a symplectic resolution of* p *on* U *if there are:*

(1) a tubular neighborhood W *of a symplectic divisor* D *in* $\tilde{X}$*.*

(2) a map $\pi : (\tilde{X}, D) \to (X, p)$ *such that* $\pi : \tilde{X} \backslash W \to X \backslash U$ *is a symplectic diffeomorphism.*

A symplectic divisor in $\tilde{X}$ is a set D of symplectically embedded codimension two manifolds which intersect transversely. We say that the divisor is closed if each of the manifolds in D are closed.

Theorem 1. *Let* U *be a neighborhood of an isolated algebraic singularity* p *in the symplectic 4-manifold* $(X \setminus \{p\}, \omega)$*. There is a symplectic resolution of* p *on* U*.*

We will require the following definitions and lemmas from contact geometry. Let (X, ω) be a symplectic manifold and let $H \subset X$ be a hypersurface in X. Suppose that ξ is a co-oriented contact structure on H. ξ is given as $ker(\alpha)$ for a 1-form α with the property that the 2-form $d\alpha$ is nondegenerate on ξ. We say that α is a contact form for ξ if α is positive on positively co-oriented tangent vectors in H transverse to ξ. A contact form for ξ is determined up to multiplication by a positive function. Since multiplication of α by a positive function causes the multiplication of $d\alpha|_\xi$ by the same function, the conformal symplectic class of the form $d\alpha|_\xi$ depends only on ξ. Denote this class by $CS(\xi)$.

Definition 3 ([E-G])**.** *We say that the symplectic form* ω *on* X *dominates the contact structure* ξ *on* H *if*

$$\omega|_\xi \in CS(\xi)$$

.

Since contact forms for ξ are well defined up to multiplication by a function, for dominated contact structures there is a canonical contact form α such that $\omega|_\xi = d\alpha|_\xi$. Note that α induces the given co-orientation of ξ. We equip H with the contact orientation induced by α and the corresponding co-orientation induced by ω. In particular, an ω-dominated hypersurface has a canonical co-orientation.

Definition 4. *We say that the contact structure* ξ *on* H *is* ω*-convex if*

$$\omega|_H = d\alpha$$

for a contact form α *for* ξ*.*

Remark 1. *For equivalent formulations of ω-convex see* [E-G]*. Clearly if ξ is ω-convex then ω dominates ξ. A simple argument shows that the converse holds when* $\dim H > 3$ *(see* [McD]*). When* $\dim H = 3$ *the converse fails (see* [E-G]*).*

Lemma 1 ([E1])**.** *Let $H \subset X$ be a closed hypersurface with a co-oriented contact structure ξ. Let ω be an exact symplectic form defined near H such that ω dominates ξ. Then there exists a symplectic structure Ω on $H \times \mathbb{R}_+$ which is equivalent to ω near $H = H \times \{1\}$ and which has the form $d(t\alpha)$ on $H \times [C, \infty)$ for some $C > 0$, where $t \in [C, \infty)$ and α is a contact form for ξ. In particular ξ is Ω-convex on $H \times \{t\}$ for $t \geq C$.*

The proof of Lemma 1 in [E1] shows that we may assume that the canonical co-orientation on $H \times \{1\}$ induced by Ω is given by $\frac{\partial}{\partial t}$. Hence, we may assume that the equivalence of the previous lemma maps the negative side of H to $H \times (0, 1)$. Note, furthermore, that the negative side of $H \times \{t\}$ corresponds to $H \times (0, t)$ for all $t \geq C$.

Lemma 2 ([E2])**.** *Let $H \subset X$ be a closed hypersurface with a co-oriented contact structure ξ. Let ω_0, ω_1 be exact symplectic forms defined near H such that ξ is ω_i-convex, $i = 0, 1$. Then ω_0, ω_1 are conformally concordant: There is a symplectic structure Ω on $H \times [0, 1]$ which is equivalent to $c_i\omega_i$ near $H = H \times \{i\}$, $i = 0, 1$, for some positive constants c_0 and c_1.*

Proof. Let α_i be the canonical contact form for ξ induced by ω_i. Since ξ is ω_i convex for $i = 0, 1$, $\alpha_1 = f\alpha_0$ for a positive function $f : H \to \mathbb{R}_+$. Scaling ω_1 by a positive constant causes the multiplication of f by the same constant. Hence, we may assume that $f(x) > 1$ for all $x \in H$.

Consider the manifold $H \times \mathbb{R}_+$. The 2-form $d(t\alpha_0)$ is a symplectic structure Ω on $H \times \mathbb{R}_+$. Let $H_0 = H \times \{1\}$. The restriction of Ω to H_0 is equal to $d\alpha_0$. By the symplectic neighborhood theorem there is a diffeomorphism ϕ_0 from a neighborhood of H in X to a neighborhood of H_0 in $H \times \mathbb{R}_+$ such that $\phi_0^*(\Omega) = \omega_0$.

Let H_1 be the graph of f. Let $\psi : H \to H_1$ be the obvious diffeomorphism from H to H_1. Then $\psi^*(\Omega) = d(f\alpha_0) = d\alpha_1$. As above, there is a diffeomorphism ϕ_1 from a neighborhood of H in X to a neighborhood of H_1 in $H \times \mathbb{R}_+$ such that $\phi_1^*(\Omega) = \omega_1$.

The restriction of Ω to the region between H_0 and H_1 gives the desired symplectic concordance. □

As in Lemma 1, we may assume that the equivalences of Lemma 2 map the sides of H as dictated by the co-orientations given by $\frac{\partial}{\partial t}$.

PROOF OF THE THEOREM: Let (V, q) be the singularity that models the singularity (X, p). We can suppose that q is the origin in $\mathbb{C}^N$. Let

$$S_\varepsilon = \{(x_1 \cdots x_N) \in \mathbb{C}^N : \sum_{i=1}^{N} |x_i|^2 = \varepsilon^2\}$$

denote the $(2N-1)$ sphere of radius ε, centered at the origin. For ε sufficiently small $H = S_\varepsilon \cap V$ is a hypersurface in $U(q)$. The distribution of complex lines defines a contact structure ξ on H. The Kähler form η on V, induced from $\mathbb{C}^N$, is exact near H and ξ is η-convex. Thus the hypersurface $\psi^{-1}(H) \subset N(p) \subset X$ has an ω-convex contact structure $\psi^*(\xi)$. Note that p lies on the negative side of $\psi^{-1}(H)$. Let $N'(p)$ denote the union of $\{p\}$ and the negative side of $\psi^{-1}(H)$.

Since $q \subset V$ is a algebraic singularity, we can resolve it in the projective category by blowing up. The resolution is a Kähler, in fact, projective manifold Y containing an analytic divisor D, called the *exceptional divisor*, with an analytic projection $\pi : Y \to V$ which restricts to a biholomorphism:

$$\begin{array}{c} Y \setminus D \\ \big\downarrow \pi \\ V \setminus \{q\}. \end{array} \tag{0.1}$$

Denote the Kähler form on Y by τ. Let $\tilde{H} = \pi^{-1}(H)$. $\tilde{H}$ is a hypersurface with contact structure $\pi^*(\xi)$. Since π is a diffeomorphism away from D, we may identify $(\tilde{H}, \pi^*(\xi))$ with (H, ξ). Since π is a biholomorphism away from D, the contact planes on $\tilde{H}$ are the distribution of complex lines contained in $T\tilde{H}$. It follows that the Kähler form τ is positive on each contact plane and hence dominates the contact structure ξ. (This step apparently fails when $\dim \tilde{H} > 3$.) Again, since π is a biholomorphism, it follows that D is on the negative side of $\tilde{H}$. Let Y' denote the union of D and the negative side of $\tilde{H}$.

By the choice of H, Y' is a regular neighborhood of the exceptional divisor D. By Grauert's criterion [B-P-V], the intersection form on the second homology group $H_2(Y')$ is negative definite and, hence, nondegenerate. Every class in $H_2(Y')$ is represented by a cycle in D. Let S be a cycle supported in $Y' \setminus D$. S has zero intersection with every cycle in D. Thus, S is homologically trivial in Y'. Since τ is a closed 2-form on Y', $\int_S \tau = 0$. Thus, the periods of τ in $Y' \setminus D$ are trivial. Since τ is closed, this implies that τ is exact in $Y' \setminus D$. In particular, τ is exact near $\tilde{H}$.

Using Lemma 1 and the subsequent remark concerning $\frac{\partial}{\partial t}$, we can extend the symplectic structure τ to a manifold W containing Y' such that on $\partial W \simeq \tilde{H}$ the contact structure ξ is τ-convex. Note that W is on the negative side of ∂W. Thus, by Lemma 2 and the subsequent remark concerning $\frac{\partial}{\partial t}$, rescaling if necessary, we can delete the neighborhood $N'(p) \subset X$ and symplectically glue into $X \setminus N'(p)$ the manifold W using a symplectic concordance. The resulting symplectic manifold is the required symplectic resolution. $\square$

Remark 2. *In the previous proof, the exceptional divisor may have singular components. However, by further resolution, we can assume that each component of D is nonsingular.*

An alternate proof can be given which avoids the use of Lemma 1. On $\pi^{-1}(U(q)) \subset Y$ consider the 2-form $\omega_\varepsilon = \pi^*(\eta) + \varepsilon\tau$; for $\varepsilon > 0$. Since $\pi^*\eta$ is nonnegative and τ is positive on complex lines, ω_ε is nondegenerate and therefore defines a symplectic form. Clearly, for any $\varepsilon > 0$, ω_ε dominates the contact structure ξ on $\tilde{H}$. In fact on $\tilde{H}$:

$$\omega_\varepsilon|_{\tilde{H}} = d\alpha + \varepsilon d\beta,$$

where α is a contact form and β is a 1-form. For ε sufficiently small the 1-form $\alpha + \varepsilon\beta$ defines a new contact structure on $\tilde{H}$ that we denote ξ_ε. Note that the contact structure ξ_ε is ω_ε-convex. By Gray's stability theorem [G] the contact structures ξ_ε and ξ are contactomorphic when ε is sufficiently small. Hence for ε sufficiently small we may assume that the contact structure ξ on $\partial Y' = \tilde{H}$ is ω_ε-convex. The proof of the theorem is completed using Lemma 2 to glue (Y', ω_ε) into $(X \setminus N'(p), \omega)$ by a symplectic concordance.

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