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DISCRETENESS AND HOMOGENEITY OF THE TOPOLOGICAL FUNDAMENTAL GROUP

JACK S. CALCUT AND JOHN D. McCARTHY

Abstract. For a locally path connected topological space, the topological fundamental group is discrete if and only if the space is semilocally simply-connected. While functoriality of the topological fundamental group, with target the category of topological groups, remains an open question in general, the topological fundamental group is always a homogeneous space.

1. Introduction

The concept of a natural topology for the fundamental group appears to have originated with Witold Hurewicz [8] in 1935. It received further attention in 1950 by James Dugundji [2] and more recently by Daniel K. Biss [1], Paul Fabel [3], [4], [5], [6], and others. The purpose of this note is to prove the following folklore theorem.

Theorem 1.1. *Let X be a locally path connected topological space. The topological fundamental group $\pi_1^{\mathrm{top}}(X)$ is discrete if and only if X is semilocally simply-connected.*

Theorem 5.1 of [1] is Theorem 1.1 without the hypothesis of local path connectedness. However, a counterexample of Fabel [6] shows that this stronger result is false. Fabel [6] also proves a weaker

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version of Theorem 1.1, assuming that X is locally path connected and a metric space. In this note we remove the metric hypothesis.

Our proof proceeds from first topological principles, making no use of rigid covering fibrations [1] nor even of classical covering spaces. We make no use of the functoriality of the topological fundamental group, a property which was also a main result in [1, Corollary 3.4] but, in fact, is unproven [5, pp. 188–189]. Beware that the misstep in the proof of Proposition 3.1 in [1], namely the assumption that the product of quotient maps is a quotient map, is repeated in Theorem 2.1 of [7].

In general, the homeomorphism type of the topological fundamental group depends on a choice of basepoint. We say that $\pi_1^{\text{top}}(X)$ is *discrete*, without reference to a basepoint, provided $\pi_1^{\text{top}}(X,x)$ is discrete for each $x \in X$. If x and y are connected by a path in X, then $\pi_1^{\text{top}}(X,x)$ and $\pi_1^{\text{top}}(X,y)$ are homeomorphic. This fact was proved in Proposition. 3.2 of [1], and a detailed proof is provided for completeness in section 4 of this paper. Theorem 1.1 now immediately implies the following.

Corollary. *Let X be a path connected and locally path connected topological space. The topological fundamental group $\pi_1^{\text{top}}(X,x)$ is discrete for some $x \in X$ if and only if X is semilocally simply-connected.*

As mentioned above, it is open whether π_1^{top} is a functor from the category of pointed topological spaces to the category of topological groups. The unsettled question is whether multiplication

$$\begin{array}{ccc} \pi_1^{\text{top}}(X,x) \times \pi_1^{\text{top}}(X,x) & \xrightarrow{\mu} & \pi_1^{\text{top}}(X,x) \\ ([f],[g]) & \longmapsto & [f] \cdot [g] \end{array}$$

is continuous. By Theorem 1.1, if X is locally path connected and semilocally simply-connected, then $\pi_1^{\text{top}}(X,x)$, and, hence, the product $\pi_1^{\text{top}}(X,x) \times \pi_1^{\text{top}}(X,x)$ are discrete and so μ is trivially continuous. Continuity of μ, in general, remains an interesting question.

Lemma 5.1 below shows that if (X, x) is an arbitrary pointed topological space, then left and right multiplication by any fixed element in $\pi_1^{\text{top}}(X, x)$ are continuous self maps of $\pi_1^{\text{top}}(X, x)$. Therefore, $\pi_1^{\text{top}}(X, x)$ acts on itself by left and right translation as a group of self homeomorphisms. Clearly, these actions are transitive. Thus, we obtain the following result.

Theorem 1.2. *Let (X, x) be a pointed topological space. Then $\pi_1^{\text{top}}(X, x)$ is a homogeneous space.*

This note is organized as follows. Section 2 contains definitions and conventions, section 3 proves two lemmas and Theorem 1.1, section 4 addresses change of basepoint, and section 5 shows left and right translation are homeomorphisms.

2. Definitions and conventions

By convention, neighborhoods are open. Unless stated otherwise, homomorphisms are inclusion induced.

Let X be a topological space and $x \in X$. A neighborhood U of x is *relatively inessential* (in X) provided $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. X is *semilocally simply-connected* at x provided there exists a relatively inessential neighborhood U of x. X is *semilocally simply-connected* provided it is so at each $x \in X$. A neighborhood U of x is *strongly relatively inessential* (in X) provided $\pi_1(U, y) \to \pi_1(X, y)$ is trivial for every $y \in U$.

The fundamental group is a functor from the category of pointed topological spaces to the category of groups. Consequently, if A and B are any subsets of X such that $x \in A \subset B \subset X$ and $\pi_1(B, x) \to \pi_1(X, x)$ is trivial, then $\pi_1(A, x) \to \pi_1(X, x)$ is trivial as well. This observation justifies the convention that neighborhoods are open.

If X is locally path connected and semilocally simply-connected, then each $x \in X$ has a path connected relatively inessential neighborhood U. Such a U is necessarily a strongly relatively inessential neighborhood of x, as the reader may verify (see for instance, [9, Exercise 5, p. 330]).

Let (X, x) be a pointed topological space and let $I = [0, 1] \subset \mathbb{R}$. The space

$$C_x(X) = \{f : (I, \partial I) \to (X, x) \mid f \text{ is continuous}\}$$

is endowed with the compact-open topology. The function

$$\begin{aligned} C_x(X) &\xrightarrow{q} \pi_1(X,x) \\ f &\longmapsto [f] \end{aligned}$$

is surjective, so $\pi_1(X,x)$ inherits the quotient topology, and one writes $\pi_1^{\text{top}}(X,x)$ for the resulting *topological fundamental group.* Let $e_x \in C_x(X)$ denote the constant map. If $f \in C_x(X)$, then f^{-1} denotes the path defined by $f^{-1}(t) = f(1-t)$.

3. Proof of Theorem 1.1

We prove two lemmas and then Theorem 1.1.

Lemma 3.1. *Let (X,x) be a pointed topological space. If $\{[e_x]\}$ is open in $\pi_1^{\text{top}}(X,x)$, then x has a relatively inessential neighborhood in X.*

Proof: The quotient map q is continuous and $\{[e_x]\} \subset \pi_1^{\text{top}}(X,x)$ is open, so $q^{-1}([e_x]) = [e_x]$ is open in $C_x(X)$. Therefore, e_x has a basic open neighborhood

$$e_x \in V = \bigcap_{n=1}^{N} V(K_n, U_n) \subset [e_x] \subset C_x(X), \tag{3.1}$$

where each $K_n \subset I$ is compact, each $U_n \subset X$ is open, and each $V(K_n, U_n)$ is a subbasic open set for the compact-open topology on $C_x(X)$. We will show that

$$U = \bigcap_{n=1}^{N} U_n$$

is a relatively inessential neighborhood of x in X. Clearly, U is open in X and, by (3.1), $x \in U$. Finally, let $f : (I, \partial I) \to (U, x)$. For each $1 \le n \le N$, we have

$$f(K_n) \subset U \subset U_n.$$

Thus, $f \in [e_x]$ by (3.1), so $[f] = [e_x]$ is trivial in $\pi_1(X,x)$. □

Lemma 3.2. *Let (X,x) be a pointed topological space and let $f \in C_x(X)$. If X is locally path connected and semilocally simply-connected, then $\{[f]\}$ is open in $\pi_1^{\text{top}}(X,x)$.*

Proof: As q is a quotient map, we must show that $q^{-1}([f]) = [f]$ is open in $C_x(X)$. So let $g \in [f]$. For each $t \in I$, let U_t be a path connected relatively inessential neighborhood of $g(t)$ in X. The sets $g^{-1}(U_t)$, where $t \in I$, form an open cover of I. Let $\lambda > 0$ be a Lebesgue number for this cover. Choose $N \in \mathbb{N}$ so that $1/N < \lambda$. For each $1 \leq n \leq N$, let

$$I_n = \left[\frac{n-1}{N}, \frac{n}{N}\right] \subset I.$$

Reindex the U_t's so that

$$g(I_n) \subset U_n \text{ for each } 1 \leq n \leq N.$$

The U_n's are not necessarily distinct, nor does the proof require this condition. For each $1 \leq n \leq N$, let W_n denote the path component of $U_n \cap U_{n+1}$ containing $g(n/N)$, so

$$g\left(\frac{n}{N}\right) \in W_n \subset (U_n \cap U_{n+1}) \subset X. \tag{3.2}$$

Consider the basic open set

$$V = \left(\bigcap_{n=1}^{N} V(I_n, U_n)\right) \cap \left(\bigcap_{n=1}^{N-1} V\left(\left\{\frac{n}{N}\right\}, W_n\right)\right) \subset C_x(X). \tag{3.3}$$

By construction, $g \in V$. It remains to show that $V \subset [f]$. So, let $h \in V$. As $[g] = [f]$, it suffices to show that $[h] = [g]$.

By (3.3) we have

$$\begin{aligned} h(I_n) &\subset U_n \quad \text{for each } 1 \leq n \leq N \text{ and} \\ h\left(\frac{n}{N}\right) &\in W_n \quad \text{for each } 1 \leq n \leq N-1. \end{aligned} \tag{3.4}$$

For each $1 \leq n \leq N-1$, let $\gamma_n : I \to W_n$ be a continuous path such that

$$\begin{aligned} \gamma_n(0) &= h\left(\frac{n}{N}\right) \quad \text{and} \\ \gamma_n(1) &= g\left(\frac{n}{N}\right), \end{aligned}$$

which exists by (3.2) and (3.4). Let $\gamma_0 = e_x$ and $\gamma_N = e_x$. For each $1 \leq n \leq N$, define

$$\begin{aligned} I &\xrightarrow{s_n} I_n \\ t &\longmapsto \tfrac{1}{N}t + \tfrac{n-1}{N} \end{aligned}$$

and let

$$\begin{aligned} g_n &= g \circ s_n \quad \text{and} \\ h_n &= h \circ s_n. \end{aligned}$$

So, g_n and h_n are affine reparameterizations of $g|_{I_n}$ and $h|_{I_n}$, respectively. For each $1 \leq n \leq N$,

$$\delta_n = g_n * \gamma_n^{-1} * h_n^{-1} * \gamma_{n-1}$$

is a loop in U_n based at $g_n(0)$ (see Figure 1). As U_n is a strongly rel-

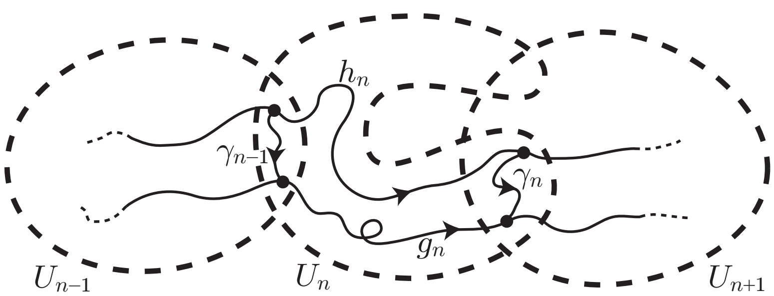

FIGURE 1. Loop $\delta_n = g_n * \gamma_n^{-1} * h_n^{-1} * \gamma_{n-1}$ in U_n based at $g_n(0)$.

atively inessential neighborhood, $[\delta_n] = 1 \in \pi_1(X, g_n(0))$. Therefore, g_n and $\gamma_{n-1}^{-1} * h_n * \gamma_n$ are path homotopic. In $\pi_1(X, x)$, we have

$$\begin{aligned} [h] &= [h_1 * h_2 * \cdots * h_N] \\ &= \left[\gamma_0^{-1} * h_1 * \gamma_1 * \gamma_1^{-1} * h_2 * \gamma_2 * \cdots * \gamma_{N-1}^{-1} * h_N * \gamma_N\right] \\ &= [g_1 * g_2 * \cdots * g_N] \\ &= [g], \end{aligned}$$

proving the lemma. □

In the previous proof, the second collection of subbasic open sets in (3.3) is essential. Figure 2 shows two loops g and h based at x in the annulus $X = S^1 \times I$. All conditions in the proof are satisfied, except $g(1/N)$ and $h(1/N)$ fail to lie in the same connected component of $U_1 \cap U_2$. Clearly, g and h are not homotopic loops.

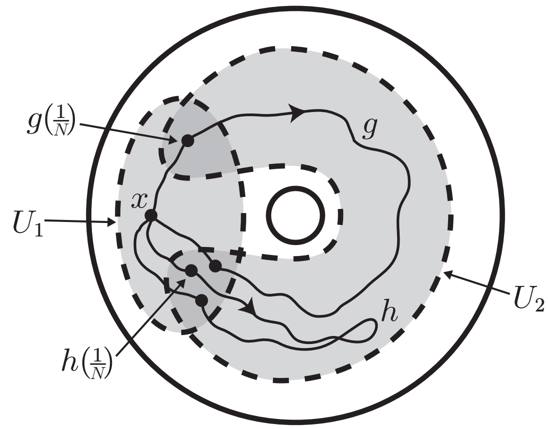

FIGURE 2. Loops g and h based at x in the annulus X.

Proof of Theorem 1.1: First, assume $\pi_1^{top}(X)$ is discrete and let $x \in X$. By definition, $\pi_1^{top}(X,x)$ is discrete, so $\{[e_x]\}$ is open in $\pi_1^{top}(X,x)$. By Lemma 3.1, x has a relatively inessential neighborhood in X. The choice of $x \in X$ was arbitrary, so X is semilocally simply-connected.

Next, assume X is semilocally simply-connected and let $x \in X$. Points in $\pi_1^{top}(X,x)$ are open by Lemma 3.2, so $\pi_1^{top}(X,x)$ is discrete. The choice of $x \in X$ was arbitrary, so $\pi_1^{top}(X)$ is discrete. □

4. Basepoint change

Lemma 4.1. *Let X be a topological space and $x, y \in X$. If x and y lie in the same path component of X, then $\pi_1^{top}(X,x)$ and $\pi_1^{top}(X,y)$ are homeomorphic.*

Proof: Let $\gamma : I \to X$ be a continuous path with $\gamma(0) = y$ and $\gamma(1) = x$. Define the function

$$C_y(X) \xrightarrow{\Gamma} C_x(X)$$
$$f \longmapsto \left(\gamma^{-1} * f\right) * \gamma\,.$$

First, we show that Γ is continuous. Let $I_1 = [0,1/4]$, $I_2 = [1/4,1/2]$, and $I_3 = [1/2,1]$. Define the affine homeomorphisms

$$\begin{array}{ccc} I_1 \xrightarrow{s_1} I & I_2 \xrightarrow{s_2} I & I_3 \xrightarrow{s_3} I \\ t \longmapsto 4t & t \longmapsto 4t-1 & t \longmapsto 2t-1 \end{array}$$

and note that

$$\begin{array}{ll} I \xrightarrow{\Gamma(f)} X & \\ t \longmapsto \gamma^{-1} \circ s_1(t) & 0 \leq t \leq \frac{1}{4} \\ t \longmapsto f \circ s_2(t) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ t \longmapsto \gamma \circ s_3(t) & \frac{1}{2} \leq t \leq 1\,. \end{array}$$

Consider an arbitrary subbasic open set

$$V = V\left(K, U\right) \subset C_x(X).$$

Observe that $\Gamma(f) \in V$ if and only if

$$\gamma^{-1} \circ s_1\left(K \cap I_1\right) \subset U, \tag{4.1}$$

$$f \circ s_2\left(K \cap I_2\right) \subset U, \text{ and} \tag{4.2}$$

$$\gamma \circ s_3\left(K \cap I_3\right) \subset U. \tag{4.3}$$

Define the subbasic open set

$$V' = V\left(s_2\left(K \cap I_2\right), U\right) \subset C_y(X).$$

Observe that $f \in V'$ if and only if (4.2) holds. As conditions (4.1) and (4.3) are independent of f, either $\Gamma^{-1}(V) = \emptyset$ or $\Gamma^{-1}(V) = V'$. Thus, Γ is continuous. Next, consider the diagram

$$\begin{array}{ccc} C_y(X) & \xrightarrow{\Gamma} & C_x(X) \\ {\scriptstyle q_y}\downarrow & & \downarrow{\scriptstyle q_x} \\ \pi_1^{\text{top}}\left(X, y\right) & \overset{\pi(\Gamma)}{\dashrightarrow} & \pi_1^{\text{top}}\left(X, x\right). \end{array}$$

The composition $q_x \circ \Gamma$ is constant on each fiber of q_y, so there is a unique set function making the diagram commute, namely $\pi\left(\Gamma\right) : [f] \mapsto [\Gamma(f)]$. As q_y is a quotient map, the universal property of quotient maps [9, Theorem 11.1, p. 139] implies that $\pi\left(\Gamma\right)$ is continuous. It is well known that $\pi\left(\Gamma\right)$ is a bijection [9, Theorem 2.1, p. 327]. Repeating the above argument with the roles of

x and y interchanged and the roles of γ and γ^{-1} interchanged, we see that $\pi(\Gamma)^{-1}$ is continuous. Thus, $\pi(\Gamma)$ is a homeomorphism as desired. □

5. Translation

Lemma 5.1. *Let (X, x) be a pointed topological space. If $[f] \in \pi_1^{\text{top}}(X, x)$, then left and right translation by $[f]$ are self homeomorphisms of $\pi_1^{\text{top}}(X, x)$.*

Proof: Fix $[f] \in \pi_1^{\text{top}}(X, x)$ and consider left translation by $[f]$ on $\pi_1^{\text{top}}(X, x)$

$$\begin{aligned} \pi_1^{\text{top}}(X, x) &\xrightarrow{L_{[f]}} \pi_1^{\text{top}}(X, x) \\ [g] &\longmapsto [f] \cdot [g] \,. \end{aligned}$$

Plainly, $L_{[f]}$ is a bijection of sets. Consider the commutative diagram

$$\begin{array}{ccc} C_x(X) & \xrightarrow{L_f} & C_x(X) \\ \downarrow q & & \downarrow q \\ \pi_1^{\text{top}}(X, x) & \xrightarrow{L_{[f]}} & \pi_1^{\text{top}}(X, x) \,, \end{array} \tag{5.1}$$

where L_f is defined by

$$\begin{aligned} C_x(X) &\xrightarrow{L_f} C_x(X) \\ g &\longmapsto f * g \,. \end{aligned}$$

First, we show L_f is continuous. Let $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. Define the affine homeomorphisms

$$\begin{aligned} I_1 &\xrightarrow{s_1} I & \qquad I_2 &\xrightarrow{s_2} I \\ t &\longmapsto 2t & t &\longmapsto 2t - 1 \end{aligned}$$

and note that

$$\begin{aligned} I &\xrightarrow{f*g} X \\ t &\longmapsto f \circ s_1(t) & \qquad 0 \le t \le \tfrac{1}{2} \\ t &\longmapsto g \circ s_2(t) & \qquad \tfrac{1}{2} \le t \le 1 \,. \end{aligned}$$

Consider an arbitrary subbasic open set

$$V = V(K, U) \subset C_x(X).$$

Observe that $f * g \in V$ if and only if

$$f \circ s_1 (K \cap I_1) \subset U \text{ and} \tag{5.2}$$

$$g \circ s_2 (K \cap I_2) \subset U. \tag{5.3}$$

Define the subbasic open set

$$V' = V(s_2(K \cap I_2), U) \subset C_x(X).$$

Observe that $g \in V'$ if and only if (5.3) holds. As condition (5.2) is independent of g, either $L_f^{-1}(V) = \emptyset$ or $L_f^{-1}(V) = V'$. Thus, L_f is continuous. The composition $q \circ L_f$ is constant on each fiber of the quotient map q and (5.1) commutes, so the universal property of quotient maps [9, Theorem 11.1, p. 139] implies that $L_{[f]}$ is continuous.

Applying the previous argument to f^{-1}, we get $L_{[f]}^{-1} = L_{[f^{-1}]}$ is continuous and $L_{[f]}$ is a homeomorphism. The proof for right translation is almost identical. □

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(Calcut) Department of Mathematics; Michigan State University; East Lansing, Michigan 48824-1027
E-mail address: jack@math.msu.edu
URL: http://www.math.msu.edu/∼jack/

(McCarthy) Department of Mathematics; Michigan State University; East Lansing, Michigan 48824-1027
E-mail address: mccarthy@math.msu.edu
URL: http://www.math.msu.edu/∼mccarthy/