SYMPLECTIC GLUING ALONG HYPERSURFACES AND RESOLUTION OF ISOLATED ORBIFOLD SINGULARITIES

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0. **Introduction**

This paper is concerned with two themes of symplectic topology. The first is the development of techniques to construct symplectic manifolds and, in particular, compact symplectic 4-manifolds. The second is the resolution of symplectic singularities and, in particular, the resolution of isolated singularities in symplectic 4-manifolds.

On the first topic we prove a theorem which allows the gluing of two symplectic manifolds along a special class of hypersurfaces that we call *ω-compatible hypersurfaces*. Let (X, ω) be a symplectic $2n$-manifold and $M \subset X$ a hypersurface with a fixed point free S^1-action. M is called ω-compatible if the orbits of the action lie in the null directions of $\omega|_M$. An ω-compatible hypersurface M has a canonical co-orientation. Hence, if M is a separating hypersurface, then M divides X into distinguished components X^- and X^+. In dimension 4, our main gluing theorem is as follows.

Theorem 4.2 . *Let Y be a Seifert 3-manifold. Let (X_i, ω_i), $i = 1, 2$, be symplectic 4-manifolds and suppose that there are ω_i-compatible embeddings $j_i : Y \to (X_i, \omega_i)$, $i = 1, 2$ such that $j_i(Y)$ is a separating hypersurface in X_i. Then there is a symplectic structure ω on X*

$$X = X_1^- \bigcup_Y X_2^+$$

obtained by gluing X_1^- to X_2^+ along Y. Moreover, there are neighborhoods $N_i(Y)$ of Y in X_i such that $\omega = \omega_2$ on $X_2^+ \setminus N_2(Y)$ and $\omega = c\omega_1$ on $X_1^- \setminus N_1(Y)$ for some constant $c > 0$.

In higher dimensions there is a similar gluing theorem but an additional assumption on the forms $\omega_i|_M$ is necessary (cf. Theorem 4.1). An ω-compatible hypersurface is a special case of the ω-convex (or ω-concave) hypersurfaces studied by Eliashberg and Gromov [E-G]. However, a gluing theorem such as given here is not possible along arbitrary ω-convex hypersurfaces. Further conditions are necessary. The choice of the ω-compatible condition is dictated by our applications and the simplicity of the condition.

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The symplectic normal connect sum described in [Go] and [M-W1] can be viewed as a simple example of gluing along ω-compatible hypersurfaces. However, the main application of the gluing theorem in this paper is to the second topic, the resolution of singularities in symplectic 4-manifolds. Let Y be a topological space and p a point of Y such that $Y \setminus p$ is a symplectic 4-manifold with symplectic form τ. Let U be a neighborhood of p. We say a symplectic 4-manifold $(\tilde{Y}, \tilde{\tau})$ is a *symplectic resolution of p on U* if there is:

(i) a tubular neighborhood W of a symplectic divisor D in $\tilde{Y}$,
(ii) a map $\pi : (\tilde{Y}, D) \to (Y, p)$ such that $\pi : \tilde{Y} \setminus W \to Y \setminus U$ is a symplectic diffeomorphism.

A *symplectic divisor* in $\tilde{Y}$ is a set D of symplectically embedded surfaces which intersect transversely. A *symplectic star* is a symplectic divisor which has the structure of a star-like graph where each surface is a vertex and each point of intersection is an edge. We prove the following result.

Theorem 6.1 . *Let U be a neighborhood of an isolated orbifold point p on a 4-dimensional symplectic orbifold (X, ω). There exists a symplectic resolution $(\tilde{X}, \tilde{\omega})$ of p on U. Morover, the symplectic divisor D in $\tilde{Y}$ is a symplectic star.*

To prove this we first observe that an isolated orbifold point p has a neighborhood U such that $\partial U = M$ is a $\tau-$compatible hypersurface. Here τ is the orbifold symplectic form. Then we construct a symplectic 4-manifold (W, ω) which is the tubular neighborhood of a symplectic divisor, such that $\partial W = M$ is ω-compatible. Applying the gluing theorem we delete U and glue in W along M to construct the resolution. The graph of the symplectic divisor of this resolution is determined by the topology of the singularity alone. Thus, topologically, the resolution is canonical.

In algebraic geometry there are many different techniques used to resolve isolated surface singularities (see, for example, Laufer [L] or Barth-Peters-Van de Ven [B-P-V]). Since we resolve the orbifold singularity by gluing to a standard model, our technique is similar, in spirit, to that of Barth-Peters-Van de Ven. This approach seems to us the most natural in symplectic geometry. Moreover, it leads to the construction (and resolution) of a large family of isolated symplectic singularities which we call symplectic star singularities. These singularities include orbifold singularities but many others as well. Unfortunately, at this time, we do not have an intrinsic geometric characterization of such singularities.

Symplectic manifolds with properties like W have been constructed by other workers. Audin [A] describes a construction similar to ours but relying on the Morse theoretic properties of the hamiltonian. Eliashberg [E] gives a plumbing construction along lagrangian submanifolds to build contact structures on spheres. Our construction relies on blowing up and the proper transform as is appropriate for resolving singularities.

Many of the ideas in this paper have their origin in a remarkable section of Gromov's book, *Partial Differential Relations* [G,§3.4.4]. The best known of these is that of blowing up and blowing down in symplectic geometry. It is not as well known that the idea of the symplectic normal connect sum, exploited by Gompf [Go] and the authors [M-W1], was also first described here. Finally, the problem of resolving symplectic singularities is posed in this section for the first time. We remark that in this section Gromov described a resolution scheme for the singularities of immersed symplectic manifolds which is not correct without imposing further conditions on the singularities. See [M-W2] for an example of an immersed symplectic surface whose singularities cannot be resolved by blowing up.

The paper is organized as follows. The main technical part of the paper, the construction of W with ω-compatible boundary is accomplished in Sections 1 and 2. The gluing theorem is given in Sections 3 and 4. The remainder of the paper combines these two constructions to give our applications. The reader interested only in the gluing theorem can skip Sections 1 and 2.

1. **Preliminaries**

In this section, we discuss background material on G-invariant symplectic structures, blow-ups and proper transforms. The reader may wish to proceed to the next section and refer back to this section as the need arises.

Let G be a compact group acting by symplectomorphisms on a symplectic 4-manifold (X, ω). Let P be a fixed point of G and (z, w) be Darboux coordinates in a G-invariant neighborhood U of P such that $P = (0, 0)$. We say that (z, w) are *normalized Darboux coordinates at* P if the action of G on U is given in these coordinates by a representation:

$$\rho : G \to \mathbf{U}(2) \quad g \cdot (z, w) = \rho(g)(z, w). \tag{1.1}$$

By the equivariant normal form theorem [W1], such coordinates always exist. Suppose that (z, w) are normalized Darboux coordinates at P. For sufficiently small ϵ, the open ball $B_\epsilon = \{(z, w)|z\overline{z} + w\overline{w} < \epsilon^2\}$ determines a G-invariant neighborhood of P contained in U. If G is abelian, then we may assume that $\rho(G)$ lies in the diagonal subgroup $\mathbf{U}(1) \times \mathbf{U}(1)$ of $\mathbf{U}(2)$. If G is a nontrivial group, then each connected component of the set of fixed points of G is either an embedded symplectic surface of fixed points of G or an isolated fixed point of G. This follows from the simple observation that the set of fixed points of a nontrivial subgroup of $\mathbf{U}(2)$ is either $(0, 0)$ or a complex line.

There is an operation of blowing up for G-invariant symplectic structures provided we blow-up at fixed points of G. The operation of blowing up was first described by Gromov [G]. This operation has a very clean description in terms of the approach of Guillemin and Sternberg, which we will follow. Let P be a fixed point of G. Suppose that (z, w) are normalized Darboux coordinates at P on a neighborhood U of P. Let ϵ be a positive number such that $B_\epsilon \subset U$ and δ be a positive number less than ϵ. Following the discussion in [G-S], we may describe the blow-up $(\tilde{X}, \tilde{\omega})$ of (X, ω) at P as follows. Let $L \subset \mathbb{CP}^1 \times \mathbb{C}^2$ be the canonical bundle of $\mathbb{CP}^1$ given by the incidence relation:

$$L = \{([z, w], (u, v)) | zv = wu\}.$$

Let Ω be the standard symplectic form on $\mathbb{CP}^1$ and τ be the standard form on $\mathbb{C}^2$. Let ω_δ be the form $i^*[\delta^2 pr_1^*(\Omega) + pr_2^* \tau]$ where i is the inclusion of L into $\mathbb{CP}^1 \times \mathbb{C}^2$ and pr_j is the projection of $\mathbb{CP}^1 \times \mathbb{C}^2$ onto its jth factor. The standard actions of $\mathbf{U}(2)$ on $\mathbb{CP}^1$ and $\mathbb{C}^2$ induce the product action on $\mathbb{CP}^1 \times \mathbb{C}^2$. This action preserves the incidence relation L. Since Ω and τ are $\mathbf{U}(2)$-invariant, ω_δ is $\mathbf{U}(2)$-invariant. On the other hand, the symplectic form ω on B_ϵ is invariant under the standard $\mathbf{U}(2)$ action on B_ϵ. Guillemin and Sternberg provide a $\mathbf{U}(2)$-equivariant symplectomorphism:

$$\phi : (B_\epsilon \setminus \overline{B}_\delta, \omega) \to (\mathcal{N}(\Sigma_\prime) \setminus \Sigma_\prime, \omega_\delta)$$

where $\mathcal{N}(\Sigma_\prime)$ is a tubular neighborhood of the zero section Σ_0 of L. Then $(\tilde{X}, \tilde{\omega})$ is obtained by gluing $X^* = X \setminus \overline{B}_\delta$ and $\mathcal{N}(\Sigma_\prime)$ along $B_\epsilon \setminus \overline{B}_\delta$ and $\mathcal{N}(\Sigma_\prime) \setminus \Sigma_\prime$ via the symplectomorphism ϕ. The resulting form $\tilde{\omega}$ is uniquely determined by the property that its restriction to X^* is ω and its restriction to $\mathcal{N}(\Sigma_\prime)$ is ω_δ. Note that the G action on X preserves X^*. Moreover, G acts on $B_\epsilon \setminus \overline{B}_\delta$ via the representation $\rho : G \to \mathbf{U}(2)$. This same representation affords an action on $\mathcal{N}(\Sigma_\prime)$ and $\mathcal{N}(\Sigma_\prime) \setminus \Sigma_\prime$. Since ϕ is $\mathbf{U}(2)$ equivariant, it is *a fortiori* G-equivariant. Hence, there is a well defined G action on $\tilde{X}$. Moreover, $\tilde{\omega}$ is invariant under this action. Note that this operation of blow-up in the context of G-invariant symplectic structures generalizes the usual operation of blow-up for which G is the trivial group and every point is a fixed point of G.

Note that the closure of X^* in X is equal to $X \setminus B_\delta$, whereas X^* is dense in $\tilde{X}$. It is easy to see, from the description of the gluing map ϕ provided by Guillemin and Sternberg, that the natural inclusion of X^* into $\tilde{X}$ given by the above construction, actually extends continuously to a map $\eta : X \setminus B_\delta \to \tilde{X}$ which maps the boundary S^3_δ of the ball B_δ onto Σ_0 by collapsing the Hopf fibers on S^3_δ. Hence, η exhibits $\tilde{X}$ as the quotient of $X \setminus B_\delta$ obtained by collapsing the Hopf fibers on S^3_δ.

Since $\tilde{X} = X^* \cup \Sigma_0$, the fixed point set of the action on $\tilde{X}$ is the union of the fixed point set on X^* and the fixed point set on Σ_0. We may determine the fixed points on Σ_0 by considering the action on $\mathcal{N}(\Sigma_\prime)$:

$$g \cdot ([z, w], (u, v)) = ([\rho(g)(z, w)], \rho(g)(u, v)).$$

Σ_0 is the locus of the equation $(u, v) = (0, 0)$. On this locus, the action is given by the rule:

$$g \cdot ([z, w], (0, 0)) = ([\rho(g)(z, w)], (0, 0)).$$

Hence, the fixed points on Σ_0 correspond to the complex eigenlines of the action of the subgroup $\rho(G)$ of $\mathbf{U}(2)$ on $\mathbb{C}^2$.

We recall that symplectic blow-up is analogous to the operation of blowing up in the complex category. Symplectic blow-up, however, fails to have some of the important features of complex blow-up. For instance, in the complex setting, there is a holomorphic map $\sigma : \tilde{X} \to X$ such that $\sigma| : \tilde{X} \setminus \Sigma_0 \to X \setminus \{P\}$ is a biholomorphism. If C is a complex curve, then the closure $\overline{C}$ in $\tilde{X}$ of $\sigma^{-1}(C \setminus \{P\})$ is also a complex curve. $\overline{C}$ is called the proper transform of C in $\tilde{X}$. If C has an ordinary double point at P, this double point is removed by passing to $\overline{C}$. As we show in [M-W2], it is not possible to define an operation of proper transform in the symplectic category for arbitrary immersed symplectic surfaces so that double point singularities are removed by passing to the proper transform. On the other hand, we can define an operation of proper transform for embedded surfaces, provided we choose suitably adapted Darboux coordinates.

Lemma 1.1. *Let C be an embedded symplectic surface in a symplectic 4-manifold (X, ω), $P \in C$, and (z, w) Darboux coordinates in a neighborhood U of P. Let $(\tilde{X}, \tilde{\omega})$ be the corresponding blow-up of (X, ω) at P. Suppose that C meets U in a complex line, l. Let F be the fiber corresponding to l of the disc bundle $\mathcal{N}(\Sigma_{\prime})$. Then the closure $\overline{C}$ in $\tilde{X}$ of $C \cap X^*$ is an embedded symplectic surface, called the proper transform of C, meeting $\mathcal{N}(\Sigma_{\prime})$ in the fiber F.*

Proof. From the description in Guillemin-Sternberg, we see that the the gluing map ϕ identifies $l \cap B_\epsilon$ with the deleted fiber $F \setminus \{Q\}$ where Q is the point of intersection of F and Σ_0. The result is immediate. □

It is easy to see that we can always choose Darboux coordinates satisfying the hypotheses of the previous lemma, provided C is an embedded symplectic surface. Hence, we can always form a proper transform of C in an appropriate blow-up of (X, ω) at P. We shall now introduce a class of immersed surfaces called graphs for which we can extend this operation of proper transform. This class will be sufficient for our purposes.

Graphs, stars, and strings. We recall from [O], that a *graph* $\mathcal{G}$ is a finite 1-dimensional, simplicial complex. Let $A_0, \dots, A_n$ denote the vertices of a graph $\mathcal{G}$. The *valence* of A_i is the number of edges of $\mathcal{G}$ which contain A_i. A_i is *extreme* if it has valence 1. A *star* is a contractible graph where at most one vertex has valence greater than 2. If there is such a vertex, say A_0, we call it the *center*. If there is no such vertex, then $\mathcal{G}$ is a linear graph. For convenience, we allow the term center to apply to any component of a linear graph. $\mathcal{G}[\lfloor_{\infty}, \dots, \lfloor_{f}]$ denotes a linear graph with s vertices $A_1, \dots, A_s$, $s-1$ edges $[A_1, A_2], \dots, [A_{s-1}, A_s]$, $g_i = 0$ and $m_i = -b_i$.

Let ω be an S^1-invariant symplectic structure on a 4-manifold X. Suppose that to each vertex A_i of a graph $\mathcal{G}$ we associate a smoothly embedded symplectic surface in (X, ω), C_i. Let Γ denote the union $\Gamma = \bigcup_{i=0}^{n} C_i$.

We say that the immersed symplectic surface Γ is a *graph of type* $\mathcal{G}$ if the following conditions are satisfied:

- C_i meets C_j if and only if A_i and A_j are connected by an edge of $\mathcal{G}$,
- C_i and C_j are transverse,
- no point in X lies in more than two of the surfaces C_i,
- C_i and C_j are disjoint or meet at one point,
- C_i is S^1-invariant,
- for each fixed point P of S^1 in Γ, there exist normalized Darboux coordinates at P on a neighborhood U of P such that Γ meets U in a complex line through P. (We say that these coordinates are Γ-normalized.)

Suppose that an immersed symplectic surface Γ is a graph. It is evident from the definition that Γ determines the simplicial complex $\mathcal{G}$. Let g_i denote the genus of C_i and m_i denote the self-intersection $C_i \cdot C_i$. In this manner, we associate a weighted graph $\mathcal{G}(\Gamma)$, in the sense of [O], to Γ. If U is a regular neighborhood of Γ in X, then U is diffeomorphic to the interior of $K(\mathcal{G}(\Gamma))$, the 4-manifold obtained by equivariant plumbing along the weighted graph $\mathcal{G}(\Gamma)$ ([O]).

Remark 1.1. Γ-normalized Darboux coordinates exist if P is a smooth point on a component Σ of Γ which is fixed pointwise by S^1. Indeed, in this case, if (z, w) are normalized Darboux coordinates at P on a neighborhood U of P such that U meets only those components of Γ which contain P, then (z, w) are Γ-normalized Darboux coordinates at P. This follows from the simple observation that the set of fixed points of a nontrivial subgroup of $\mathbf{U}(2)$ is either $(0, 0)$ or a complex line. Let P be a double point of Γ. If there exist Γ-normalized Darboux coordinates (z, w) at P, then the two components of Γ containing P have positive intersection at P. This property of Γ is a necessary, but not sufficient, condition for the existence of Γ-normalized Darboux coordinates at P. It is easy to construct counterexamples based upon the discussion in section 3 of [McD2] concerning transverse symplectic planes.

Let Γ be a graph. We refer to C_i as a *component of* Γ. Note that Γ is an immersed surface with only double point singularities. A point $P \in \Gamma$ is a *double point of* Γ if P lies on more than one component of Γ. Otherwise, P is a *smooth point of* Γ. C_i is *extreme* if A_i is extreme. Γ is a *string* if $\mathcal{G}(\Gamma)$ is a linear graph. Γ is a *star* if $\mathcal{G}(\Gamma)$ is a star. If Γ is a star, then C_0 is a *center for* Γ if A_0 is a center for $\mathcal{G}(\Gamma)$ and C_0 is fixed pointwise by S^1. If Γ is not a string, then Γ has a unique center. Otherwise, there is at most one center for Γ. Henceforth, we assume that all stars have a center. If C_0 is the center for a star Γ, we may write Γ as a union:

$$\Gamma = C_0 \cup \Big(\bigcup_{1 \le j \le r} C^j \Big)$$

where $\bigcup_{1\leq j\leq r} C^j$ is a disjoint union of strings C^j and C^j meets C_0 at a single point P_j on an extreme component $C_{j,1}$ of C^j. We say that C^j is a *branch of* Γ *emanating from* C_0. Note that there is a well-defined ordering to the components of each branch of Γ emanating from C_0.

Symplectic S^1 actions and fixed points. Suppose that ω is an S^1-invariant symplectic structure on a 4-manifold X, P is a fixed point of S^1 and the action of S^1 is diagonalized at P by the Darboux coordinates (z, w). Then the associated representation ρ is the diagonal representation of type (p, q) for some relatively prime integers p and q:

$$t \cdot (z, w) = (t^p z, t^q w).$$

We say that P is a fixed point of *type* (p, q) and refer to (z, w) as (p, q) *Darboux coordinates at* P. If P is a fixed point of S^1, then we can linearize the action of S^1 at P, (i.e. we can consider the induced action on the tangent space to X at P). Note that in (p, q) Darboux coordinates the action at P is naturally identified with its linearization. The coordinate lines of the (p, q) Darboux coordinate system correspond to a symplectic splitting of $T_P(X)$. Hence, in order to compute the type of a fixed point P, it suffices to consider the linearization of the action of S^1 at P and find an equivariant symplectic splitting of $T_P(X)$. This observation will be useful for computing the type of a fixed point.

If p and q are both nonzero, then P is an isolated fixed point of the action. On the other hand, if $p = 0$ or $q = 0$, then P lies on a surface of fixed points which meets B_ϵ in one of the complex coordinate lines. In any case, the action on B_ϵ is Hamiltonian with Hamiltonian $-pz\overline{z} - qw\overline{w}$, (up to a constant). Hence, the index of P is 0, if $p, q < 0$; 4, if $p, q > 0$; and 2 if p and q have opposite signs. Otherwise, P is a degenerate fixed point with Hessian of rank 2. In these cases, P lies on a surface of minima or maxima of H.

Suppose that (z, w) are (p, q) Darboux coordinates at P. Let $(\tilde{X}, \tilde{\omega})$ be the corresponding blow-up of (X, ω) at P. As explained above, the fixed points on Σ_0 correspond to the complex eigenlines of the diagonal action of type (p, q) on $\mathbb{C}^2$. To compute the types of fixed points on Σ_0 we consider the standard charts covering $\mathcal{N}(\Sigma_\prime)$, $N_1 = \{([1, w], (u, wu)) \in \mathcal{N}(\Sigma_\prime)|\sqcap, \sqsupseteq \in \mathbb{C}\}$ and $N_2 = \{([z, 1], (zv, v)) \in \mathcal{N}(\Sigma_\prime)|\ddagger, \sqsubseteq \in \mathbb{C}\}$. N_1 is S^1-invariant and the action on N_1 is given by:

$$t\cdot([1, w], (u, wu)) = ([t^p, t^q w], (t^p u, t^q wu)) = ([1, t^{q-p} w], (t^p u, (t^{q-p} w)(t^p u))).$$

Suppose that $Q = ([1, w_0], (0, 0))$ is a fixed point on Σ_0 lying in N_1. Σ_0 corresponds to the equation $u = 0$ and $F_{[1,w_0]}$ corresponds to the equation $w = w_0$. While (u, w) is not a Darboux coordinate system in a neighborhood of Q, Σ_0 and $F_{[1,w_0]}$ are symplectically orthogonal at Q. Hence, the surfaces $u = 0$ and $w = 0$ give rise to an equivariant symplectic splitting of $T_Q(L)$. Thus, by the above description of the action in N_1, the linearization of the action of S^1 at Q is a diagonal action of type $(p, q - p)$. Hence, if Q is any fixed point on Σ_0 lying in N_1, then Q is a fixed point of type $(p, q-p)$. Likewise, if Q is any fixed point on Σ_0 lying in N_2, then Q is a fixed point of type $(p-q, q)$. If $p = q$, then every complex line is a complex eigenline and, hence, Σ_0 is a surface of fixed points of type $(p, 0)$. Otherwise, there are precisely two complex eigenlines of the action, namely $w = 0$ and $z = 0$. Hence, if $p \neq q$, then there are exactly two fixed points on Σ_0, one of type $(p-q, q)$ and the other of type $(p, q-p)$.

If p and q are both nonzero, then P is the only fixed point in $\overline{B}_\delta$ of the action on X. Hence, in this case, the fixed point set on X^* corresponds to the complement of P in the fixed point set of X. In passing to the blow-up, therefore, we have effectively replaced P by the fixed points on Σ_0. If $p = q$, then we have replaced P by the surface Σ_0 of fixed points of type $(p, 0)$. Otherwise, we have replaced P by two fixed points, one of type $(p-q, q)$ and the other of type $(p, q-p)$. Note that P has index 2 if and only if p and q have opposite signs. Hence, if P has index 2, the two new fixed points also have index 2.

If $p = 0$ or $q = 0$, then the component Σ of the fixed point set on X which passes through P is a surface meeting B_ϵ in one of the coordinate lines D_ϵ. We may assume that $p = \pm 1$ and $q = 0$, so that D_ϵ is given by the equation $z = 0$. Let Q be the point $([0,1],(0,0))$ on Σ_0 corresponding to the coordinate line, $z = 0$, and F be the fiber of $\mathcal{N}(\Sigma_{,})$ passing through Q. The fixed point set on X^* corresponds to the complement of the closed disc $\overline{D}_\delta$ in the fixed point set of X. The gluing map ϕ identifies $D_\epsilon \setminus \overline{D}_\delta$ with the deleted fiber $F \setminus \{Q\}$. In this manner, we see that the proper transform $\overline{\Sigma}$ of Σ is a surface of fixed points. Hence, the fixed point set in $\tilde{X}$ is obtained from the fixed point set on X by replacing the surface of fixed points Σ of type $(p, 0)$ by its proper transform $\overline{\Sigma}$, a surface of fixed points of type $(p, 0)$, together with the new isolated fixed point $([1,0],(0,0))$ on Σ_0 of type $(p, -p)$. Note that this new isolated fixed point is of index 2.

We have established the following lemma.

Lemma 1.2. *Let X be a 4-manifold equipped with an S^1-invariant symplectic form ω. Let $P \in X$ be a fixed point of S^1 and $(\tilde{X}, \tilde{\omega})$ be the blow-up of (X, ω) at P. (a) If P lies on a surface Σ of fixed points of S^1, then the fixed point set in $\tilde{X}$ consists of the fixed points in X^*, the proper transform $\overline{\Sigma}$ and an isolated fixed point of index 2 on Σ_0. (b) If P is an isolated fixed point of index 2, then the fixed point set in $\tilde{X}$ consists of the fixed points in X^* and two isolated fixed points of index 2 on Σ_0.*

Suppose, furthermore, that ω is Hamiltonian with Hamiltonian H. Let H^* be the restriction of H to X^*. Since X^* is S^1-invariant, H^* is an S^1-Hamiltonian on X^*. Since $\mathcal{N}(\Sigma_{,})$ is simply connected, the action on $\mathcal{N}(\Sigma_{,})$ is Hamiltonian with Hamiltonian H_δ. Since $X^* \cap \mathcal{N}(\Sigma_{,})$ is connected, H^* and H_δ must differ by a constant on the overlap. By adjusting H_δ by this constant, therefore, we may assume that H^* and H_δ agree on the overlap. Hence, there is a well-defined Hamiltonian $\tilde{H}$ for the action on

$\tilde{X}$. Since X^* is dense in $\tilde{X}$, $\tilde{H}$ is the unique continuous extension of H^*. The critical points of $\tilde{H}$ are the fixed points of the action on $\tilde{X}$. Suppose that Q is a critical point of $\tilde{H}$. If Q lies in X^*, then $\tilde{H}(Q) = H(Q)$. Otherwise, $Q \in \Sigma_0$. In this case Q corresponds to an invariant circle α on the boundary S^3_δ of $\overline{B}_\delta$. Since $\tilde{H}$ is the unique continuous extension of H^*, $\tilde{H}(Q) = H(\alpha)$. On U, the Hamiltonian H satisfies the equation $H(z,w) = H(P) - pz\overline{z} - qw\overline{w}$. If $p = q$, then $\tilde{H}(Q) = H(\alpha) = H(P) - p\delta^2$. If $p \neq q$, then either $\alpha = S^3_\delta \cap \{(z,w) \in U | w = 0\}$ or $\alpha = S^3_\delta \cap \{(z,w) \in U | z = 0\}$. In the first case, $\tilde{H}(Q) = H(\alpha) = H(P) - p\delta^2$. In the second case, $\tilde{H}(Q) = H(\alpha) = H(P) - q\delta^2$. Note that the critical value of Q can be made arbitrarily close to the critical value of P by choosing a sufficiently small δ.

2. Graphs, Stars and Seifert 3-Manifolds

Let S be a weighted star with every vertex $A_{j,i}$, except the center, of genus $g_{j,i} = 0$ and self-intersection $m_{j,i} \leq -2$. Then every Seifert 3-manifold Y of type $\{b; (o, g, 0, 0); (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)\}$ is isomorphic to the boundary of an equivariant plumbing according to such a weighted star S [O]. The weighted star S is determined from the Seifert invariants of Y as follows. For each integer j with $1 \leq j \leq r$, consider the continued fraction expansion:

$$\frac{\alpha_j}{\alpha_j - \beta_j} = [b_{j,1}, \dots, b_{j,s_j}]. \tag{2.1}$$

Then let S be the weighted star with center A_0 of genus g and weight $-b-r$ and r linear branches, $\mathcal{G}[\lfloor_{|,\infty}, \dots, \lfloor_{|,\int_|}]; \infty \leq | \leq \nabla$. The equivariant plumbing is determined by the condition that the action on the base corresponding to A_0 is trivial. Note that, conversely, S determines the Seifert invariants of Y. We shall say that Y is a Seifert 3-manifold of type S.

Let Y be a Seifert 3-manifold of type S. In equivariant plumbing, Y is produced as the boundary of a tubular neighborhood of the zero sections of appropriate bundles. We show that an analogous construction is possible in the symplectic category using blow-up instead of plumbing. We would like to produce the desired star by blowing-up from a basic object. However, blow-up necessarily produces stars Γ with overgrown branches. (In particular, branches produced by blow-up always have at least one -1-curve, whereas the branches we wish to realize have no such curves.) Hence, we shall have to prune the resulting branches by restricting the neighborhood of Γ. This construction needs to be carried out in such a way that we have a complete understanding of the symplectic structure in a neighborhood of Y. Based on the results of our previous paper, [M-W1], the natural strategy is to construct (X, ω) so that ω is an S^1-Hamiltonian symplectic structure and Y is a regular level set of the Hamiltonian H. To carry out this strategy, we must overcome some complications which arise at the pruning stage of the construction.

Blowing-up graphs. To construct both stars and graphs we iterate the procedure of blowing up and taking proper transforms as described in Section 1. In fact everything works easily because our definition of a graph Γ (of type $\mathcal{G}$) includes normalized Darboux coordinates at the points where we blow up. However this means that we must show the existence of such coordinates on the proper transform of Γ if we wish to iterate the procedure. This technical result is done in the lemmas of this subsection. First we define the total transform of a graph.

Let Γ be a graph and let $P \in \Gamma$ be a fixed point of S^1. Suppose that (z, w) are Γ-normalized Darboux coordinates at P. Let $(\tilde{X}, \tilde{\omega})$ be the blow-up of (X, ω) at P and Σ_0 be the corresponding -1-curve. Let $\overline{C_i}$ be the proper transform of the component C_i of Γ. We define the *total transform of* Γ to be the union $\tilde{\Gamma}$ of the surfaces $\overline{C_0}, \dots, \overline{C_n}$ and Σ_0.

Lemma 2.1. *Let (x, y) be normalized Darboux coordinates at Q for the action of the stabilizer $\mathcal{S}_Q$ of Q in $U(2)$ on an $\mathcal{S}_Q$-invariant neighborhood of Q in $\mathcal{N}(\Sigma_\prime)$. Let F be the fiber of L through Q. Then, Σ_0 and F meet B_ϵ in complex lines.*

Proof. The standard action of $\mathbf{U}(2)$ on L is given by :

$$A \cdot ([z, w], (u, v)) = ([A(z, w)], A(u, v)). \tag{2.2}$$

Since Σ_0 is the locus of the equation $(u, v) = (0, 0)$, it is $\mathbf{U}(2)$-invariant. The action on this locus is given by :

$$A \cdot ([z, w], (0, 0)) = ([A(z, w)], (0, 0)). \tag{2.3}$$

From this, it is clear that the stabilizer $\mathcal{S}_Q$ of Q (with respect to the action of $\mathbf{U}(2)$ on L) is the subgroup of $\mathbf{U}(2)$ which preserves the complex line $[z_0, w_0] \subset \mathbb{C}^2$. Hence, $\mathcal{S}_Q$ is conjugate in $\mathbf{U}(2)$ to the diagonal subgroup $\mathbf{U}(1) \times \mathbf{U}(1)$. In particular, $\mathcal{S}_Q$ is isomorphic to $S^1 \times S^1$.

Let $\rho : \mathcal{S}_Q \to \mathbf{U}(\in)$ be the representation associated to the normalized Darboux coordinates (x, y) as in equation 1.1. If $A \in \mathbf{U}(2) \setminus \{I\}$, then the fixed point set of A in L lies in Σ_0. Hence, no nontrivial element of $\mathbf{U}(2)$ fixes every point of an open set. So the action of $\mathcal{S}_Q$ on U is effective and, hence, the representation ρ is faithful. It follows that $\rho(\mathcal{S}_Q)$ is an abelian subgroup of $\mathbf{U}(2)$ isomorphic to $S^1 \times S^1$. Hence, $\rho(\mathcal{S}_Q)$ is conjugate in $\mathbf{U}(2)$ to $\mathbf{U}(1) \times \mathbf{U}(1)$. It follows that we may assume that $\rho(\mathcal{S}_Q) = \mathbf{U}(\infty) \times \mathbf{U}(\infty)$ and $\rho : \mathcal{S}_Q \to \mathbf{U}(\infty) \times \mathbf{U}(\infty)$ is an isomorphism.

The action of $\mathbf{U}(2)$ on $\mathcal{N}(\Sigma_\prime)$ is given by automorphisms of the complex line bundle L. Hence $\mathcal{S}_Q$ preserves Σ_0 and F. Let $V = T_Q(\Sigma_0)$ and $W = T_Q(F)$. Then V and W are a pair of distinct 2-dimensional planes in $T_Q(\mathcal{N}(\Sigma_\prime))$ invariant under the induced action of $\mathcal{S}_Q$ on $T_Q(\mathcal{N}(\Sigma_\prime))$. Since the action of $\mathcal{S}_Q$ is linear in the coordinates (x, y), V and W determine a pair of 2-dimensional planes in U invariant under the diagonal group $\mathbf{U}(1) \times \mathbf{U}(1)$. In particular, since $(i, i) \cdot (x, y) = i(x, y)$, V and W are complex lines in U. Since the only complex lines invariant under the diagonal group are those given by $y = 0$ and $x = 0$, we may assume without loss of generality that $V = \{(x, y) \in U | y = 0\}$ and $W = \{(x, y) \in U | x = 0\}$.

Let $D_\epsilon = \{x \in \mathbb{C} | x\overline{x} < \epsilon^2\}$. Since Σ_0 is tangent to V at Q, for sufficiently small δ there exists a smooth function $f : D_\delta \to \mathbb{C}$ such that $\Sigma_0 \cap B_\delta \subset \Gamma_f \subset \Sigma_0$, where Γ_f is the graph of $f : D_\delta \to \mathbb{C}$. Since $(0, 0) \in \Sigma_0 \cap B_\delta$, $f(0) = 0$. Hence, since f is continuous, we can choose $\epsilon < \delta/2$ such that $f(D_\epsilon) \subset D_{\delta/2}$. Let (x, y) be a point in $\Sigma_0 \cap B_\epsilon$. Since $\epsilon < \delta/2$, $(x, y) \in \Sigma_0 \cap B_\delta$. Hence, $y = f(x)$. Since Σ_0 and B_δ are both invariant under $\mathcal{S}_Q$, $(\alpha x, \beta y) \in \Sigma_0 \cap B_\delta$ for all α and β in S^1. Hence, $\beta f(x) = \beta y = f(\alpha x)$ for all α and β in S^1. Let $\beta = -1$ and $\alpha = 1$. Then $-f(x) = f(x)$ and we conclude that $y = f(x) = 0$. This proves that $\Sigma_0 \cap B_\epsilon \subset \{(x, y) \in B_\epsilon | y = 0\}$.

Suppose, on the other hand, that $(x, 0) \in B_\epsilon$. Then $x \in D_\epsilon$. Since $\epsilon < \delta$, $x \in D_\delta$ and, hence, $(x, f(x)) \in \Gamma_f \subset \Sigma_0$. Since $x \in D_\epsilon$, $x, f(x) \in D_{\delta/2}$. Hence, $(x, f(x)) \in B_\delta$. Since $(x, f(x))$ is also in Σ_0, the previous argument, shows that $f(x) = 0$. Hence, $(x, 0) = (x, f(x)) \in \Sigma_0$. This proves that $\{(x, y) \in B_\epsilon | y = 0\} = \Sigma_0 \cap B_\epsilon$. Likewise, $\{(x, y) \in B_\epsilon | x = 0\} = F \cap B_\epsilon$.

This completes the proof. □

Lemma 2.2. *Let Γ be a graph. Let $P \in \Gamma$ be a fixed point of S^1 and $\tilde{\Gamma}$ be the total transform of Γ in the blow-up $(\tilde{X}, \tilde{\omega})$ of (X, ω) at P. Then $\tilde{\Gamma}$ is a graph.*

Proof. Let $\tilde{C}_i = \overline{C_i}$ if $0 \leq i \leq n$ and $\tilde{C}_{n+1} = \Sigma_0$. Consider the graph $\tilde{\mathcal{G}}$ whose vertices are $A_0, \ldots, A_n, A_{n+1}$ such that A_i is connected to A_j by an edge of $\tilde{\mathcal{G}}$ if and only if $\tilde{C}_i$ meets $\tilde{C}_j$. It is easy to see that $\tilde{\Gamma}$ satisfies the intersection properties of a graph. Hence, the only significant issue is the existence of $\tilde{\Gamma}$-normalized Darboux coordinates at each fixed point Q of S^1 in $\tilde{\Gamma}$. If Q is in X^*, then we may obtain $\tilde{\Gamma}$-normalized Darboux coordinates at Q by restricting Γ-normalized Darboux coordinates at Q to X^*. Hence, we may assume that Q lies on Σ_0. Let (x, y) be normalized Darboux coordinates at Q as in Lemma 2.1. If Q is a smooth point of $\tilde{\Gamma}$, then $\tilde{\Gamma} \cap B_\epsilon = \Sigma \cap B_\epsilon$. On the other hand, if Q is a double point of $\tilde{\Gamma}$, then $\tilde{\Gamma} \cap B_\epsilon = (\Sigma \cap B_\epsilon) \cup (F \cap B_\epsilon$. Therefore, by Lemma 2.1, the chosen Darboux coordinates (x, y) are $\tilde{\Gamma}$-normalized Darboux coordinates at Q. □

Attaching overgrown branches to a star. The stars Γ which we wish to construct have centers C_0 of arbitrary genus and weight. The branches, however, are very special. They are linear graphs of type $\mathcal{G}[\lfloor_\infty, \ldots, \lfloor_f]$ where $b_i \geq 2$ for all i. In other words, the components of each branch have genus 0 and self-intersection less than -1. In this section, we shall show how to use blowing up to attach a linear branch of the above type to a graph Γ by blowing up at a smooth fixed point P of Γ lying on C_0. In the process of creating the desired branch, we introduce extra components. The main result of this subsection is the following lemma.

Lemma 2.3. *Let Γ be a star with center C_0 and $P \in C_0$. Suppose that P is a smooth point of Γ and $b = (b_1, \ldots, b_s)$ with $b_i \geq 2$. By blowing up at P we* *may attach a branch B of type $\mathcal{G}[\lfloor_{\infty},\dots,\lfloor_{\int},\infty,\dots]$ to Γ at P, forming a star $\tilde{\Gamma}$ with center $\overline{C_0}$, $\tilde{m_0} = m_0 - 1$ and $\tilde{m_i} = m_i$ if $i \neq 0$. Moreover, there are exactly 2 fixed points on each component of B. If Q is a fixed point on a component of B, then $Q \in C_0$ or Q has index 2.*

Proof. Our construction involves two stages. The first stage is to blow up once at P. Let $\tilde{\Gamma}$ be the corresponding total transform of Γ. Note that the genera and weights of the components of Γ remain unchanged, in passing to their proper transforms in $\tilde{\Gamma}$, with one exception: $\overline{C_0} \cdot \overline{C_0} = C_0 \cdot C_0 - 1$. By blowing up we have constructed a new branch B of type $\mathcal{G}[\infty]$ emanating from $\overline{C_0}$. We denote the single component of this new branch by Σ_0.

By Lemma 1.2 (a), since P lies on the surface C_0 of fixed points of S^1, the fixed point set in the blow-up of X at P consists of the fixed points in X^*, the proper transform $\overline{C}_0$ and an isolated fixed point Q_2 of index 2 on Σ_0. Let Q_1 be the double point of $\tilde{\Gamma}$ lying on the components $\overline{C}_0$ and Σ_0 of $\tilde{\Gamma}$. Then there are exactly two fixed points on the single component Σ_0 of B, the fixed point $Q_1 \in \overline{C}_0$ and the fixed point Q_2 of index 2. The second stage of our construction is to enlarge this new branch B of $\tilde{\Gamma}$ by blowing-up at appropriate index 2 fixed points on B. This restriction ensures that the changes in $\tilde{\Gamma}$ are confined to B. Moreover, by Lemma 1.2 (b), all the fixed points introduced have index 2. Hence, every fixed point on B lies on C_0 or has index 2.

Since B is a total transform, B has at least one -1-curve. If at each stage of our construction, we only blow up at points lying on a -1 component of B, we will ensure that B has exactly one -1 component, E. Henceforth, we assume this limitation. Let B^* denote the subbranch of B consisting of all the components of B preceding E. E has exactly two critical points, Q_1 and Q_2, where Q_1 lies on B^*, so that Q_1 precedes Q_2 on B. Blowing up at Q_1 has the effect of decreasing the weight m of the last component of B^*. Blowing up at Q_2, on the other hand, has the effect of introducing a new -2-curve as the last component of B^*. In either case, by Lemma 1.2 (b), blowing up at Q_i preserves the hypothesis that there are exactly two fixed points on each component of B.

These two operations allow us to create a branch B with an initial subbranch of type b. We start with the branch Σ_0 for which B^* is the empty branch. We then blow-up at Q_2 so that B^* is of type $\mathcal{G}[\in]$. Then we blow-up at Q_1 a finite number of times until B^* is of type $\mathcal{G}[\lfloor_{\infty}]$. Then we blow-up at Q_2 so that B^* is of type $\mathcal{G}[\lfloor_{\infty},\in]$. Then we blow-up at Q_1 a finite number of times until B^* is of type $\mathcal{G}[\lfloor_{\infty},\lfloor_{\in}]$. Continuing in this fashion, blowing up at Q_2 once and then at Q_1 an appropriate number of times, we reach the desired goal. □

Pruning overgrown branches. Since the blow-up of an S^1-Hamiltonian symplectic structure is an S^1-Hamiltonian symplectic structure, the above constructions can be done in the category of S^1-Hamiltonian symplectic structures. As we have mentioned above, our construction requires us to prune the overgrown branches produced by blowing-up. Since C_0 is fixed pointwise by S^1, every point $P \in C_0$ is a fixed point of type $(\pm 1, 0)$. For simplicity, we assume that P is a fixed point of type $(-1, 0)$. (This ensures that C_0 is a surface of minima of the Hamiltonian H rather than a surface of maxima of H.) This will suffice for our purposes. The pruning procedure will involve the following lemma.

Lemma 2.4. *Let B be a branch as in Lemma 2.3. Suppose that P is a fixed point of type $(-1, 0)$ and H is a Hamiltonian for the S^1 action with $H(P) = 0$. Let Q_1 and Q_2 be the fixed points on the -1-curve E of B and c be a regular value of H with $H(Q_1) < c < H(Q_2)$. Then B^* is the union of all components of B which are contained in $H^{-1}([0, c))$. Moreover, the critical points of H on B lying in $H^{-1}([0, c))$ are the critical points of H on B^*. (We say that B^* is obtained by restriction of B to $H^{-1}([0, c))$.)*

Proof. Let $B_1, \dots, B_n$ be the components of B. Denote by P_0 the point lying on the center $\overline{C_0}$ and B_1 and by P_i the point of B lying on B_i and B_{i+1}. Since $\overline{C_0}, B_1, \dots, B_n$ are S^1-invariant, $P_0, \dots, P_{n-1}$ are fixed points of S^1 on B. There are exactly two fixed points on each component of B, so this accounts for all but one remaining fixed point P_n on B_n.

Since each B_i is S^1-invariant, the S^1-Hamiltonian H restricts to a Hamiltonian H_i for the induced action of S^1 on B_i. Hence, the critical points of H_i on B_i are the fixed points P_{i-1} and P_i. One of these points must be a minimum and the other a maximum of H_i. Since $P_0 \in C_0$, P_0 is the minimum of H_1 and P_1 is the maximum of H_1. Since $P_1 \notin C_0$, P_1 is a critical point of H of index 2. Hence, P_1 is the minimum of H_2 and P_2 is the maximum of H_2. Continuing in this fashion, we see that P_{i-1} is the minimum of H_i and P_i is the maximum of H_i. It follows, that H is monotone increasing along the branch B. The result follows immediately. $\square$

Controlling the Hamiltonian on an overgrown branch. The previous lemma allows us to prune an overgrown branch. However, if we have more than one overgrown branch, it may be impossible to prune all of them at once. There may be no regular value c of H satisfying the restrictions $H(Q_1) < c < H(Q_2)$ for all branches. In order to ensure that such a value exists, we need to control the behavior of H on the branches. The following lemma provides the desired control.

Lemma 2.5. *Let Γ, C_0 and P be as in lemma 2.3. Suppose that P is a fixed point of type $(-1, 0)$ and H is a Hamiltonian for the S^1 action with $H(P) = 0$. Let ϵ be a positive number such that there exist Γ-normalized Darboux coordinates on a Darboux ball $B_\epsilon(P)$ and let α and β be numbers such that $0 < \alpha < \beta < \epsilon^2$. Then we may attach a branch B as in Lemma 2.3 so that the fixed points on B of index 2 lie in $H^{-1}(\alpha, \beta)$.*

Proof. We follow the two stage construction in the proof of lemma 2.3. Let δ be a positive number with $\delta^2 \in (\alpha, \beta)$, so that $\delta < \epsilon$. Let (z, w) be Γ-normalized Darboux coordinates on a Darboux ball $B_\epsilon(P)$ and $(\tilde{X}, \tilde{\omega})$ be the blow-up of (X, ω) of weight δ corresponding to the inclusion $\overline{B}_\delta \subset B_\epsilon$. Since P is a fixed point of type $(-1, 0)$, $H(S^3_\delta) = [0, \delta^2]$. Let Q_1 and Q_2 be the two critical points on Σ_0. Then $H(Q_1) = 0$ and $H(Q_2) = \delta^2$. We may continue with the second stage of the construction, blowing up at appropriate index 2 fixed points on B. By choosing sufficiently small Darboux balls at these points, we may keep the values of all the fixed points of index 2 on B arbitrarily close to δ^2. This proves the result.

□

Regular level sets and Seifert 3-manifolds. Let M be a Seifert 3-manifold. M is isomorphic as a Seifert 3-manifold to the boundary of the 4-manifold obtained by equivariant plumbing along a weighted star S as in Corollary 5 of Chapter 2 of [O]. We will realize M as a regular level set $H^{-1}(c)$ of an S^1-Hamiltonian where $H^{-1}([0, c))$ is a tubular neighborhood of a star Γ of type S. This tubular neighborhood will be our model for resolving certain singularities.

Theorem 2.1. *Let S be a weighted star such that all vertices other than the center have genus 0 and weight less than -1. There exists a blow-up (X, ω) of a ruled surface with an S^1-Hamiltonian symplectic structure and a star Γ of type S in (X, ω) such that $H^{-1}([0, c])$ is a compact tubular neighborhood of Γ with boundary $H^{-1}(c)$ for some regular value c of the Hamiltonian H. Moreover,*

(1) *the critical points of H in $H^{-1}([0, c))$ lie on Γ,*
(2) *C_0 is the critical level set corresponding to the minimum value 0 of H,*
(3) *there are exactly two critical points on each component of the branches of Γ,*
(4) *if Q is a critical point of H in $H^{-1}([0, c))$, then $Q \in C_0$ or Q has index 2.*

Proof. Let g and m be the genus and weight of the center A_0 of S. Let $b = -m - r$, where r is the number of branches of S emanating from A_0. Let S_{-b} be a ruled surface over a Riemann surface of genus g with a zero section Z_0 with $Z_0 \cdot Z_0 = -b$ and an infinity section Z_∞ with $Z_\infty \cdot Z_\infty = b$. Consider the standard action of S^1 on S_{-b} so that the fixed point set of S^1 consists of Z_0 and Z_∞. Let τ be an S^1-Hamiltonian symplectic structure on S_{-b} with Hamiltonian H such that $H(Z_0) = 0$ and $H(Z_\infty) = 1$ ([M-W1]). Since Z_0 is a surface of fixed points, Z_0 is a star with center Z_0. On the other hand, since Z_0 is the level set corresponding to the minimum value of H, every point $P \in Z_0$ is a fixed point of type $(-1, 0)$.

Let $P_1, \ldots, P_r$ be r distinct points in S_{-b} lying on Z_0. Let (z^j, w^j) be Z_0-normal Darboux coordinates at P_j. Choose ϵ sufficiently small such that $B_\epsilon(P_1), \ldots, B_\epsilon(P_r)$ are pairwise disjoint. Set $b^j = (-m_{j,1}, \ldots, -m_{j,s_j})$ and let $0 < \alpha < \beta < \epsilon^2$. Using Lemma 2.5, attach a branch B^1 of type

b^1 to Z_0 at P_1 by blowing-up $B_\epsilon(P_1)$ so that all the critical points of index 2 on B^1 lie in $H^{-1}(\alpha_1, \beta_1)$. Note that the points $P_2, \dots, P_r$ lie on $\overline{Z_0}$ and the balls $B_\epsilon(P_2), \dots, B_\epsilon(P_r)$ are contained in $\tilde{S_{-b}}$. Let Q_1^1 and Q_2^1 be the critical points of H on the -1-curve E^1 of B^1 with $H(Q_1^1) < H(Q_2^1)$. Since $(H(Q_1^1), H(Q_2^1)) \subset (\alpha, \beta)$, $0 < H(Q_1^1) < H(Q_2^1) < \epsilon^2$. Again, using Lemma 2.5, attach a branch B^2 of type b^2 to $\overline{Z_0}$ at P_2 by blowing-up $B_\epsilon(P_2)$ so that the values of critical points of index 2 on B^2 lie in $(H(Q_1^1), H(Q_2^1))$. Continuing in this manner, we attach branches $B^1, \dots, B^r$ to Z_0 at the points $P_1, \dots, P_r$ so that all the values of critical points of index 2 on B^{i+1} lie in $(H(Q_1^i), H(Q_2^i))$. Hence, the sequence of intervals, $(H(Q_1^1), H(Q_2^1)), \dots, (H(Q_1^r), H(Q_2^r))$, is a nested decreasing sequence of intervals. The total transform $\tilde{Z_0}$ is a star with center $\overline{Z_0}$ of genus g. Since $Z_0 \cdot Z_0 = -b$, $m = -b - r$ and $\tilde{Z_0}$ is obtained by attaching r branches to Z_0, it follows from Lemma 2.3 that $\overline{Z_0} \cdot \overline{Z_0} = m$. Let $\Gamma = \overline{Z_0} \cup \bigcup_{j=1}^r (B^j)^*$. Since $(B^j)^*$ is a branch of type b^j, Γ is a star of type S. Note that the set of critical points of H consists of the surface $\overline{Z_0}$ of fixed points of type $(-1, 0)$ and critical value 0; the surface $\overline{Z_\infty}$ of fixed points of type $(0, 1)$ and critical value 1; and two critical points on each component of the branches of $\tilde{Z_0}$. Moreover, each critical point in the complement of $\overline{Z_0} \cup \overline{Z_\infty}$ has index 2.

Let c be a regular value of H in $(H(Q_1^r), H(Q_2^r))$. Then c is a regular value of H in $(H(Q_1^j), H(Q_2^j))$ for all j. By Lemma 2.4, $(B^j)^*$ is obtained by restriction of B^j to $H^{-1}([0, c))$ for all j. Hence, Γ is the union of all components of $\tilde{Z_0}$ which are contained in $H^{-1}([0, c))$. Moreover, the critical points of H lying in $H^{-1}([0, c))$ are the critical points of H on Γ.

It follows that $H^{-1}([0, c])$ is a compact tubular neighborhood of Γ with boundary $H^{-1}(c)$.

□

We have the following corollary.

Corollary 2.1. *Let M be a Seifert 3-manifold of type S and $H^{-1}([0, c])$ be a compact tubular neighborhood of a graph Γ of type S as in Theorem 2.1. Then M is isomorphic to the regular level set $H^{-1}(c)$ at the boundary of $H^{-1}([0, c])$.*

Proof. Since $H^{-1}[0, c]$ is compact, the regular level set $H^{-1}(\lambda)$ for any regular value λ of H with $0 < \lambda < c$ is a Seifert 3-manifold. For sufficiently small λ, $H^{-1}(\lambda)$ is a principal bundle of euler class m over a Riemann surface of genus g. As we pass through the critical points of H, $H^{-1}(\lambda)$ changes by various surgeries. By the conditions of Theorem 2.1, these surgeries are uniquely determined by the weighted graph S. Hence, the type of $H^{-1}(c)$ is determined by the weighted graph S. Indeed, it is determined in exactly the same way as the type of the boundary $K(S)$ of the equivariant plumbing according to the weighted graph S. Hence, $H^{-1}(c)$ is isomorphic to $K(S)$. □

3. Duistermaat-Heckmann and the uniqueness of S^1-invariant symplectic forms

Let (X,ω) be a symplectic $2n$-manifold which admits a hamiltonian S^1 action with hamiltonian h. Assume that h is proper and let I be an interval of regular values of h. In this section we will use the work of Duistermaat-Heckman [D-H] to analyze the symplectic structure ω on the manifold $h^{-1}(I)$.

Let λ be a regular value of h and denote $h^{-1}(\lambda) = Y_\lambda$. Y_λ is a smooth compact submanifold of X. Since Y_λ is a level set of h the S^1 action on X preserves Y_λ and induces an S^1 action on Y_λ. The orbits of this action are the integrals of the null-spaces of the degenerate form $i^*_\lambda\omega$, where $i_\lambda : Y_\lambda \to X$ denotes the inclusion. It follows that the orbit space $\Sigma_\lambda = Y_\lambda/S^1$, called the *reduced space*, is equipped with a unique symplectic form τ_λ which satisfies $i^*_\lambda\omega = p^*_\lambda\tau_\lambda$ where $p_\lambda : Y_\lambda \to \Sigma_\lambda$ is the projection. If the orbits on Y_λ form a principal S^1 bundle then Σ_λ is a smooth compact manifold. This is the case discussed in [M-W1]. However, in general, the orbits of the S^1 action do not form a fibration because there are points $y \in Y_\lambda$ with finite nontrivial stabilizer groups $G_y = \{g \in S^1 : gy = y\}$. Consequently, Σ_λ is, in general, not a smooth compact manifold but rather a compact orbifold.

Following [D-H] we can describe the topology of the situation as follows. Let Γ_λ be the finite subgroup of S^1 generated by all $G_y,\ y \in Y_\lambda$. Then we have:

$$\begin{array}{ccc} Y_\lambda & \xrightarrow{r_\lambda} & Z_\lambda = Y_\lambda/\Gamma_\lambda \\ & p_\lambda \searrow & \downarrow\ \pi_\lambda \\ & & \Sigma_\lambda = Y_\lambda/S^1 \end{array} \tag{3.1}$$

where $r_\lambda : Y_\lambda \to Z_\lambda$ is a finite branched covering and $\pi_\lambda : Z_\lambda \to \Sigma_\lambda$ is a principal $S^1/\Gamma_\lambda-$ bundle. Here both Z_λ and Σ_λ are orbifolds, and Y_λ is a $(2n-1)$-manifold with a fixed point free action of S^1. The orbifolds and maps of 3.1 are determined uniquely by this S^1 action on Y. Another important fact we will use is that orbifolds carry differential objects like differential forms, Riemannian metrics, etc. For example, the 2-form τ_λ is a well-defined symplectic form on Σ_λ. Further, the de Rham theorem and the Hodge theorem hold on orbifolds. The de Rham theorem for V-manifolds (the original term for orbifolds) was proved by Satake [S] and the Hodge theorem was proved by Baily [B].

Let λ_0 be a fixed regular value of h and suppose $\lambda \in I$ are regular values in a neighborhood of λ_0. Using an S^1-invariant connection on the fibration $h^{-1}(I) \to I$ it is easy to show that there is a diffeomorphism $\psi : h^{-1}(I) \to Y_{\lambda_0} \times I$ satisfying:

(i) $pr_1 \cdot \psi = h$ where $pr_1 : Y_{\lambda_0} \times I \to I$ is the projection onto the second factor
(ii) $\psi\,|_{Y_\lambda} : Y_\lambda \to Y_{\lambda_0}$ is an S^1 equivariant diffeomorphism for all $\lambda \in I$.

Set $Y = Y_{\lambda_0}$. It follows that for all $\lambda \in I$ we can identify $\Sigma_\lambda = Y_\lambda/S^1$ with $\Sigma = Y/S^1$, and Γ_λ with $\Gamma = \Gamma_{\lambda_0}$. Thus, for all λ, $Z_\lambda = Y_\lambda/\Gamma_\lambda$ are identified with $Z = Y/\Gamma$, a principal S^1 bundle over Σ. The symplectic forms τ_λ, however, still depend on λ and satisfy $p^*\tau_\lambda = i^*_\lambda\omega$ where p is the projection $Y \to \Sigma$ and $i_\lambda : Y \to Y \times I$ is given by $i_\lambda(y) = (\lambda, y)$. Denote by $[\tau_\lambda] \in H^2(\Sigma, \mathbb{R})$ the cohomology class represented by τ_λ. The main result of the above discussion is a beautiful theorem of Duistermaat-Heckman [D-H].

Theorem 3.1. *Let $\lambda, \eta \in I$ lie in an interval of regular values of the hamiltonian h. Then:*

$$[\tau_\lambda] = [\tau_\eta] + (\lambda - \eta)c \tag{3.2}$$

where $c \in H^2(\Sigma, \mathbb{R})$ denotes the Chern class of the principal S^1 bundle $\pi : Z \to \Sigma$.

The theorem in [D-H] is more general then the one stated here, but this result is sufficient for our purposes.

McDuff [McD1] has given a converse to this result in the case when Y is a principal S^1 bundle. We adapt her proof to the general case.

Given a symplectic $2n$-manifold (X, ω) and a proper hamiltonian h on X, we have seen that if λ_0 is a regular value of h, the compact manifold $Y = h^{-1}(\lambda_0)$ is a $(2n-1)$-manifold with a fixed point free action of S^1. Y determines orbifolds Z and Σ so that $\pi : Z \to \Sigma$ is a principal S^1 bundle with Chern class c, and $r : Y \to Z$ is a finite branched covering.

Theorem 3.2. *Let I be an interval of regular values of h around λ_0. Then the symplectic form ω on $h^{-1}(I)$ is uniquely determined up to an S^1−equivariant diffeomorphism preserving the level sets of h by the $(2n-1)$-manifold Y, the fixed point free action of S^1 on Y, and a family $\{\tau_\lambda\}$, $\lambda \in I$, of (orbifold) symplectic forms on Σ which satisfy 3.2.*

Proof. As noted above we can suppose that ω is a symplectic form on $Y \times I$ and that the hamiltonian function h is given by projection onto the second factor. Recall the inclusion:

$$\begin{aligned} i_\lambda : Y_\lambda &\to Y_\lambda \times I \\ y &\mapsto (y, \lambda). \end{aligned}$$

Consider the S^1-invariant 1-form on Y_λ:

$$\alpha_\lambda = i^*_\lambda\left(\frac{\partial}{\partial\lambda} \lrcorner\, \omega\right). \tag{3.3}$$

Let X be the hamiltonian vector field, so:

$$X \lrcorner\, \omega = dh. \tag{3.4}$$

X is tangent to the level sets of h and generates the S^1-action on Y_λ. Thus we have:

$$\begin{aligned} X \lrcorner\, \alpha_\lambda &= X \lrcorner\, i_\lambda^*(\frac{\partial}{\partial\lambda} \lrcorner\, \omega) \\ &= i_\lambda^*(X \lrcorner\, \frac{\partial}{\partial\lambda} \lrcorner\, \omega) \\ &= \omega(\frac{\partial}{\partial\lambda}, X) \\ &= -dh(\frac{\partial}{\partial\lambda}) = -1. \end{aligned} \tag{3.5}$$

The last equality follows since h is projection onto the second factor. Thus:

$$\alpha_\lambda(X) = -1. \tag{3.6}$$

Since α_λ is S^1-invariant, and therefore $\Gamma-$invariant, there is a 1-form β_λ on Z_λ which is $S^1/\Gamma-$invariant and satisfies:

$$r^*\beta_\lambda = -\alpha_\lambda. \tag{3.7}$$

From 3.7 we have $\beta_\lambda(\hat{X}) = 1$, where $\hat{X}$ generates the S^1/Γ action on Z_λ, and therefore β_λ is a connection 1-form for the principal $(S^1/\Gamma \simeq)S^1$ bundle $Z \to \Sigma$. Thus on $Y \times I,\ \omega$ can be written:

$$\omega = p^*(\tau_\lambda) + r^*(\beta_\lambda) \wedge d\lambda. \tag{3.8}$$

We have shown that any S^1-invariant symplectic form ω on $Y \times I$ with hamiltonian h given by projection onto the second factor has the form 3.8. The only ambiguity in ω is in the choice of a family of connection 1-forms $\beta_\lambda, \lambda \in I$. Since β_λ is a connection 1-form, $-d\beta_\lambda$ is a 2-form on Σ_λ representing the Chern class c of the fibration $Z_\lambda \to \Sigma_\lambda$. Accordingly β_λ can be changed only by the addition of 1-forms $\pi^*(\sigma_\lambda)$, where σ_λ are closed (orbifold) 1-forms on Σ_λ. Following McDuff we set:

$$\omega_t = p^*(\tau_\lambda) + r^*(\beta_\lambda) \wedge d\lambda + tp^*(\sigma_\lambda) \wedge d\lambda. \tag{3.9}$$

The forms ω_t are S^1-invariant symplectic forms for $0 \le t \le 1$. We have:

$$\frac{d}{dt}(\omega_t) = p^*(\sigma_\lambda) \wedge d\lambda = d[p^*(\mu_\lambda)] \tag{3.10}$$

where $\mu_\lambda = \int^\lambda \sigma_s ds$. Let ξ_λ be the vector field on Σ such that:

$$\xi_\lambda \lrcorner\, \tau_\lambda = \mu_\lambda.$$

Let $\xi_{\lambda,t}$ be the unique lift of ξ_λ to Z_λ determined by the connection $\beta_\lambda + t\pi^*(\sigma_\lambda)$. Since the vector field ξ_λ respects the orbifold structure of Σ, for each $t,\ \xi_{\lambda,t}$ is an orbifold vector field on Z_λ and hence has a unique lift $\tilde{\xi}_{\lambda,t}$

to Y_λ. The vector field $\tilde{\xi}_{\lambda,t}$ thus determines, for each t, a vector field $\tilde{\xi}_t$ on $Y \times I$ such that:

$$\frac{d}{dt}(\omega_t) = d(\tilde{\xi}_t \lrcorner \omega_t). \tag{3.11}$$

Let g_t be the flow of $\tilde{\xi}_t$. Then g_t is a family of S^1 equivariant diffeomorphisms of $Y \times I$ preserving the level sets Y_λ of h and such that:

$$g_t^*(\omega_t) = \omega_0.$$

□

When X is a symplectic 4-manifold, Theorem 2 can be strengthened. We need the following orbifold version of Moser's theorem [M].

Theorem 3.3. *Let X be a compact orbifold. Suppose that $\{\rho_t\},\ 0 \le t \le 1$, is a family of (orbifold) symplectic forms on X such that $[\rho_t] \in H^2(X;\mathbb{R})$ is independent of t. Then there is a family of (orbifold) diffeomorphisms $g_t : X \to X,\ 0 \le t \le 1$ such that $g_t^*(\rho_t) = \rho_0$.*

Proof. Since Hodge theory holds on orbifolds Moser's proof applies. □

Corollary 3.1. *Let Σ be a compact surface orbifold. Suppose that ρ_0 and ρ_1 are (orbifold) symplectic forms on Σ such that*

$$\int_\Sigma \rho_0 = \int_\Sigma \rho_1.$$

Then there is an (orbifold) diffeomorphism g such that $g^*\rho_1 = \rho_0$. g is pathwise connected to the identity by (orbifold) diffeomorphisms.

Proof. Apply the theorem to the family of symplectic forms $\rho_t = t\rho_1 + (1 - t)\rho_0,\ 0 \le t \le 1$. □

When the reduced space Σ is a surface orbifold the corollary implies that the symplectic forms $\{\tau_\lambda\},\ \lambda \in I$, are determined, up to symplectomorphism, by their area. Hence the symplectic form ω on $h^{-1}(I)$ is determined up to S^1 equivariant diffeomorphism by the Seifert 3-manifold Y and a family of scalars, $[\tau_\lambda] = t_\lambda$, satisfying:

$$t_\lambda = t_\eta + c(\lambda - \eta),\ \lambda, \eta \in I. \tag{3.12}$$

(c is determined by Y.) Thus we have:

Theorem 3.4. *Let (X,ω) be a symplectic 4-manifold which admits a hamiltonian S^1 action with a proper hamiltonian h. Let λ_0 be a regular value of h. The compact manifold $Y = h^{-1}(\lambda_0)$ is a Seifert 3-manifold. Let I be an interval of regular values of h around λ_0. Then the symplectic form ω on $h^{-1}(I)$ is uniquely determined up to an S^1-equivariant diffeomorphism preserving the level sets of h by the Seifert 3-manifold Y and a family of scalars t_λ which satisfy 3.12.*

4. **The Gluing Theorem**

Let (X, ω) be a symplectic $2n$-manifold (not necessarily compact). Let Y be a $(2n-1)$-manifold with a fixed point free action of S^1, and $j : Y \to X$ be an embedding. We say j is an ω-*compatible embedding* if the null-spaces of the degenerate 2-form $j^*\omega$ coincide with the orbits of the S^1-action on Y. Our first result is:

Proposition 4.1. *Suppose Y is a $(2n-1)$-manifold with a fixed point free action of S^1 and $j : Y \to (X, \omega)$ is an ω-compatible embedding. Then, for ϵ sufficiently small, there exists an S^1-invariant symplectic form ω_0 on $Y \times (-\epsilon, \epsilon)$ with hamiltonian h given by projection onto the second factor; a tubular neighborhood $N(Y)$ of Y in X and a symplectomorphism ϕ*

$$\phi : (N(Y), \omega) \to (Y \times (-\epsilon, \epsilon), \omega_0)$$

such that $\phi\,|_Y : Y \to Y \times \{0\}$ is an S^1 equivariant diffeomorphism.

The proposition allows us to symplectically model a neighborhood $N(Y)$ of Y by the symplectic manifold $(Y \times (-\epsilon, \epsilon), \omega_0)$. To prove the proposition we need:

Lemma 4.1. *Let Y be a $(2n-1)$-manifold with a fixed point free action of S^1 and $j : Y \to X$ an ω-compatible embedding. Then the 2-form $j^*\omega$ is invariant under the S^1 action on Y.*

Proof. Let N denote the vector field which generates the S^1-action on Y. Then the Lie derivative of $j^*\omega$ in the direction N is:

$$\mathcal{L}_N\, j^*\omega = d(N \lrcorner\, j^*\omega) + N \lrcorner\, dj^*\omega = 0.$$

Since $N \lrcorner\, j^*\omega = 0$ by assumption and $j^*\omega$ is closed. The lemma follows. $\square$

Proof of the Proposition. As a consequence of the lemma the form $j^*\omega$ induces a non-degenerate 2-form τ on the quotient space $Y/S^1 = \Sigma$. Σ is, as observed above, an orbifold and τ is an orbifold symplectic form. Choose a family $\tau_\lambda,\ \lambda \in (-1, 1)$, of symplectic forms on Σ satisfying 3.2 (where c is determined uniquely by Y) and such that $\tau_0 = \tau$. Denote the S^1-invariant form on $Y \times (-1, 1)$ with reduced spaces $(\Sigma, \tau_\lambda),\ \lambda \in (-1, 1)$, by ω_0. ω_0 is uniquely determined, up to S^1-equivariant diffeomorphism. Changing ω_0 by such a diffeomorphism, if necessary, we can suppose that:

$$\omega\,|_Y = \omega_0|_{\{0\} \times Y}\,.$$

It is now easy to find a diffeomorphism ψ from a neighborhood U of Y in X to a neighborhood W of $Y \times \{0\}$ in $Y \times (-1, 1)$ so that:

(i) $\psi\,|_Y : Y \to Y \times \{0\}$ is the identity
(ii) $\psi^*(\omega_0)\Big|_{T_p(x)} = \omega\Big|_{T_p(x)}$ for all $p \in Y$.

We can now apply the Moser-Weinstein technique (see [W1] or [McD-S]) to $\psi^*(\omega_0)$ and ω along $Y \subset X$ to conclude that there is a tubular neighborhood $N(Y)$ of Y and a diffeomorphism $\varphi : N(Y) \to N(Y)$, with $\varphi\,|_Y = id$, such that $\varphi^*(\psi^*(\omega_0)) = \omega$ on $N(Y)$. The proposition follows by taking ϵ sufficiently small. $\square$

Remark 4.1. Note that the proposition holds regardless of the choice of family of symplectic forms $\{\tau_\lambda\},\ \lambda \in (-1,1)$, satisfying 3.2 and $\tau_0 = \tau$. This is because, by Moser's theorem, two symplectic forms σ and σ' which have the same periods and are sufficiently close are isotopic. Thus if $\{\tau_\lambda\}$ and $\{\tau'_\lambda\},\ \lambda \in (-1,1)$, are two families of symplectic forms both satisfying 3.2 and $\tau_0 = \tau = \tau'_0$ then for ϵ sufficiently small the families $\{\tau_\lambda\}$ and $\{\tau'_\lambda\},\ \lambda \in (-\epsilon,\epsilon)$, are isotopic.

Suppose that $j : Y \to X$ is ω-compatible. Suppose that Y divides X into two components. These components can be canonically distinguished as follows. The vector field N which generates the S^1-action on Y is null for $j^*\omega$, that is, for any vector $T \in T_pY,\ p \in Y,\ \omega(N,T) = 0$. Consequently, for any vector $V \in T_pX,\ p \in Y$, transverse to $T_pY,\ \omega(N,V) \neq 0$. For otherwise, ω would be degenerate on T_pX. Clearly if $\omega(N,V) > 0$ then $\omega(N,-V) < 0$. Thus T_pY divides T_pX into two components: $T_p^+X = \{V \in T_pX : \omega(N,V) > 0\}$ and $T_p^-X = \{V \in T_pX : \omega(N,V) < 0\}$. We use this to distinguish the components of $X \setminus Y$ by setting X^+ to be the component with inward pointing normal vector $V \in T_p^+X$, for all $p \in Y$, and X^- to be the component with inward pointing normal vector $V \in T_p^-X$, for all $p \in Y$. Thus we can write:

$$X = X^- \bigcup_Y X^+.$$

Theorem 4.1. *Let Y be a $(2n-1)$-manifold with a fixed point free action of S^1. Let (X_i,ω_i) $i = 1,2$ be symplectic $2n$-manifolds and suppose there are ω_i-compatible embeddings $j_i : Y \to (X_i,\omega_i)$ $i = 1,2$ such that $j_i(Y)$ is a separating hypersurface in X_i. Suppose that on the orbifold Y/S^1 the quotient symplectic forms τ_i are symplectomorphic. Then there is a symplectic structure ω on the $2n$-manifold*

$$X = X_1^- \bigcup_Y X_2^+$$

obtained by gluing X_1^- to X_2^+ along Y. Moreover, there are neighborhoods $N_i(Y)$ of Y in X_i such that $\omega = \omega_2$ on $X_2^+ \setminus N_2(Y)$ and $\omega = \omega_1$ on $X_1^- \setminus N_1(Y)$.

Proof. By Proposition 5.1 there are tubular neighborhoods $N_i(Y)$ of Y in X_i and symplectomorphisms:

$$\phi_i : (N_i(Y),\omega_i) \to (Y \times (-\epsilon,\epsilon),\omega_0),\ i = 1,2$$

where ω_0 is the S^1-invariant symplectic form constructed in the proof of the proposition. The symplectic gluing map is then given by $\phi_2^{-1} \circ \phi_1$. $\square$

Theorem 4.2. *Let Y be a Seifert 3-manifold. Let $(X_i,\omega_i),\ i = 1,2,$ be symplectic 4-manifolds and suppose that there are ω_i-compatible embeddings*

$j_i : Y \to (X_i, \omega_i),\ i = 1, 2$ such that $j_i(Y)$ is a separating hypersurface in X_i. Then there is a symplectic structure ω on X

$$X = X_1^- \bigcup_Y X_2^+$$

obtained by gluing X_1^- to X_2^+ along Y. Moreover, there are neighborhoods $N_i(Y)$ of Y in X_i such that $\omega = \omega_2$ on $X_2^+ \setminus N_2(Y)$ and $\omega = c\omega_1$ on $X_1^- \setminus N_1(Y)$ for some constant $c > 0$.

Proof. By rescaling the symplectic form ω_1 on X_1 we can suppose that the quotient symplectic forms $\tau_i,\ i = 1, 2$, on Y/S^1 have equal area and hence, by Moser's theorem, are symplectomorphic. The theorem now follows from the previous result. □

These theorems can also be formulated as gluing symplectic manifolds along their boundaries.

Theorem 4.3. *Let Y be a $(2n-1)$-manifold with a fixed point free action of S^1 and let $(X_i, \omega_i),\ i = 1, 2$, be symplectic manifolds with boundary Y. Suppose that the inclusion $Y \hookrightarrow X_i$ is ω_i-compatible and that the quotient symplectic forms τ_i on Y/S^1 are symplectomorphic. Suppose also that the inward pointing normal of Y in X_1 lies in $T^-(X_1)$ and the inward pointing normal of Y in X_2 lies in $T^+(X_2)$. Then there is a symplectic form ω on $X = X_1 \bigcup_Y X_2$ obtained by gluing X_1 to X_2 along Y. Moreover, there are neighborhoods $N_i(Y)$ of Y in X_i such that $\omega = \omega_2$ on $X_2^+ \setminus N_2(Y)$ and $\omega = \omega_1$ on $X_1^- \setminus N_1(Y)$.*

5. Isolated Orbifold Singularities

This section is devoted to describing the symplectic structure in a deleted neighborhood of an isolated symplectic orbifold singularity. Let (X, ω) be a four dimensional symplectic orbifold (i.e. a symplectic V-manifold ([W2])). Let $P \in X$ be an isolated orbifold singularity of X. Let $p : (\tilde{U}, \tilde{\omega}) \to (U, \omega)$ be a local symplectic orbifold uniformization for (X, ω) in a neighborhood U of P. By definition, $(\tilde{U}, \tilde{\omega})$ is a symplectic manifold and there is a finite group G acting on $(\tilde{U}, \tilde{\omega})$ so that p is G equivariant and induces an orbifold diffeomorphism $\tilde{U}/G \cong U$. In addition, of course, p is a local symplectic diffeomorphism in the complement of the branch locus of p, the set of points in $\tilde{U}$ with nontrivial stabilizer in G. Since P is an isolated orbifold singularity, we may assume that the branch locus of P consists of a single point $\tilde{P}$ with stabilizer G. Let $\tilde{U}^* = \tilde{U} \setminus \{\tilde{P}\}$ and $U^* = U \setminus \{P\}$. G acts freely and properly discontinuously on $\tilde{U}^*$ so that $q = p| : \tilde{U}^* \to U^*$ is a regular covering map. Hence, (U^*, ω) is a symplectic manifold and $q^*(\omega) = \tilde{\omega}$.

Since $\tilde{P}$ is a fixed point of the finite group G of symplectomorphisms of $(\tilde{U}, \tilde{\omega})$, we may choose normalized Darboux coordinates (z, w) at $\tilde{P}$ for the action of G. For sufficiently small ϵ, the open ball $B_\epsilon = \{(z, w) | z\overline{z} + w\overline{w} < \epsilon^2\}$ determines a G-invariant neighborhood of $\tilde{P}$ contained in $\tilde{U}$. By the definition of normalized coordinates, the action of G on B_ϵ is given by a representation $\rho : G \to \mathbf{U}(2)$. Consider the scalar action of S^1 on B_ϵ with Hamiltonian $\tilde{H}$ given by the rule $\tilde{H}(z,w) = z\overline{z} + w\overline{w}$. $\tilde{H}$ has a single critical point $\tilde{P}$ of index 0. S^1 acts freely on the regular level sets of $\tilde{H}$, so that the regular level sets of $\tilde{H}$ are Seifert 3-manifolds with no singular fibers. Indeed, the regular levels sets of $\tilde{H}$ are 3-spheres and the orbits of the S^1 action are the fibers of the Hopf fibration. Since G acts as a subgroup of $\mathbf{U}(2)$, G preserves the Hamiltonian $\tilde{H}$. Moreover, the action of G commutes with the scalar action of S^1. Hence, $\tilde{H}$ descends to a Hamiltonian H for an induced action of S^1 on (U^*, ω) and the level sets of H are the quotients of the level sets of $\tilde{H}$ by the action of G. Since $\tilde{H}$ has no critical points in $\tilde{U}^*$, H has no critical points in U^*. On the other hand, since H is a Hamiltonian for the induced action of S^1 on U^*, the fixed points of S^1 on U^* correspond to the critical points of H on U^*. Therefore, the action of S^1 on U^* is fixed point free, (though not necessarily free). Hence, the level sets of H are Seifert 3-manifolds. The singular fibers correspond to the eigenspaces of nonscalar elements of $G \subset \mathbf{U}(2)$. Hence we have:

Proposition 5.1. *Let P be an isolated orbifold singularity on a four dimensional symplectic orbifold. Let U be a neighborhood of P. Then there is a neighborhood V of P with $\overline{V} \subset U$ and ∂V is an ω-compatible hypersurface in U.*

Orbifold singularities arise naturally in symplectic geometry. Two classical ways they occur are from finite group actions on symplectic manifolds and from symplectic reduction. In the spirit of this paper we describe a third way. Consider a symplectically embedded compact surface Σ in a symplectic 4-manifold (X, ω) with self-intersection $\Sigma \cdot \Sigma = -k$, $k \in \mathbb{Z}^+$. Let (R_{-k}, σ_{-k}) denote the ruled surface with zero section Z_0 such that $Z_0 \cdot Z_0 = -k$ and such that the symplectic form σ_{-k} is S^1-invariant under the standard action of S^1 (i.e., the action that rotates the fiber two-spheres leaving the zero and infinity sections fixed). Suppose moreover that there is a symplectomorphism $\varphi : (\Sigma, \omega) \to (Z_0, \sigma_{-k})$. By the symplectic neighborhood theorem there is a neighborhood $N(\Sigma)$ of Σ in X, a neighborhood $N(Z_0)$ of Z_0 in R_{-k} and a symplectomorphism $\phi : (N(\Sigma), \omega) \to (N(Z_0), \sigma_{-k})$ such that $\phi|_\Sigma = \varphi$. Using the symplectomorphism ϕ we can suppose that there is a hamiltonian function h on $N(\Sigma)$ with minimum value λ_0 along Σ. If λ is a regular value of h denote $h^{-1}(\lambda)$ by Y_λ. Y_λ / S^1 is the reduced space (Σ, ω_λ) as described in §3. Moreover by 3.2 the areas $[\omega_\lambda]$ are decreasing as $\lambda \to \lambda_0$. We can clearly extend the interval of regular values of h by extending $N(\Sigma) \setminus \Sigma$ by a product manifold $(\lambda_1, \lambda_0] \times Y_\lambda$ with S^1 invariant symplectic form. (Otherwise said: " by stretching the neighborhood of Σ".) The areas $[\omega_\lambda]$ continue to decrease until they reach zero. At that point X has an isolated orbifold singularity. In fact any isolated orbifold singularity in 4-dimensions can be introduced by applying this construction to the neighborhood of some symplectic star.

Notice that by repeating this construction on any symplectic star with negative chern class, we may introduce symplectic singularities which in general need not be orbifold singularities. However, clearly, these singularities can be resolved by the methods of this paper. We call any symplectically equivalent singularity a *symplectic star singularity.*

6. **Applications**

This section is devoted to applications of the gluing theorem to symplectic 4-manifolds.

Let (X, ω) be a 4-dimensional symplectic orbifold and let $p \in X$ be an isolated orbifold singularity. Let U be a neighborhood of p. Recall the definition of *a symplectic resolution of* X *at* p *on* U given in the introduction.

Theorem 6.1. *Let U be a neighborhood of an isolated orbifold point p on a 4-dimensional symplectic orbifold (X, ω). There exists a symplectic resolution $(\tilde{X}, \tilde{\omega})$ of p on U. Morover, the symplectic divisor D in $\tilde{Y}$ is a symplectic star.*

Proof. In §5 it is shown that a neighborhood V of p admits an ω-compatible embedding of a Seifert 3-manifold Y uniquely determined by the orbifold singularity. Using the results of §2 we can construct a symplectic star Γ and a neighborhood $(N(\Gamma), \omega_\Gamma)$ such that $\partial N(\Gamma)$ is isomorphic to Y as a Seifert 3-manifold and such that $\partial N(\Gamma)$ is ω_Γ-compatible. Rescaling ω_Γ, if necessary, we can use Theorem 4.2 to symplectically glue $N(\Gamma)$ to $X \setminus U(p)$ along Y, where $U(p)$ is a neighborhood of p in X. Then $\tilde{X} = N(\Gamma) \cup_Y (X \setminus U(p))$ is the required resolution. □

While it is not clear how abundant ω-compatible embeddings of Seifert 3-manifolds are, using the construction of §2 we can find many examples of such embeddings in smooth symplectic 4-manifolds. Let (Σ, η) be a compact surface with area form η symplectically embedded in the symplectic 4-manifold (X, ω). Suppose that the self-intersection $\Sigma \cdot \Sigma$ of Σ is $k \in \mathbb{Z}$. Let (R_k, σ_k) denote the ruled surface with zero section Z_0 satisfying:

(i) The symplectic form σ_k is S^1-invariant under the standard action of S^1 (i.e., the action that rotates the fiber two-spheres leaving the zero and infinity sections fixed).
(ii) There is a symplectomorphism $\varphi : (\Sigma, \eta) \to (Z_0, \sigma_k)$.
(iii) $Z_0 \cdot Z_0 = k$.

Then, by the symplectic neighborhood theorem there is a neighborhood $N(\Sigma)$ of Σ in X, a neighborhood $N(Z_0)$ of Z_0 in R_k and a symplectomorphism $\phi : (N(\Sigma), \omega) \to (N(Z_0), \sigma_k)$ such that $\phi|_\Sigma = \varphi$. Via the symplectomorphism ϕ we can use the method of §2 to successively blow-up X in the neighborhood $N(\Sigma)$ to construct any symplectic star and corresponding symplectic star neighborhood with center Σ. This construction obviously yields many examples of ω-compatible Seifert 3-manifolds in blow-ups of symplectic 4-manifolds.

The above construction can be utilized to construct symplectic 4-manifolds as follows. Suppose that (X_0, ω_0) is a 4-dimensional symplectic manifold with an isolated singularity $p \in X_0$. Suppose that there is a neighborhood $U(p) \subset X_0$ of p such that $\partial U(p)$ is an ω_0-compatible Seifert 3-manifold Y with negative chern class. For instance, suppose that p is an isolated orbifold singularity or a symplectic star singularity. Let $\hat{Y}$ denote the dual Seifert 3-manifold. $\hat{Y}$ is isomorphic to Y with the S^1-action reversed. Then $\hat{Y}$ can be realized as the boundary of a symplectic star neighborhood $N(\hat{\Gamma})$ with symplectic star $\hat{\Gamma}$. (Likewise, Y can be realized as the boundary of a symplectic star neighborhood $N(\Gamma)$ with symplectic star Γ.) We call $\hat{\Gamma}$ the *dual star* to the star Γ of the singularity p. Denote the center of $\hat{\Gamma}$ by $\hat{C}$. Suppose $\hat{\Gamma}$ has r strings and $\hat{C} \cdot \hat{C} = m \in \mathbb{Z}$. Suppose now that (X_1, ω_1) is a symplectic 4-manifold and Σ is a symplectically imbedded surface in X_1 such that:

(i) Σ is diffeomorphic to $\hat{C}$
(ii) $\Sigma \cdot \Sigma = m + r$.

By blowing up X_1 we can construct a symplectic 4-manifold $(\tilde{X}_1, \tilde{\omega}_1)$ containing the symplectic star $\hat{\Gamma}$ and symplectic star neighborhood $N(\hat{\Gamma})$. Thus $\tilde{X}_1$ contains an $\tilde{\omega}_1$-compatible Seifert 3-manifold $\partial N(\hat{\Gamma}) = \hat{Y}$. Applying Theorem 4.2 we can symplectically glue $\tilde{X}_1 \setminus N(\hat{\Gamma})$ to $X_0 \setminus U(p)$ along Y. If the genus of the center C_0 of Γ is positive, this construction shows that it is possible to fill the deleted singularity p by essentially different smooth symplectic manifolds with boundary Y. (These fillings may not be symplectic resolutions of p as defined in the introduction.) On the other hand, suppose that C_0 is rational. Since Y has negative chern class, $\hat{Y}$ has positive chern class. This implies that $m + r > 0$. Hence, the results of [McD3] show that this construction does not yield essentially different fillings of the deleted singularity p. In any case, we have the following questions:

Question . Is the symplectic resolution of symplectic orbifold singularities or, more generally, symplectic star singularities unique up to blowing up and blowing down?

This question can be rephrased as a question about whether certain ω-compatible hypersurfaces have a unique s-filling, in the sense of Eliashberg [E].

Question . Is there an intrinsic geometric characterization of the class of symplectic star singularities?

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