Simplicial actions of mapping class groups

John D. McCarthy and Athanase Papadopoulos

Michigan State University
Mathematics department, East Lansing, MI 48824-1027, USA
email: mccarthy@math.msu.edu
Institut de Recherche Mathématique Avancée
CNRS et Université de Strasbourg
7 rue René Descartes - 67084 Strasbourg Cedex, France
email: athanase.papadopoulos@math.unistra.fr

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1 Introduction

In this chapter, $S = S_{g,b}$ is a connected compact orientable surface of genus $g \geq 0$ with $b \geq 0$ boundary components and ∂S denotes the boundary of S. The *mapping class group of* S, $\Gamma = \Gamma_{g,b} = \Gamma(S)$, is the group of isotopy classes of orientation-preserving self-homeomorphisms of S. The *extended mapping class group of* S, $\Gamma^* = \Gamma^*_{g,b} = \Gamma^*(S)$, is the group of isotopy classes of self-homeomorphisms of S. Note that Γ is a normal subgroup of index 2 in Γ^*. The study of these groups has used their action on various abstract simplicial complexes, each of which encodes combinatorial information about the relationship which certain subspaces of S (curves, arcs, cut systems, ideal triangulations, etc.) bear to one another. The aim of this chapter is to review some of these actions, and to study in detail some actions on recently defined complexes, namely, the complex of domains and the truncated complex of domains.

The first action of the (extended) mapping class group on a complex appears in the work of A. Hatcher and W. P. Thurston, published in [19]. In this work, there are two actions on complexes that are described, namely, the cut system complex and the pants decomposition complex. We note that these complexes are not simplicial complexes, but CW complexes. If we want to consider only simplicial action, then we could stick to the actions of the mapping class groups on the one-dimensional skeleta of these complexes, and the actions on these graphs are already extremely interesting.

The actions on the cut system and the pants decomposition complexes were used by Hatcher and Thurston to obtain an explicit finite presentation of the mapping class group. Both complexes are defined in the paper [19] for closed surfaces of genus $g \geq 2$. The vertices of the cut system graph are systems of g closed curves that cut the surface into a sphere with $2g$ holes. The vertices of the pants decompositon graph are systems of $3g - 3$ closed curves that cut the surface into $2g - 2$ pairs of pants, or spheres with three holes. In both cases, two vertices are joined by an edge if and only if these vertices are represented by systems of curves on the surface that are related by an *elementary move*. (There are two different notions of elementary moves, one for each setting.) In each case, an elementary move consists of replacing a curve in a cut system (respectively a pants decomposition) by a new curve which has minimal intersection number with the old curve, and such that the new curve system is a cut system (respectively pants decomposition). The elementary moves between pants decompositions are represented in Figure 8 below. The cut system complex and the pants decomposition complex, which we denote respectively $HT(S)$ and $P(S)$, and as well as their 1-skeleta, which we denote respectively $HT_1(S)$ and $P_1(S)$ and which are called the *cut system graph* and the *pants decomposition graph* respectively, are both rigid in the sense that the natural image of the mapping class group in the simplicial automorphism group of these complexes is an isomorphism, except for a few number of elementary surfaces. We shall elaborate on this rigidity phenomenon below.

The curve complex, $C(S)$, is a flag complex that was introduced shortly after the work of Hatcher and Thurston by W. Harvey. This complex captures the combinatorics of disjointness vs. intersection of the set of isotopy classes of essential unoriented simple closed curves on S. For $n \geq 0$, an n-simplex of $C(S)$ is a set of $n+1$ distinct isotopy classes of essential closed curves on S which can be represented by disjoint curves on the surface.

The curve complex was then studied by various people, from different points of view, namely, by Ivanov, Korkmaz, Luo, Masur, Minsky, Bowditch, Klarreich, Hamenstädt, and others. For instance, Ivanov proved his famous result (completed by Korkmaz and Luo) stating that the simplicial automorphism group of the curve complex coincides with the natural image of the extended mapping class group in that group. Ivanov then used the action on the curve complex to give a new and geometric proof of the famous theorem by Royden (completed by Earle and Kra) saying that the automorphism group of the Teichmüller metric (except for a few speclal surfaces) is the extended mapping class group. Ivanov's proof of that result made a relation between the curve complex and some boundary structure, a relation that was initially suspected by Harvey. Masur and Minsky studied the curve complex, endowed with its natural simplicial metric, from the point of view of large-scale geometry; they showed that this complex is Gromov hyperbolic. Klarreich identified the Gromov boundary of the curve complex with the subspace of minimal laminations, in the quotient space of measured lamination space obtained by forgetting the transverse measure (see Chapter 10 by Hamenstädt in Volume I of this Handbook [14]).

The arc complex, $A(S)$, is defined in analogy with the curve complex, with essential closed curves replaced by essential properly embedded arcs in S. A rigidity theorem for the arc complex, analogous to the theorem by Ivanov-Korkmaz-Luo, was obtained quite recently by Irmak and McCarthy (see Theorem 4.7 below). The analogous rigidity theorems for the cut system complex and for the pants decomposition complex were obtained in the meantime by Margalit and by Irmak and Korkmaz respectively (see Theorems 4.13 and 4.21).

A recurrent theme in the theory of simplicial complexes associated to surfaces is that in general, these actions are rigid in the sense tmentione above, that is, there are no simplicial actions on these complexes other than those arising from the homeomorphisms of surfaces. The fact that the natural homomorphism from the extended mapping class group to the simplicial automorphism group is onto is usually expressed by saying that all the automorphisms of the complex are geometric (that is, they are induced by a homeomorphism of the surface). We shall give several examples of rigid actions. Athough the statements of these rigidity results are similar, the proofs are in general different, and they use the properties of the specific complexes that are involved.

The aim of this chapteris to review the actions of the mapping class group on these complexes and on several others. Besides the cut system graph, the pants decomposition graph and the curve complex, we shall consider the arc complex, the ideal triangulation graph, the arc and curve complex, the Schmutz graph of nonseparating curves, the complex of nonseparating curves, the complex of separating curves, the Torelli complex, and a new complex $D(S)$ on which Γ^* acts simplicially, the *complex of domains of* S. A *domain on* S is a nonempty connected compact embedded surface in S which is not equal to S and each of whose boundary components is either contained in ∂S or is essential on S. The vertex set $D_0(S)$ of $D(S)$ is the set of isotopy classes of domains on S. An n-simplex of $D(S)$ is a set of $n+1$ distinct vertices of $D(S)$ that can be represented by disjoint domains of S.

Another important theme in the study of these complexes is that simplicial information at the vertices of these complexes recognizes the topological data on the surface that these vertices represent. This is well illustrated in the detailed study of the complex of domains and the truncated complex of domains that we make in what follows.

One principle guiding our study of $D(S)$ is that its vertices are essentially copies of the curve complex of the subsurfaces they represent, together with the boundary components of these subsurfaces which are contained in the interior of S.

There are several reasons for which we study the actions of mapping class group on the complex of domains.

First of all, this complex is the complex which is naturally associated to the Thurston theory of surface diffeomorphisms. Indeed, the various pieces of the Thurston decomposition of a surface diffeomorphism, thick domains and annular or thin domains, appear as the vertices of this flag complex.

Secondly, this complex contains, as subcomplexes, a certain number of flag complexes, each of which is induced from a particular subset of the 0-skeleton $D_0(S)$ of $D(S)$. We mention some of these subcomplexes:

• The truncated complex of domains, $D^2(S)$, whose vertices are the isotopy classes of domains that are not pairs of pants with two boundary components contained in ∂S.

• The complex of elementary domains, $E(S)$, the induced subcomplex of $D(S)$ whose vertices are represented by domains that are either annuli or pairs of pants.

• The complex of annular domains, $R(S)$, the induced subcomplex of $D(S)$ whose vertices are represented by domains that are annuli. This complex is naturally isomorphic to the curve complex $C(S)$

• The complex of pairs of pants, $P(S)$, which is the induced subcomplex of $D(S)$ whose vertices are represented by domains that are pairs of pants.

• The complex of peripheral pairs of pants, $P_\partial(S)$, which is the induced subcomplex of $D(S)$whose vertices are represented by pairs of pants with at least one boundary component contained in ∂S.

• The complex of thick domains, $TD(S)$, which is the induced subcomplex of $D(S)$ whose vertices are represented by domains on S that are not annuli.

• The complex of boundary graphs, $B(S)$, whose vertices are isotopy classes of graphs on S that are the union of an essential arc on S together with the boundary components of S which contain at least one endpoint of this arc. $B(S)$ is naturally identified with $P_\partial(S)$. We shall see that it is also naturally identified with a certain subcomplex of $A(S)$ with the same vertex set as $A(S)$ but with, in general, fewer simplices.

Each of these complexes and sub-complexes has some special combinatorial features, and there are interesting questions that are particular to each of them. Furthermore, besides the study of the individual complexes, there are intreseting questions about natural maps between them.

Except for a small number of exceptional surfaces (which in general belong to the followig classes: spheres with at most four holes, tori with at most two holes and the closed surface of genus 2), the natural homomorphism of the extended mapping class group in the automorphism group of any one of the complexes that we study is an isomorphism, except for the case of the complex of domains, for a surface with at least two boundary components. Indeed, a special feature of the complex of domains of of such a surface is that its automorphism group is uncountable. This group includes automorphisms that send biperipheral pairs of pants (that is, pairs of pants with two of their boundary components being boundary components of S) to the corresponding biperipheral annuli (that is, annuli homotopic to the boundary component of a biperipheral pair of pants which is not a boundary component of S) and that fix all the other vertices of $D(S)$. Such automorphisms are not geometric, since a homeomorphism of the surface cannot send an annulus to a pair of pants. This phenomenon disappears if instead of the complex of domains we consider the truncated complex of domains, and we shall see that the natural homomorphism of the extended mapping class group in the automorphism group of the truncated complex of domains is an isomorphism (except, as usual, for a small number of exceptional surfaces).

We note that all the simplicial complexes that we consider can be seen as induced subcomplexes of a "universal" simplicial complex $U_{\mathcal{C}}(S)$ whose simplices are finite collections of isotopy classes of a class $\mathcal{C}$ of closed subspaces of S which is invariant under the action of the group $\mathrm{Homeo}(S)$ of self-homeomorphisms of S on closed subspaces of S. This ensures that there is a natural action $\Gamma^*(S) \times U_{\mathcal{C}}(S) \to U_{\mathcal{C}}(S)$ of $\Gamma^*(S)$ on $U_{\mathcal{C}}(S)$. Note that the induced subcomplex $U_{\mathcal{D}}(S)$ of $U_{\mathcal{C}}(S)$ corresponding to any subcollection $\mathcal{D}$ of $\mathcal{C}$ which is also invariant under this action affords a similar simplicial representation $\rho : \Gamma^*(S) \to \mathrm{Aut}(U_{\mathcal{D}}(S))$ of $\Gamma^*(S)$.

The plan of this chapter is the following.

Section 2 contains some basic principles on surfaces, subsurfaces, curves, and geometric intersection numbers that will be used in later sections. We have included proofs of a few facts that are used later on in the text. Of course, a reader who is an expert in the theory of surfaces will skip these proofs.

Section 3 contains a short introduction to abstract simplicial complexes. In this section, we define some basic simplicial notions that we use later in the chapter. Some of the notions are standard notions, and others are new. In particular, we introduce the notion of an exchangeable pair of vertices and of an exchange automorphism of a simplicial complex K. We give necessary and sufficient conditions for a pair of vertices to be exchangeable. We define certain special subgroups of the automorphism group of K, which we call Boolean subgroups. Such a group is isomorphic to the Boolean algebra of a collection of subsets of K consisting of exchangeable pairs of vertices. The exchange automorphisms and the Boolean subgroups will be used in an essential way in the section on the automorphism group of the complex of domains, 8. We also develop the theory of quotient complexes.

Section 4 is a survey on several simplicial complexes associated to a surface S: the curve complex $C(S)$, the arc complex $A(S)$, the arc and curve complex $AC(S)$, the pants decomposition graph $P_1(S)$, the ideal triangulation graph $T(S)$, the Schmutz graph of nonseparating curves $G(S)$, the complex of nonseparating curves $N(S)$, the cut system graph $HT_1(S)$, the complex of separating curves $CS(S)$, and the Torelli complex $TC(S)$. We state the rigidity results without proofs, and we sometimes elaborate on some special cases of surfaces that are excluded by the hypotheses of the theorems. It is usually pleasant and instructive to work out the details of the special cases.

In Section 5, we introduce the complex of domains, $D(S)$, and several of its complexes, in particular the truncated complex of domains, $D^2(S)$, the complex of boundary graphs $B(S)$ and the complex of peripheral pairs of pants $P_\partial(S)$. There are natural inclusion maps among these complexes and the complexes introduced in the preceding sections, and there is a natural projection map $D(S) \to D^2(S)$.

In Section 6, we show on a series of examples that combinatorial information at rhe vertices of the simplicial complexes $D(S)$ and $D^2(S)$ can be used to characterize the subsurfaces that are represented by these vertices. To be specific, we show for instance that a vertex x of $D(S)$ is elementary (that is, it represents either an annulus or a pair of pants) if and only if we have $\mathrm{Lk}(\mathrm{Lk}(x)) = \{x\}$. Some of the characterizations of vertices of $D(S)$ and of $D^2(S)$ that are obtained in this section will be used in the following sections, where the rigidity results are proved.

In Section 7, we prove that if S is not a sphere with four holes, a torus with at most two holes, or a closed surface of genus two, then the natural homomorphism $\rho : \Gamma^*(S) \to \mathrm{Aut}(D^2(S))$ corresponding to the action of $\Gamma^*(S)$ on $D^2(S)$ is an isomorphism. In other words, every automorphism of $D^2(S)$ is induced by a homeomorphism $S \to S$ which is uniquely defined up to isotopy on S.

In Section 8, we prove the rigidity result for the complex of domains. This involves the notion of an *exchange automorphism*. We prove thatif S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two, then every automorphism of $D(S)$ is a composition of an exchange automorphism of $D(S)$ with a geometric automorphism of $D(S)$.

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2 Surfaces

Let $S = S_{g,b}$ be a connected compact orientable surface of genus g with b boundary components. We say that *S is a surface of genus g with b holes*. Note that $S_{g,b}$ is a closed surface of genus g if and only if $b = 0$. Some surfces have special names; for instance, $S_{0,1}$ is a disc, $S_{0,2}$ is an annulus, $S_{0,3}$ is a pair of pants, $S_{0,b}$ is a sphere with b holes and $S_{1,b}$ is a torus with b holes. When talking about a torus (respectively sphere) with no holes, we shall use the term *closed* torus (respectively sphere) in order to avoid ambiguity.

Let ∂S denote the boundary of S. We shall sometimes index the b boundary components of S by ∂_i, $1 \leq i \leq b$.

For each collection C of subsets of S, the *support of* C *on* S is the union in S of the subsets of S in the collection C. We denote the support of C on S by $|C|$.

Throughout this chapter, all isotopies between subspaces of S will be ambient isotopies. More precisely, if X and Y are subspaces of S, an *isotopy from* X *to* Y is a map $\varphi : S \times [0,1] \to S$ such that the maps $\varphi_t : S \to S$, $0 \leq t \leq 1$, are homeomorphisms of S, $\varphi_0 = id_S : S \to S$, and $\varphi_1(X) = Y$.

2.1 Curves

A *curve* on S is an embedded connected closed one-dimensional submanifold of the interior of S. (Thus, a curve is homeomorphic to a circle.)

Let α be a curve on S. We say that α *is k-peripheral* if there exists a sphere with $k+1$ holes X on S such that α is a boundary component of X and every other boundary component of X is a boundary component of S. We say that α *is essential* if it is neither 0-peripheral nor 1-peripheral on S. In other words, α is essential on S if and only if there does not exist a disc on S whose boundary is equal to α or an annulus A on S whose boundary is equal to the union of α with a boundary component of S.

If S is a sphere with at most three holes, then there are no essential curves on S. Otherwise, there are infinitely many pairwise nonisotopic essential curves on S.

Suppose that α and β are disjoint essential curves on S. Then α is isotopic to β on S if and only if there exists an annulus A on S such that the boundary of A is equal to $\alpha \cup \beta$. This is a classical result, see e.g. [10].

A collection of pairwise disjoint essential curves on S is called a *system of curves* if the curves in the collection are pairwise nonisotopic. Note that every subcollection of a system of curves on S is also a system of curves.

In the rest of this chapter, unless otherwise indicated, all simple closed curves will be assumed to be essential, and we shall denote them by the name *curve*.

If C is a finite collection of curves on S then S_C will denote the compact surface obtained from S by cutting S along $|C|$.

A *pants decomposition of* S is a collection of curves C on S such that each component of S_C is a pair of pants. Note that every pants decomposition of S is a system of curves on S.

The surface S has a pants decomposition if and only if S is not a sphere with at most two holes nor a closed torus. Moreover, on such a surface S, if C is a system of curves, then there exists a pants decomposition on S containing C.

A nonempty system of curves C on S is a *maximal* system of curves on S if and only if one of the following two situations occurs:

(1) S is a closed torus and C consists of exactly one nonseparating curve on S.

(2) S is not a closed torus and C is a pants decomposition of S. In this case, the cardinality of C is equal to $3g - 3 + b$.

Suppose that C is a pants decomposition of S. Let R be a regular neighborhood of $|C|$. The closure of the complement of R on S is a disjoint union of pairs of pants on S. These pairs of pants are called *pairs of pants of* C. Note that the pairs of pants of a pants decomposition are defined only up to isotopy relative to C.

Suppose that P is a pair of pants of a pants decomposition C of S. Then P is contained in a unique component Q of S_C. We say that P *is an embedded pair of pants of* C if the natural quotient map $\pi : S_C \to S$ embeds the pair of pants Q in S.

Let C be a pants decomposition of S. We say that C *is an embedded pants decomposition of* S if every pair of pants of C is embedded. For example, if S is a closed surface of genus two and C is a disjoint union of three nonisotopic nonseparating curves on S, then C is an embedded pants decomposition of S.

2.2 Arcs

An *arc* α on S is a subspace of S which is homeomorphic to the interval $[0, 1]$. The *endpoints of* α are the images of 0 and 1 under a homeomorphism from $[0, 1]$ to α. We say that an arc α is *properly embedded* if α intersects the boundary of S precisely at its endpoints. Unless otherwise indicated, all arcs on S will be assumed to be properly embedded.

Let α be an arc on S. Suppose that one endpoint of α lies on ∂_i and the other endpoint of α lies on ∂_j. Then we say that α *joins* ∂_i *to* ∂_j, and that α *is an arc of type* $\{i, j\}$.

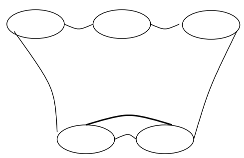
Let α be an arc on S of type $\{i, j\}$ and β be an arc on S of type $\{k, l\}$. Note that if α is isotopic to β, then $\{i, j\} = \{k, l\}$.

An arc α on S is *essential* if there does not exist an embedded closed disk on S whose boundary is equal to the union of α with an arc contained in the boundary of S.

Any arc on S joining two distinct boundary components of S is essential. Likewise, if $g > 0$, $b = 1$, then any arc on S that intersects a simple closed curve on S transversely and at exactly one point is essential.

2.3 Boundary graphs

Let α be an arc on S joining ∂_i to ∂_j. The *boundary graph of* α, G_α, is the union $\partial_i \cup \alpha \cup \partial_j$ (see Figure 1). We refer to ∂_i and ∂_j as the *boundary components of* G_α. Note that G_α has one boundary component if α joins a boundary component of S to itself (i.e. if $i = j$) and two boundary components if α joins distinct boundary components of S (i.e. if $i \neq j$). Since arcs are, by definition, properly embedded, $\partial_i \cup \partial_j = G_\alpha \cap \partial S$. A *boundary graph* is a boundary graph of an arc and an *essential boundary graph* is a boundary graph of an essential arc.



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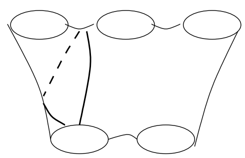

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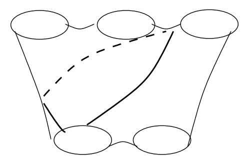

(3)

Figure 1. The three types of boundary graphs: (1) a boundary graph of an arc joining two distinct boundary components of S; (2) a boundary graph of an arc joining a boundary component of S to itself with at least one of the two boundary components of a regular neighborhood of the boundary graph being a boundary component of S; (3) a boundary graph of an arc joining a boundary component of S to itself with a regular neighborhood of the boundary graph having two essential curves on S on its boundary.

In the rest of this chapter, unless otherwise indicated, all boundary graphs will be assumed to be essential.

2.4 Geometric intersection number

Definition 2.1. Let α and β be curves on S. The *geometric intersection number of* α *and* β *on* S is the minimum number $i(\alpha, \beta) = i_S(\alpha, \beta)$ of points in $\alpha' \cap \beta'$ where α' and β' are curves on S that are isotopic to α and β respectively.

Definition 2.2. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a collection of curves on S. We say that $\{\alpha_1, \ldots, \alpha_n\}$ *is in minimal position on* S if for all $1 \leq i < j \leq n$, the geometric intersection number of α_i and α_j on S is equal to the number of points in $\alpha_i \cap \alpha_j$.

We recall that if $\{\alpha_1, \ldots, \alpha_n\}$ is a finite collection of curves on S, then there exists a collection $\{\beta_1, \ldots, \beta_n\}$ of curves on S such that α_i is isotopic to β_i on

S, $1 \leq i \leq n$ and $\{\beta_1, \ldots, \beta_n\}$ is in minimal intersection position on S. This fact is often useful. For a proof, we can assume without loss of generality that the curves $\{\alpha_1, \ldots, \alpha_n\}$ intersect transversely, and then apply an innermost disk elimination argument, to eliminate non-essential intersection points. For the existence of such diks, see e.g. [10]. Alternatively, we can prove this fact by equipping S with a hyperbolic structure and replacing each curve α_i by the unique closed geodesic β_i which is homotopic to it, see [13].

The following proposition will also be useful.

Proposition 2.3. *Let C be a collection of disjoint essential curves on S and let $\alpha \in C$. Then there exists a curve γ on S such that for each $\beta \in C$, $i(\gamma, \beta) \neq 0$ if and only if β is isotopic to α on S.*

Proof. Since S contains an essential curve, S is not a sphere with at most three holes.

Suppose, on the one hand, that S is a closed torus. Then any two disjoint essential curves on S are isotopic, β is isotopic to α for each β in C. Since any essential curve on S is nonseparating on S, α is nonseparating. Hence, S_α is an annulus and there exists a curve γ on S that intersects α transversely and has exactly one point of intersection with α. It follows that $i(\gamma, \beta) = i(\gamma, \alpha) = 1$ for each β in C.

If S is not a sphere with at most three holes or a closed torus, we can find a pants decomposition C_1 of S such that each curve in C_1 is homotopic to a curve in C and each curve in C is homotopic to a curve in C_1.

If α_1 is a curve of the pants decomposition C_1 of S, there exists a curve γ on S such that $i(\gamma, \alpha_1) \neq 0$ and $i(\gamma, \delta) = 0$ for each curve δ in $C_1 \setminus \{\alpha_1\}$ [13].

It follows that for each β in C, $i(\gamma, \beta) = i(\gamma, \beta_1) \neq 0$ if and only if β is isotopic to α on S (i.e. if and only if $\beta_1 = \alpha_1$). $\square$

Geometric intersection number is an effective obstruction for isotoping curves to one another. We shall use this obstruction repeatedly throughout this work. More precisely, we shall use the following two propositions.

Proposition 2.4. *Let α and β be essential curves on S. Then α is not isotopic to β if and only if there exists a curve γ on S such that $i(\alpha, \gamma) = 0$ and $i(\beta, \gamma) \neq 0$.*

Proof. Suppose, on the one hand, that α is isotopic to β on S. Then $i(\alpha, \gamma) = i(\beta, \gamma)$ for every curve γ on S. Hence, there does not exist a curve γ on S such that $i(\alpha, \gamma) = 0$ and $i(\beta, \gamma) \neq 0$.

Suppose, on the other hand, that α is not isotopic to β on S.

First, suppose that $i(\beta, \alpha) \neq 0$. Then, α is a curve on S such that $i(\alpha, \alpha) = 0$ and $i(\beta, \alpha) \neq 0$.

Now, suppose that $i(\beta, \alpha) = 0$. Then there exists a curve β_1 on S which is isotopic to β and disjoint from α. Therefore, α and β_1 are disjoint nonisotopic essential curves on S. Let $C = \{\alpha, \beta_1\}$. It follows from Proposition 2.3 that there exists a curve γ on S such that $i(\alpha, \gamma) = 0$ and $i(\beta, \gamma) \neq 0$.

Hence, in any case, there exists a curve γ on S such that $i(\alpha, \gamma) = 0$ and $i(\beta, \gamma) \neq 0$.

□

Proposition 2.5. *Suppose that S is not a sphere with four holes or a torus with at most one hole. Let α and β be nonisotopic essential curves on S. Then there exists a curve γ on S such that γ is not isotopic to α, $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) \neq 0$.*

Proof. Since S contains an essential curve, S is not a sphere with at most three holes. Since S is also not a closed torus, we can extend α to a pants decomposition $\mathcal{F}$ of S. For each curve δ in $\mathcal{F} \setminus \{\alpha\}$ choose a curve δ' on S such that δ' is disjoint from every curve in $\mathcal{F} \setminus \{\delta\}$ and has nonzero geometric intersection with δ [13]. Let $\mathcal{F}'$ be the union of $\mathcal{F} \setminus \{\alpha\}$ with $\{\delta' \mid \delta \in \mathcal{F} \setminus \{\alpha\}\}$.

Suppose that $i(\tau, \beta) = 0$ for each curve τ of $\mathcal{F}'$.

Without loss of generality, we may assume that $\mathcal{F}' \cup \{\beta\}$ is in minimal position.

Since S is not a sphere with four holes or a torus with one hole, then each component of the surface obtained from S by cutting S along $\mathcal{F}'$ is either a disk or an annulus on S containing a boundary component of S, or an annulus on S containing α, or a pair of pants on S containing α such that two of its boundary components are boundary components of S.

Since $i(\tau, \beta) = 0$ for each curve τ of $\mathcal{F}'$ and $\mathcal{F}' \cup \{\beta\}$ is in minimal position, β is contained in the interior of one of the components M of the cut surface. Since β is an essential curve on S contained in M, M cannot be a disk on S or an annulus on S containing a boundary component of S. Hence, M is either an annulus on S containing α, or a pair of pants on S containing α such that two of its boundary components are boundary components of S. Since α and β are essential curves on S contained in M, it follows that β is isotopic to α. This is a contradiction.

Hence, $i(\gamma, \beta) \neq 0$ for some curve γ of $\mathcal{F}'$. Note that each curve τ of $\mathcal{F}'$ is disjoint from α and not isotopic to α on S. Hence, γ is not isotopic to α on S, $i(\gamma, \alpha) = 0$, and $i(\gamma, \beta) \neq 0$.

□

2.5 Orientation

Proposition 2.6. *Suppose that S is not a sphere with at most three holes and let $H : S \to S$ be a homeomorphism of S. If H preserves the isotopy class of every essential curve on S, then H is orientation-preserving.*

Proof. First, consider the case where the genus of S is zero. Since S is not a sphere with at most three holes, there exists a sphere with four holes X embedded in S such that three of the four boundary components of X are boundary components C_1, C_2, and C_3 of S and the remaining boundary component C_0 of X is either a boundary component of S or an essential curve on S.

As in Section 4.2 of [31], we recall the lantern relation discovered by M. Dehn [9] and rediscovered and popularized by D. Johnson [32]. To do this, we choose an orientation on S and we let $\alpha_{ij} = \alpha_{ji}$, $1 \leq i < j \leq 3$ be an arc on S joining C_i to C_j. We can suppose that α_{12}, α_{23}, and α_{31} are disjoint. The surface obtained from X by cutting along $\alpha_{12} \cup \alpha_{23} \cup \alpha_{31}$ contains a unique component D which is a disc. Suppose that D is on the left of α_{12} as we travel along α_{12} from C_1 to C_2, as in Figure 2. Let $C_{ij} = C_{ji}$ be the unique essential boundary component of a regular neighborhood $P_{ij} = P_{ji}$ in X of $C_i \cup \alpha_{ij} \cup C_j$. Let $T_i : S \to S$ and $T_{jk} = T_{kj} : S \to S$ denote right Dehn twist maps supported on regular neighborhoods on S of C_i and C_{jk}. Let t_i and $t_{jk} = t_{kj}$ be the isotopy classes of T_i and $T_{jk} = T_{kj}$. Then, we have the following relation, usually called the *lantern relation*:

$$t_0 \cdot t_1 \cdot t_2 \cdot t_3 = t_{12} \cdot t_{23} \cdot t_{31} = t_{23} \cdot t_{31} \cdot t_{12} = t_{31} \cdot t_{12} \cdot t_{23}.$$

Since C_1, C_2, and C_3 are boundary components of S, it follows that for $i = 1, 2, 3$, t_i is equal to the identity element of $\Gamma^*(S)$. Hence:

$$t_0 = t_{12} \cdot t_{23} \cdot t_{31} = t_{23} \cdot t_{31} \cdot t_{12} = t_{31} \cdot t_{12} \cdot t_{23}.$$

Let $h \in \Gamma^*(S)$ be the isotopy class of $H : S \to S$. Suppose for contradiction that $H : S \to S$ is orientation-reversing. The mapping class $h \cdot t_{ij} \cdot h^{-1}$ is equal to s_{ij} where s_{ij} is a left Dehn twist about $H(C_{ij})$. On the other hand, since H preserves the isotopy class of every essential curve on S, $H(C_{ij})$ is isotopic on S to C_{ij}. It follows that $s_{ij} = t_{ij}^{-1}$. Likewise, if C_0 is essential on S, then $h \cdot t_0 \cdot h^{-1} = t_0^{-1}$. On the other hand, if C_0 is a boundary component of S, then t_0 is equal to the identity element of $\Gamma^*(S)$ and, hence, $h \cdot t_0 \cdot h^{-1} = t_0^{-1}$. In any case, $h \cdot t_0 \cdot h^{-1} = t_0^{-1}$. We conclude that

$$t_0^{-1} = h \cdot t_0 \cdot h^{-1} = h \cdot (t_{12} \cdot t_{23} \cdot t_{31}) \cdot h^{-1} = t_{12}^{-1} \cdot t_{23}^{-1} \cdot t_{31}^{-1}.$$

This implies:

$$t_0 = t_{31} \cdot t_{23} \cdot t_{12}.$$

It follows from the above equations that:

$$t_{31} \cdot t_{12} \cdot t_{23} = t_{31} \cdot t_{23} \cdot t_{12}$$

which implies:

$$t_{12} \cdot t_{23} = t_{23} \cdot t_{12}.$$

Hence, the right Dehn twists t_{12} and t_{23} about the essential curves C_{12} and C_{23} on S commute. It follows from Lemma 4.3 of [45] that the geometric intersection $i(C_{12}, C_{23}) = 0$. Since this geometric intersection number is equal to two, this is a contradiction. Hence, H is orientation-preserving.

Now the case where the genus of S is positive is handled similarly, by using a torus with one hole on S instead of a sphere with four holes on S.

Suppose that S has positive genus. Then there exist transverse essential curves α and β on S such that α and β have exactly one point of intersection. Let $T_\alpha : S \to S$ be a homeomorphism representing t_α and γ be the image of β under T_α. Then, since T_α is orientation-preserving:

$$t_\alpha \cdot t_\beta \cdot t_\alpha^{-1} = t_\gamma.$$

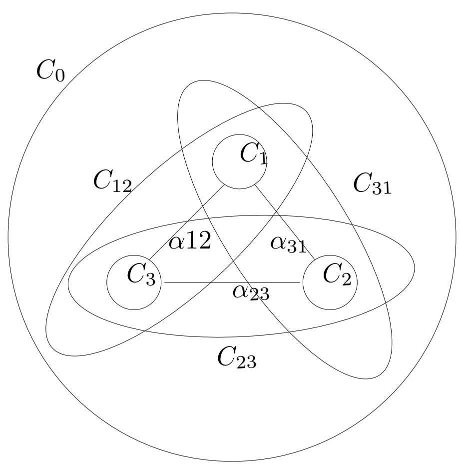

Figure 2. The lantern relation: $t_0 \cdot t_1 \cdot t_2 \cdot t_3 = t_{12} \cdot t_{23} \cdot t_{31}$, where t_i (t_{jk}) is the isotopy class of a twist map about C_i (C_{jk}).

Since β is essential on S and $T_\alpha : S \to S$ is a homeomorphism, γ is essential on S. Assume that H is orientation-reversing. Then, as before, by conjugating by h, we conclude that

$$t_\alpha^{-1} \cdot t_\beta^{-1} \cdot t_\alpha = t_\gamma^{-1}.$$

This implies:

$$t_\alpha^{-1} \cdot t_\beta \cdot t_\alpha = t_\gamma.$$

It follows from the above equations that:

$$t_\alpha \cdot t_\beta \cdot t_\alpha^{-1} = t_\alpha^{-1} \cdot t_\beta \cdot t_\alpha$$

which implies:

$$t_\alpha^2 \cdot t_\beta = t_\beta \cdot t_\alpha^2.$$

As before, it follows from Lemma 4.3 of [45] that the geometric intersection $i(\alpha, \beta) = 0$. Since this geometric intersection number is equal to one, this is a contradiction. Hence, H is orientation-preserving.

In any case, H is orientation-preserving.

$\square$

2.6 Domains

A *subsurface of* S is a surface with boundary X contained in S such that every boundary component of X is either a boundary component of S or disjoint from the boundary of S.

A *domain on* S is a connected compact subsurface X of S which is not equal to S and each of whose boundary components is either contained in ∂S or is essential on S. The *peripheral boundary components of* X are those which are contained in ∂S.

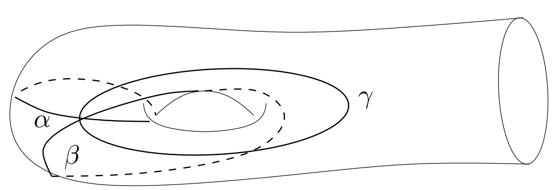

Figure 3. A conjugation relation: $t_\alpha \cdot t_\beta \cdot t_\alpha^{-1} = t_\gamma$.

The following properties of domains follow easily from the definitions:

Proposition 2.7. *Let X be a domain on S. Then:*

- *X is not a disk;*
- *no boundary component of X bounds a disk on S;*
- *there does not exist an annulus on S whose boundary is equal to the union of a boundary component of X with a boundary component of S;*
- *X has at least one essential boundary component on S.*

Let C be a curve on S. A *regular neighborhood* of C on S is an annulus R in the interior of S such that C is a curve on R which is essential on R.

A regular neighborhood of a curve on S is a domain on S if and only if the curve is an essential curve on S.

We say that a domain on S is *peripheral* if it has at least one peripheral boundary component. We say that a domain on S is *monoperipheral* if it has exactly one peripheral boundary component, and *biperipheral* if it has exactly two peripheral boundary components. More generally, for $k \geq 0$, we say that a domain on S is *k-peripheral* if it has exactly k peripheral boundary components.

Let X be a domain on S. The *inside of* X, denoted by X^*, is the complement in X of the union of the essential boundary components of X.

Let $\{\partial_i \mid 1 \leq i \leq n\}$ be the collection of all essential boundary components of X on S. Let $\mathcal{A} = \{A_i \mid 1 \leq i \leq n\}$ be a collection of disjoint annuli on X such that $A_i \cap \partial X = \partial_i$, $1 \leq i \leq n$. Let Y be the closure of the complement of $|\mathcal{A}|$ in X. We say that Y *is obtained from X by shrinking X on S*. Note that Y is a domain on S which is contained in the inside X^* of X on S and that Y is isotopic to X on S.

Proposition 2.8. *Let X be a domain on S and Y be a subsurface of X. If Y is a domain on X, then Y is a domain on S.*

Proof. By Proposition 2.7, S is not a disk and no boundary component of X bounds a disk on S.

Clearly Y is a compact, connected, orientable subsurface of S which is not equal to S. Let ∂ be a boundary component of Y. Since Y is a domain on X, ∂ is either a boundary component of X or an essential curve on X.

Suppose, on the one hand, that ∂ is a boundary component of X. Since X is a domain on S, it follows that ∂ is either a boundary component of S or an essential curve on S.

Suppose, on the other hand, that ∂ is an essential curve on X. In particular, ∂ is in the interior of X. Hence, ∂ is in the interior of S.

Suppose that ∂ is not an essential curve on S. Then ∂ either bounds a disk D on S or cobounds an annulus A on S with a boundary component ϵ of S.

Suppose that ∂ bounds a disk D on S. Since no boundary component of X bounds a disk on S, no boundary component of X can be contained in D. Hence, D is disjoint from the boundary of X but intersects the interior of X, since it contains ∂. Since D is connected, this implies that D is contained in X. Since ∂ is an essential curve on X, this is a contradiction. Hence, ∂ does not bound a disk on S.

Hence, ∂ cobounds an annulus A on S with a boundary component ϵ of S. Since no boundary component of X can bound a disk on S or cobound an annulus with the boundary component ϵ of S, no boundary component of X can be contained in the interior of A. Hence, $A \setminus \epsilon$ is disjoint from the boundary of X but intersects the interior of X, since it contains ∂. Since $A \setminus \epsilon$ is connected and X is compact, this implies that A is contained in X. As before, this is a contradiction, and, hence, ∂ is an essential curve on S.

This shows that each boundary component of Y is either a boundary component of S or an essential curve on S, completing the proof that Y is a domain on S. □

Proposition 2.9. *Let X be a domain on S. Let α and β be curves on X. Then the geometric intersection number of α and β in X is equal to the geometric intersection number of α and β in S.*

Proof. Let m be equal to the geometric intersection number of α and β in X and n be equal to the geometric intersection number of α and β in S. Without loss of generality, we may assume that α and β are transverse with exactly m points of intersection. Then there does not exist a disk on X whose boundary is the union of an arc of α and an arc of β. Since α and β meet transversely at m points, we have$m \geq n$. Suppose that $m > n$. Then there exists a disk D on S whose boundary ∂_D is the union of an arc of α and an arc of β.

Let C be a component of ∂X. Since C is connected and disjoint from ∂D, C is either contained in the interior of D or the complement of D in S. Since C does not bound a disk on S, it follows that C is in the complement of D. It follows that D is disjoint from ∂X. Since D is connected and disjoint from ∂X, D is either contained in the interior of X or the complement of X on S. Since ∂D is contained in the interior of X, it follows that D is contained in the interior of X. Hence, D is a disk on X whose boundary ∂_D is the union of an arc of α and an arc of β. This is a contradiction. Thus, $m \leq n$ and, hence, $m = n$.

□

Note that one can also prove Proposition 2.9, using hyperbolic geometry.

Proposition 2.10. *Let X be a domain on S and α be an essential curve on X. Then there exists an essential curve β on X such that the geometric intersection number of α and β on S is not equal to zero.*

Proof. Since α is an essential curve on X, then X is not a sphere with at most three holes. Then, there exists a domain X' on X which is a sphere with four holes or a torus with one hole and such that α is an essential curve on X. We can find on X' a curve with the required properties, and use Proposition 2.9. □

Proposition 2.11. *Let X and Y be domains on S. Suppose that X is isotopic on S to a domain on Y. Then Y is not isotopic on S to a domain on X.*

Proof. Let X_1 be a domain on Y such that X is isotopic to X_1 on S. Suppose that Y is isotopic on S to a domain Y_1 on X. It follows that X is isotopic to a domain X_2 on S such that Y is a domain on X_2.

Since X_1 is a domain on Y and Y is a domain on X_2, it follows from Proposition 2.8 that X_1 is a domain on X_2. Thus, X_1 has an essential boundary component α on X_2. Since α is an essential curve on X_2, it follows from Proposition 2.10 that there exists an essential curve β on X_2 such that the geometric intersection number of α and β on S is not equal to zero.

Since X_1 is isotopic to X_2 on S, the essential boundary component α of X_1 on S is isotopic to an essential boundary component of X_2 on S. Hence, α is isotopic to a curve α_1 on S which is disjoint from X_2. It follows that $i(\alpha, \beta) = i(\alpha_1, \beta)$. Since α_1 is disjoint from X_2 and β is contained in X_2, α_1 is disjoint from β and, hence, $i(\alpha_1, \beta) = 0$. Hence, $i(\alpha, \beta) = 0$, which is a contradiction.

Hence, Y is not isotopic on S to a domain on X.

□

Proposition 2.12. *Let X be a domain on S. Let Y be the complement of the interior of X in S. Then:*

(1) *if α is an essential curve on X, then α is an essential curve on S;*
(2) *if α is an essential curve on X, then α is not isotopic to any curve on S contained in Y;*
(3) *if U and V are distinct components of Y and α is an essential curve on U, then α is not isotopic to any curve on V;*
(4) *if U is a component of Y and U is isotopic to X, then U and X are annuli meeting along their common boundary, $Y = U$, and S is a closed torus;*
(5) *if U is a component of Y and U is isotopic to X, then U and X are annuli meeting along their common boundary, $Y = U$, and S is a closed torus.*

The proof is easy.

The following is a weak converse for Proposition 2.8.

Proposition 2.13. *Let X be a domain on S and Y be a subsurface of X. If Y is a domain on S, then Y is isotopic on S to either X, or a domain on X, or a regular neighborhood of an essential boundary component of X.*

Proof. Assume that Y is not isotopic on S to either X or a regular neighborhood of an essential boundary component of X.

Let ∂ be a boundary component of Y. We may assume that ∂ is not a boundary component of X. Hence, ∂ is not a boundary component of S. Since Y is a domain on S, it follows that ∂ is an essential curve on S.

Suppose that ∂ is not an essential curve on X.

Since ∂ is an essential curve on S it cannot bound a disk on S. Hence, it cannot bound a disk on X. Hence, ∂ must cobound an annulus A on X with a boundary component ϵ of X.

Since ∂ is essential on S and A is an annulus on S, ϵ is not a boundary component of S. Hence, since X is a domain on S, ϵ is an essential boundary component of X on S.

Since Y is contained in X and the interior of Y is disjoint from the complement of $\partial \cup \epsilon$ in A, the interior of Y is contained in either the interior of A or the complement of A in X. Hence, Y is contained in either A or the closure of the complement of A in X.

Suppose that Y is contained in A. Since Y is a domain on S and A is contained in the interior of S, it follows that Y is isotopic on S to A. Since Y is not isotopic to a regular neighborhood of an essential boundary component of X on S, this is a contradiction.

Hence, Y is contained in the closure of the complement of A in X.

Let $Y' = Y \cup A$. Note that Y' is a domain on S which is isotopic to Y on S, is contained in X, and has one less boundary component in the interior of X than Y. It follows, by induction, that Y is isotopic to a domain on S which is contained in X and which has all of its boundary components in the boundary of X. In other words, Y is isotopic on S to X.

$\square$

The converse of Proposition 2.13 follows immediately from Propositions 2.8 and 2.13. Hence, we have the following equivalence.

Proposition 2.14. *Let X be a domain on S and Y be a subsurface of X. Then, Y is a domain on S if and only if Y is isotopic on S to either X, or a domain on X, or a regular neighborhood of an essential boundary component of X.*

Definition 2.15. Let X, J, and Y be domains on S. We say that X *is tied to* Y *by* J if X and J have exactly one common boundary component, J and

Y have exactly one boundary component, and the interior of J is disjoint from $X \cup Y$.

As an example, suppose that S is a closed torus. Then every domain on S is an annulus and any two disjoint domains on S, X and Y, are tied to one another by exactly two annuli on S, J and K. Moreover, $S = X \cup J \cup Y \cup K$.

Proposition 2.16. *Let X and Y be disjoint domains on S. Then X is isotopic to Y on S if and only if X and Y are annuli and X is tied to Y by an annulus on S.*

Definition 2.17. Let X be a domain on S. We say that X is *elementary* if it is either an annulus or a pair of pants on S. Otherwise, we say that X is *nonelementary.*

Proposition 2.18. *Let X be a domain on S. Then X is a nonelementary domain on S if and only if there exist curves α and β on S such that $i(\alpha, \beta) \neq 0$ and α and β are contained in the interior of X.*

Proof. Suppose, on the one hand, that X is elementary and α and β are curves on S contained in the interior of X. Then α is sotopic on S to a boundary components ∂ of X. Since β is in the interior of X, ∂ and β are disjoint. Hence, $i(\alpha, \beta) = i(\partial, \beta) = 0$.

Suppose, on the other hand, that X is nonelementary. Since X is a domain on S, X has at least one essential boundary component on S.

Suppose that the genus of X is positive. Then there exists an embedded torus with one hole Y in X which is either equal to X or is a domain on X. Let α and β be curves on Y which intersect transversely and have exactly one point of intersection. Then, by Proposition 2.9, $i_S(\alpha, \beta) = i_X(\alpha, \beta) = i_Y(\alpha, \beta) = 1$.

Suppose now that the genus of X is zero. Since X is a domain on S, X has at least one essential boundary component on S. In particular, X is not a disc. Since X is nonelementary, X is not an annulus or a pair of pants. Hence, X has at least four boundary components. Thus there exists an embedded sphere with four holes Y in X which is either equal to X or is a domain on X. Let α and β be curves on Y which intersect transversely, have exactly two points of intersection, and are such that the complement of $\alpha \cup \beta$ in Y has exactly four components, each of which contains exactly one boundary component of Y. Then, by Proposition 2.9, $i_S(\alpha, \beta) = i_X(\alpha, \beta) = i_Y(\alpha, \beta) = 2$.

□

Definition 2.19. Let $\mathcal{F}$ be a collection of pairwise disjoint domains on S. Let Y be the closure of the complement of $|\mathcal{F}|$ in S. A *codomain of* $\mathcal{F}$ is a components of Y.

Note that the codomains of a collection of pairwise disjoint domains on S are themselves domains on S.

Proposition 2.20. *Let X be a domain on S and Y be a codomain of X on S. If Y is an annulus, then Y is a nonseparating annulus on S and both boundary components of Y are essential boundary components of X.*

Proof. Since Y is a codomain of X, every essential boundary component of Y on S is a boundary component of X, and Y has at least one boundary component which is also a boundary component of X. Let ∂_1 be such a boundary component of Y and let ∂_2 be the other boundary component of Y. Since ∂_1 is a boundary component of X, it is essential on S, which implies that ∂_2 is not a boundary component of S. Thus, ∂_2 is also a boundary component of X, and therefore both ∂_1 and ∂_2 are essential boundary components of X. Since X is a connected subset of S and Y is a codomain of X on S, there is a path in S connecting the two boundary components of Y and whose image is in the complement of the interior of Y in S. This shows that Y is nonseparating on S. □

Corollary 2.21. *Suppose that X is a domain on S and X is an annulus. Then X has a codomain which is an annulus if and only if S is a closed torus.*

Proof. Suppose that X is a domain on S, that X is an annulus, and that X has a codomain Z which is an annulus. By Proposition 2.20, X and Z have their two boundary components in common. Since S is orientable, the union of X and Z is a closed torus. Since S is connected, S is equal to that torus.

The converse is clear. □

A nonempty collection of pairwise disjoint domains on S is called a *system of domains on S* if the domains in the collection are pairwise disjoint.

The following is an immediate corollary of Proposition 2.16.

Corollary 2.22. *Let $\mathcal{F}$ be a collection of pairwise disjoint domains on S. Then the following are equivalent:*

(1) *$\mathcal{F}$ is a system of domains on S;*

(2) *no two distinct annuli in $\mathcal{F}$ are isotopic on S;*

(3) *no two distinct annuli in $\mathcal{F}$ are tied to one another by an annulus on S.*

The collection of codomains of a system of domains on S is a collection of pairwise disjoint domains. However, the collection of codomains of a system of domains on S is not necessarily a system of domains, since two distinct such codomains may be isotopic.

Proposition 2.23. *Suppose that $\mathcal{F}$ is a collection of disjoint domains on S. Then $\mathcal{F}$ is a system of domains on S if and only if there does not exist two distinct annular domains of $|\mathcal{F}|$ which are joined by an annular codomain of $|\mathcal{F}|$.*

3 Simplicial complexes

3.1 Abstract simplicial complexes

In this section, we introduce some standard terminology from the theory of abstract simplicial complexes, and we complement it for later use in this chapter. A reference for this classical material is Munkres [48].

Definition 3.1. Let V be a set. An *abstract simplicial complex K with vertex set V* is a collection of finite subsets of V such that:

(1) if $v \in V$, then $\{v\} \in K$;

(2) if σ is an element of K and τ is a subset of σ, then τ is an element of K.

In this chapter, the term *simplicial complex* shall refer to an abstract simplicial complex, unless otherwise specified.

Let K be a simplicial complex with vertex set V. If x is an element of V, then we say that x is a *vertex of K*. If σ is an element of K and x is an element of σ, then we say that σ is a *simplex of K* and x is a *vertex of σ*. Note that each vertex of each simplex of K is a vertex of K. If σ has $k+1$ vertices, then we say that σ is a *k-simplex of K*. If x is an element of V, then we also say that the corresponding 0-simplex $\{x\}$ of K is a vertex of K. If e is a 1-simplex of K, then we say that *e is an edge of K*. If Δ is a 2-simplex of K, then we say that *Δ is a triangle of K*.

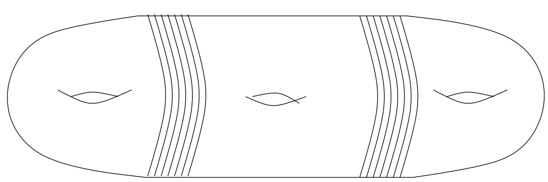

Figure 4. A system of domains (the non-shaded pieces) and their codomains(shaded). Two distinct codomains of a system of domains might be isotopic, as in this figure.

Definition 3.2. Let K be a simplicial complex and let F be a subcollection of K. We say that F *is a subcomplex of* K if each subset τ of an element of F is an element of F.

If F is a subcomplex of an abstract simplicial complex K, then F is itself an abstract simplicial complex, and the vertex set of F is a subset of the vertex set of K.

Proposition 3.3. *Let K be a simplicial complex with vertex set V and W be a subset of V. Let K_W be the set of all simplices of K that have all of their vertices in W. Then K_W is a subcomplex of K with vertex set W.*

Definition 3.4. Let K, W, and K_W be as in Proposition 3.3. We say that K_W *is the subcomplex of K induced by the subset W of the set of vertices V of K*.

Note that the subcomplex of a simplicial complex induced by a subset of its vertices is itself a simplicial complex. Moreover, it is completely determined by the simplicial complex K and its vertex set W.

Let K be a simplicial complex. For each nonnegative integer n, the *n-skeleton K_n of K* is the subcomplex of K consisting of all k-simplices of K with $k \leq n$. Note that V is equal to the support $|K_n|$ of K_n (i.e. the union of all the k-simplices of K with $k \leq n$).

The 1-skeleton of a simplicial complex K is a graph, sometimes called the *underlying graph* of K.

Note that if F is a subcomplex of an abstract simplicial complex K and n is any nonnegative integer, then the n-skeleton F_n of F is a subcomplex of the n-skeleton K_n of K.

If $\tau \subset \sigma \in K$, then we say that τ *is a face of* σ. A *maximal simplex of a simplicial complex* K is a simplex which is not a proper face of any simplex of K.

The simplicial complex K is *finite-dimensional* if there exists an integer N such that every simplex of K is a k-simplex for some $k \leq N$. If K is finite-dimensional, then the *dimension of* K is the minimum such integer N. If the dimension of K is N, then a *top-dimensional simplex of* K is an N-simplex of K.

A simplicial complex of dimension one is a *simplicial graph*, or, more briefly, a *graph* .

Definition 3.5. A simplicial complex K is a *flag complex* if the following holds:

> If $\{x_0, \ldots, x_n\}$ is a subset of K_0 such that $\{x_i, x_j\}$ is an edge of K for all $0 \leq i < j \leq n$, then $\{x_0, \ldots, x_n\}$ is a simplex of K.

Definition 3.6. Let α, β, and δ be simplices of a simplicial complex K. We say that α is *joined to* β by δ if $\delta = \alpha \cup \beta$.

Note that if α is joined to β by simplices δ and ϵ of K, then $\delta = \epsilon$.

Note also that two simplices of a simplicial complex are joined by a vertex of that simplicial complex if and only if they are both equal to that vertex.

Definition 3.7. Let α and β be simplices of a simplicial complex K which are joined in K by a simplex δ of K. Then we say that δ *is the join of* α *and* β *in* K.

Definition 3.8 (The star of a simplex). Let α be a simplex of a simplicial complex K. The *star of* α *in* K is the subcomplex $\mathrm{St}(\alpha) = \mathrm{St}(\alpha, K)$ of K whose simplices are the simplices of K which contain the simplex α together with all the faces of such simplices of K.

In particular, if x is a vertex of a simplicial complex K, then the *star of* x *in* K is the subcomplex $\mathrm{St}(x, K)$ of K whose simplices are the simplices of K which contain the vertex x together with all the faces of such simplices of K.

Let K be a simplicial complex and α be a simplex of K. Note that the 0-skeleton $St_0(\alpha, K)$ of $\mathrm{St}(\alpha, K)$ is the set of all vertices w of K such that $\alpha \cup \{w\}$ is a simplex of K.

Proposition 3.9. *Let* K *be a flag complex. Let* α *and* β *be simplices of* K. *Then the following are equivalent:*

(1) $\mathrm{St}(\alpha, K) = \mathrm{St}(\beta, K)$.

(2) $St_0(\alpha, K) = St_0(\beta, K)$.

Proof. Clearly (1) implies (2). We shall now show that (2) implies (1). To this end, suppose that (2) holds and let τ be a simplex of $\mathrm{St}(\alpha, K)$. We need to show that τ is a simplex of $\mathrm{St}(\beta, K)$. In other words, we need to show that $\beta \cup \tau$ is a simplex of K.

Since K is a flag complex, it suffices to show that any two distinct vertices of $\beta \cup \tau$ are joined by an edge of K. To this end, let x and y be distinct vertices of $\beta \cup \tau$. If x and y are both vertices of the simplex β of K, then $\{x, y\}$ is an edge of the simplex β of K and, hence, an edge of K. Likewise, if x and y are both vertices of the simplex τ of K, then $\{x, y\}$ is an edge of the simplex τ of K and, hence, an edge of K.

Suppose that x is a vertex of β and y is a vertex of τ. Since τ is a simplex of $\mathrm{St}(\alpha, K)$, y is a vertex of $\mathrm{St}(\alpha, K)$ and, hence, of $\mathrm{St}(\beta, K)$. Since y is a vertex of $\mathrm{St}(\beta, K)$, $\beta \cup \{y\}$ is a simplex of K. Since x is a vertex of β, it follows that $\{x, y\} \subset \beta \cup \{y\}$. Since $\{x, y\}$ is a face of the simplex $\beta \cup \{y\}$ of K, it follows that $\{x, y\}$ is a simplex of K.

Hence, every two distinct vertices of $\beta \cup \tau$ are joined by an edge of K. Since K is a flag complex, this implies that $\beta \cup \tau$ is a simplex of K. In other words, τ is a simplex of $\mathrm{St}(\beta, K)$.

This proves that $\mathrm{St}(\alpha, K)$ is a subcomplex of $\mathrm{St}(\beta, K)$. By a symmetric argument, it follows that $\mathrm{St}(\beta, K)$ is a subcomplex of $\mathrm{St}(\alpha, K)$.

This proves that $\mathrm{St}(\alpha, K) = \mathrm{St}(\beta, K)$.

□

Definition 3.10 (The link of a simplex). Let σ be a simplex of a simplicial complex K. The *link of σ in K* is the subcomplex $\mathrm{Lk}(\sigma) = \mathrm{Lk}(\sigma, K)$ of K whose simplices are the simplices of $\mathrm{St}(\sigma, K)$ which have empty intersection with σ.

The following proposition is proved by an argument similar to that given in the proof of Proposition 3.9.

Proposition 3.11. *Let K be a flag complex. Let α and β be simplices of K. Then the following are equivalent:*

(1) $\mathrm{Lk}(\alpha, K) = \mathrm{Lk}(\beta, K)$.

(2) *For each vertex x of K, x is a vertex of* $\mathrm{Lk}(\alpha, K)$ *if and only if x is a vertex of* $\mathrm{Lk}(\beta, K)$.

Note that if $\sigma = \emptyset$, then $\mathrm{Lk}(\sigma, K) = \mathrm{St}(\sigma, K) = K$.

Suppose that σ is a simplex of a simplicial complex K. Note that σ is joined to each of the simplices of $\mathrm{Lk}(\sigma)$; and the simplices of $\mathrm{St}(\sigma)$ are precisely the faces of the joins of σ with the simplices of $\mathrm{Lk}(\sigma)$.

In particular, if x is a vertex of a simplicial complex K, then the *link of x in K* is the subcomplex $\mathrm{Lk}(x, K)$ of K whose simplices are the simplices of $\mathrm{St}(x, K)$ that do not have x as a vertex.

Suppose that x is a vertex of a simplicial complex K. Note that $\{x\}$ is joined to each of the simplices of $\mathrm{Lk}(x, K)$, and the simplices of $\mathrm{St}(x, K)$ are precisely the faces of the joins of $\{x\}$ with the simplices of $\mathrm{Lk}(x, K)$.

Definition 3.12 (The link of a subcomplex). Let F be a subcomplex of a simplicial complex K. The *link of F in K*, denoted by $\mathrm{Lk}(F, K)$, is equal to $\cap\{\mathrm{Lk}(\sigma, K) \mid \sigma \in F\}$.

Remark 3.13. Suppose that σ is a simplex of K. Let $F(\sigma)$ be the subcomplex of K consisting of σ and all the faces of σ. Note that $\mathrm{Lk}(F(\sigma), K) \subset \mathrm{Lk}(\sigma, K)$. In general, this inclusion is strict, since a simplex which is joined to a face of σ in K is not necessarily joined to σ in K.

Note that if σ is the empty simplex of K, then $F(\sigma)$ is the empty subcomplex of K and $\mathrm{Lk}(\sigma, K) = K = \mathrm{Lk}(F(\sigma), K)$. Hence, there is no ambiguity regarding the "link of the empty set".

Remark 3.14. Let F be a subcomplex of K and σ be a simplex of F. Then $\mathrm{Lk}(F, K) \subset \mathrm{Lk}(\sigma, K)$.

Remark 3.15. Let F and G be subcomplexes of a simplicial complex K. If $F \subset G$, then $\mathrm{Lk}(G, K) \subset \mathrm{Lk}(F, K)$.

Proposition 3.16. *Let x be a vertex of a simplicial complex K. If there exists a simplex Δ in K such that* $\mathrm{Lk}(\Delta) = \{x\}$*, then* $\mathrm{Lk}(\mathrm{Lk}(x)) = \{x\}$.

Proof. Suppose that there exists a simplex Δ in K such that $\mathrm{Lk}(\Delta) = \{x\}$. Then, in particular, $x \in \mathrm{Lk}(\Delta)$. Hence, Δ is a simplex of $\mathrm{Lk}(x)$. This implies that $\mathrm{Lk}(\mathrm{Lk}(x)) \subset \mathrm{Lk}(\Delta)$. Hence, $\mathrm{Lk}(\mathrm{Lk}(x)) \subset \{x\}$. On the other hand, $x \in \mathrm{Lk}(\mathrm{Lk}(x))$. Thus, $\mathrm{Lk}(\mathrm{Lk}(x)) = \{x\}$. $\square$

Definition 3.17. Let K be a simplicial complex with vertex set V and L be a simplicial complex with vertex set W. A *simplicial map from K to L* is a map $\varphi : V \to W$ such that, for each simplex σ of K, $\varphi(\sigma)$ is a simplex of L.

Let $\varphi : V \to W$ be a simplicial map from a simplicial complex K with vertex set V to a simplicial complex L with vertex set W. The rule $\sigma \mapsto \varphi(\sigma)$ determines a map from K to L. We denote this map by $\varphi : K \to L$ and we say that *$\varphi : K \to L$ is a simplicial map from K to L*. If we need to distinguish between $\varphi : V \to W$ and $\varphi : K \to L$, then we shall say that $\varphi : V \to W$ is the *vertex correspondence associated to $\varphi : K \to L$*.

Note that the map $\varphi : K \to L$ is both determined by and determines the map $\varphi : V \to W$. The map $\varphi : K \to L$ is injective if and only if $\varphi : V \to W$ is injective. If $\varphi : K \to L$ is surjective, then $\varphi : V \to W$ is surjective. The converse, however, is not necessarily true. For instance, if L has at least one edge e and K is equal to the zero skeleton L_0 of L, then the vertex set V of K is equal to the vertex set W of L, the identity map $\varphi : V \to W$ is surjective, but the corresponding map $\varphi : K \to L$ is not surjective, since the edge e of L is not in the image of $\varphi : K \to L$.

If $\varphi : K \to L$ is a simplicial map, then $\varphi(K)$ is a subcomplex of L.

Definition 3.18. Let K and L be abstract simplicial complexes. A *simplicial isomorphism* $\varphi : K \to L$ is a simplicial map $\varphi : K \to L$ for which there exists a simplicial map $\psi : L \to K$ such that $\varphi : K \to L$ and $\psi : L \to K$ are inverse functions. In the case where $K = L$, we call a simplicial isomorphism $\varphi : K \to L$ a *simplicial automorphism* of K.

Note that a simplicial map $\varphi : K \to L$ is a simplicial isomorphism if and only if $\varphi : K \to L$ is bijective. By the previous observations, if $\varphi : K \to L$ is bijective, then $\varphi : V \to W$ is also bijective. The converse need not be true.

The following naturality peoperty follows easily from the definitions:

Proposition 3.19. *Let K be a simplicial complex, $\varphi \in \mathrm{Aut}(K)$, and x be a vertex of K. Then $\varphi(\mathrm{St}(x, K)) = \mathrm{St}(\varphi(x), K)$ and $\varphi(\mathrm{Lk}(x, K)) = \mathrm{Lk}(\varphi(x), K)$.*

3.2 Exchange automorphisms

We shall use the following notion of exchange automorphisms of abstract simplicial complexes, and the related results.

Definition 3.20. Let K be a simplicial complex, $\{x, y\}$ be a pair of vertices of K, and $\varphi : K \to K$ be an automorphism of K. We say that *φ is a simple exchange of K exchanging the vertices x and y of K* if $\varphi(x) = y$, $\varphi(y) = x$, and $\varphi(z) = z$ for every vertex z of K which is neither equal to x nor equal to y.

Let $\varphi : K \to K$ be a simple exchange of a simplicial complex K exchanging the vertices x and y of K. Note that $\varphi : K \to K$ is equal to the identity map $id_K : K \to K$ of K if and only if $x = y$. In this case, we say that *φ is a trivial simple exchange*. Otherwise, φ is said to be a *nontrivial simple exchange*.

Let K be a simplicial complex, $\varphi : K \to K$ be a simple exchange of K exchanging the vertices x and y of K, and $\psi : K \to K$ be a simple exchange of K exchanging the vertices u and v of K. Then $\varphi = \psi$ if and only if either $x = y$ and $u = v$ or $\{x, y\} = \{u, v\}$. In particular, a nontrivial simple exchange of K exchanges a unique pair of distinct vertices of K.

Example 3.21. Let $K(n)$ denote the simplicial complex of all subsets of the set $\{1, \ldots, n\}$. Then, for every pair of distinct vertices i and j of $K(n)$, the standard transposition (i, j) in the group of permutations of $\{1, \ldots, n\}$ extends to a simple exchange of $K(n)$ which exchanges i and j. These simple exchanges generate the group of simplicial automorphisms of $K(n)$, which is naturally isomorphic to the symmetric group Σ_n, the group of permutations of the vertex set $\{1, \ldots, n\}$ of $K(n)$.

Definition 3.22 (Exchangeable vertices). Let K be a simplicial complex and let x and y be two vertices of K. We say that *x and y are exchangeable in K* if there exists a simple exchange of K exchanging x and y.

Example 3.23. Let $V = \mathbb{Z} \times \{-1, 0, 1\}$ and K be the one-dimensional simplicial complex on V, illustrated in Figure 6, whose edges are the pairs $\{(m, 0), (m+1, 0)\}$ and $\{(m, 0), (m, \epsilon)\}$ with $m \in \mathbb{Z}$ and $\epsilon \in \{-1, 1\}$. Then two distinct vertices x and y of K are exchangeable if and only if $\{x, y\} = \{(m, -1), (m, 1)\}$ for some $m \in \mathbb{Z}$.

Note that for any subset W of $\mathbb{Z}$ there is a unique automorphism $\varphi_W : K \to K$ such that $\varphi_W(m,t)$ is equal to $(m,-t)$ if $m \in W$ and (m,t) otherwise. In particular, $\varphi_\emptyset = id_K : K \to K$. If U and V are subsets of $\mathbb{Z}$, then $\varphi_U \circ \varphi_V = \varphi_{U \triangle V}$. In particular, $\varphi_W \circ \varphi_W = id_K : K \to K$. It follows that the collection $\{\varphi_W | W \subset \mathbb{Z}\}$ of automorphisms of K is a subgroup B_K of the group of automorphisms of K, $Aut(K)$, naturally isomorphic to the Boolean algebra $\mathcal{B}(\mathbb{Z})$ of all subsets of $\mathbb{Z}$.

Let D_K be the group of automorphisms of K generated by the translation $(m,n) \mapsto (m+1,n)$ and the involution $(m,n) \mapsto (1-m,n)$. Note that this involution has no fixed vertices in K. The subgroup D_K of $Aut(K)$ is naturally isomorphic to the infinite dihedral group D_∞ of isometries of $\mathbb{Z}$ equipped with its standard metric.

The group of automorphism of K, $Aut(K)$, is a split extension of its subgroup D_K by its normal subgroup B_K, and we have the following commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{B}(\mathbb{Z}) & \longrightarrow & \mathcal{B}(\mathbb{Z}) \rtimes D_\infty & \longrightarrow & D_\infty = Isom(\mathbb{Z}) & \longrightarrow & 1 \\
 & & \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow & & \\
1 & \longrightarrow & B_K & \longrightarrow & \mathrm{Aut}(K) & \longrightarrow & D_K & \longrightarrow & 1
\end{array}$$

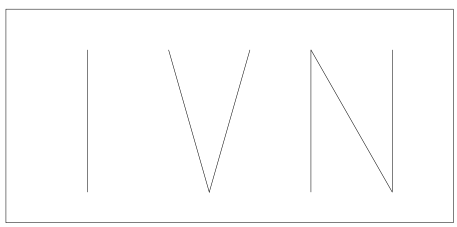

Figure 5. Three complexes: The one to the left has all its vertices exchangeable. The one in the middle has exactly one pair of distinct vertices that are exchangeable. The one to the right has no distince vertices exchangeable.

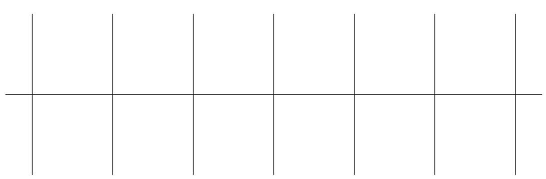

Figure 6. The complex used in example3.23: a line of edges.

On the set of vertices of any simplicial complex, we define a relation, $\sim$, called the *exchange relation*, in which $x \sim y$ if and only if x and y are exchangeable.

Proposition 3.24. *Let K be a simplicial complex. Then the exchange relation $\sim$ on K is an equivalence relation on K_0.*

Proof. Let $x \in K_0$. Then the identity map $id_K : K \to K$ is a simplicial automorphism of K exchanging x and x. Hence, $x \sim x$.

Suppose that $x, y \in K_0$ and $x \sim y$. Let $\varphi : K \to K$ be a simplicial automorphism of K which exchanges x and y. Then the same automorphism $\varphi : K \to K$ is a simplicial automorphism of K which exchanges y and x. Hence, $y \sim x$.

Suppose that $x, y, z \in K_0$, $x \sim y$, and $y \sim z$. Let $\varphi : K \to K$ be a simplicial automorphism of K which exchanges x and y and $\psi : K \to K$ be a simplicial automorphism of K which exchanges y and z. Then the conjugate $\varphi \circ \psi \circ \varphi : K \to K$ of $\psi : K \to K$ by $\varphi : K \to K$ is a simplicial automorphism of K which exchanges x and z. Hence, $x \sim z$.

□

The following result gives a basic necessary and sufficient condition for two vertices of a simplicial complex to be exchangeable.

Proposition 3.25. *Let K be a simplicial complex and x and y be vertices of K. Let F be the subcomplex of K consisting of all simplices of K that have neither x nor y as a vertex. Then the following are equivalent:*

(1) *x and y are exchangeable in K.*

(2) $\mathrm{St}(x, K) \cap F = \mathrm{St}(y, K) \cap F$.

Proof. First, we prove that (1) implies (2). To this end, suppose that x and y are exchangeable in K. Then, there is a unique automorphism $\varphi : K \to K$ such that $\varphi(x) = y$, $\varphi(y) = x$, and $\varphi(z) = z$ for every vertex z of F.

Suppose that σ is a simplex of $\mathrm{St}(x, K) \cap F$. In other words, suppose that $\{x\} \cup \sigma$ is a simplex of K and σ is a simplex of F. Then $\varphi(\{x\} \cup \sigma) = \{\varphi(x)\} \cup \varphi(\sigma) = \{y\} \cup \sigma$ is a simplex of K. Since $\{y\} \cup \sigma$ is a simplex of K and σ is a simplex of F, it follows that σ is a simplex of $\mathrm{St}(y, K) \cap F$. This proves that $\mathrm{St}(x, K) \cap F \subset \mathrm{St}(y, K) \cap F$. Likewise, $\mathrm{St}(y, K) \cap F) \subset \mathrm{St}(x, K) \cap F$ and, hence, $\mathrm{St}(x, K) \cap F = \mathrm{St}(y, K) \cap F$. This proves that (1) implies (2).

Now we prove that (2) implies (1). To this end, suppose that $\mathrm{St}(x, K) \cap F = \mathrm{St}(y, K) \cap F$.

Consider the bijection $\varphi : K_0 \to K_0$ defined by the rule $\varphi(x) = y$, $\varphi(y) = x$, and $\varphi(z) = z$ for every vertex z of F. Since $\varphi : K_0 \to K_0$ is an involution of

K_0, it suffices to prove that φ extends to a simplicial map $\varphi : K \to K$. In other words, it suffices to prove that $\varphi(\tau)$ is a simplex of K for every simplex τ of K.

To this end, suppose that τ is a simplex of K. If x and y are both vertices of τ, then $\varphi(\tau)$ is equal to the simplex τ of K. Likewise, if neither x nor y is a vertex of τ, then $\varphi(\tau)$ is equal to the simplex τ of K.

Suppose that x is a vertex of τ and y is not a vertex of τ. Let $\sigma = \tau \setminus \{x\}$. Since x is a vertex of τ and $\sigma = \tau \setminus \{x\}$, it follows that $\tau = \{x\} \cup \sigma$. Since τ is a simplex of K, this implies that σ is a simplex of $\mathrm{St}(x, K)$. Since y is not a vertex of τ and $\sigma = \tau \setminus \{x\}$, σ is a simplex of F. This implies that σ is a simplex of $\mathrm{St}(x, K) \cap F$ and, hence, of $\mathrm{St}(y, K) \cap F$. Since σ is a simplex of $\mathrm{St}(y, K)$, $\{y\} \cup \sigma$ is a simplex of K.

Since σ is a simplex of F and $\tau = \{x\} \cup \sigma$, it follows that $\varphi(\tau) = \{\varphi(x)\} \cup \varphi(\sigma) = \{y\} \cup \sigma$. Hence, $\varphi(\tau)$ is a simplex of K.

This shows that if x is a vertex of τ and y is not a vertex of τ, then $\varphi(\tau)$ is a simplex of K. Likewise, if y is a vertex of τ and x is not a vertex of τ, then $\varphi(\tau)$ is a simplex of K.

In any case, $\varphi(\tau)$ is a simplex of K.

This proves that $\varphi : K_0 \to K_0$ extends to a simplicial map $\varphi : K \to K$. Since $\varphi : K_0 \to K_0$ is an involution, its extension $\varphi : K \to K$ is an involution. Hence, this extension $\varphi : K \to K$ is a simplicial automorphism exchanging x and y.

This proves that (2) implies (1).

□

Remark 3.26. Note that $\mathrm{St}(x, K)$ joins x to the subcomplex $F \cap \mathrm{Lk}(x, K)$ of F and, in the case where $\{x, y\}$ is an edge of K, also to y. Likewise, $\mathrm{St}(y, K)$ joins y to the subcomplex $F \cap \mathrm{Lk}(y, K)$ of F and, in the case where $\{x, y\}$ is an edge of K, also to x. Roughly speaking, the above *exchangeability condition*, Condition (2) in Proposition 3.25, states that F is a sort of hyperplane of reflection across which the vertices x and y of K are able to be reflected since they have been symmetrically joined to F along a subcomplex G of F (i.e. along $F \cap \mathrm{Lk}(x, K) = F \cap \mathrm{Lk}(y, K)$) and, in the case where $\{x, y\}$ is an edge of K, to one another.

The following propositions are refinements of Proposition 3.25 corresponding to the situations where $\{x, y\}$ is or is not an edge of K.

Proposition 3.27. *Let K be a simplicial complex and x and y be distinct vertices of K that are not connected by an edge of K. Then the following are equivalent:*

(1) *x and y are exchangeable in K.*

(2) $\mathrm{Lk}(x, K) = \mathrm{Lk}(y, K)$.

Proof. Let F be the subcomplex of K consisting of all simplices of K which have neither x nor y as a vertex. Since x and y are not joined by an edge of K, it follows that $\mathrm{Lk}(x, K) = \mathrm{St}(x, K) \cap F$ and $\mathrm{Lk}(y, K) = \mathrm{St}(y, K) \cap F$. □

Proposition 3.28. *Let K be a simplicial complex and x and y be vertices of K which are connected by an edge of K. Suppose that K is a flag complex. Then the following are equivalent:*

(1) *x and y are exchangeable in K.*

(2) $\mathrm{St}(x, K) = \mathrm{St}(y, K)$.

Proof. Let F be the subcomplex of K consisting of all simplices of K which have neither x nor y as a vertex. Suppose that $\mathrm{St}(x, K) = \mathrm{St}(y, K)$. Then $\mathrm{St}(x, K) \cap F = \mathrm{St}(y, K) \cap F$. It follows from Proposition 3.25 that x and y are exchangeable. This proves that (2) implies (1).

We shall now show that (1) implies (2). Suppose that x and y are exchangeable in K. It follows from Proposition 3.25 that $\mathrm{St}(x, K) \cap F = \mathrm{St}(y, K) \cap F$. We must show that $\mathrm{St}(x, K) = \mathrm{St}(y, K)$. To this end, suppose that τ is a simplex of $\mathrm{St}(x, K)$. Let $\sigma = \tau \cup \{x\}$. Since τ is a simplex of $\mathrm{St}(x, K)$, t σ is also a simplex of K.

Let $\rho = \sigma \cup \{y\}$. We shall show that ρ is a simplex of K. Since K is a flag complex, it suffices to show that any two distinct vertices of ρ are joined by an edge of K. To this end, let w and z be vertices of ρ. If neither w nor z is equal to y, then w and z are vertices of the simplex σ of K and, hence, are joined by an edge of K. Hence, we may assume that $z = y$. This implies that w is not equal to y. It follows that x and w are both vertices of the simplex σ of K. Hence, w is a vertex of $\mathrm{St}(x, K)$. If $w = x$, then w and z are vertices of the simplex $\{x, y\}$ of K. Hence, we may assume that w is not equal to x.

Since w is a vertex of $\mathrm{St}(x, K)$ and w is not equal to x or y, it follows that w is a vertex of $\mathrm{St}(x, K) \cap F$ and, hence, of $\mathrm{St}(y, K) \cap F$. It follows that w is a vertex of $\mathrm{St}(y, K)$. Since w is not equal to y, this implies that w and y are joined by an edge of K. In other words, w and z are joined by an edge of K.

In any case, w and z are joined by an edge of K.

This shows that ρ is a simplex of K. Since τ is a face of the simplex ρ of K and y is a vertex of ρ, it follows that τ is a simplex of $\mathrm{St}(y, K)$.

This shows that $\mathrm{St}(x, K) \subset \mathrm{St}(y, K)$. Likewise, $\mathrm{St}(y, K) \subset \mathrm{St}(x, K)$. Hence, $\mathrm{St}(x, K) = \mathrm{St}(y, K)$.

This proves that (1) implies (2).

□

Proposition 3.29. *Let K be a simplicial complex. Let $\mathcal{E}$ be a collection of exchangeable pairs of distinct vertices of K with the property that no two distinct pairs in $\mathcal{E}$ have a common vertex. Then there exists a unique automorphism*

$\varphi_{\mathcal{E}} : K \to K$ such that (i) for each pair $\{x, y\}$ in $\mathcal{E}$, $\varphi_{\mathcal{E}}(x) = y$ and $\varphi_{\mathcal{E}}(y) = x$ and (ii) $\varphi_{\mathcal{E}}(z) = z$ for every vertex z of K which is not an element of some pair in $\mathcal{E}$.

Proof. Let $\varphi : K_0 \to K_0$ be the unique involution which exchanges the two vertices in each pair in $\mathcal{E}$ and fixes every other vertex of K. Let τ be a simplex of K. We shall now show that $\varphi(\tau)$ is a simplex of K. Let τ_0 be the set of all vertices x of τ such that there does not exist a vertex y of K such that $\{x, y\} \in \mathcal{E}$, let τ_1 be the set of all vertices x of τ such that there exists a vertex y of K such that $\{x, y\} \in \mathcal{E}$ and $\{x, y\} \cap \tau = \{x\}$ and let τ_2 be the set of all vertices x of τ such that there exists a vertex y of K such that $\{x, y\} \in \mathcal{E}$ and $\{x, y\} \cap \tau = \{x, y\}$. Note that $\tau = \tau_0 \cup \tau_1 \cup \tau_2$. From the definition of φ, $\varphi(\tau_i) = \tau_i$, $i = 0, 1, 2$.

Let n be the number of elements of τ_1.

Suppose, on the one hand, that $n = 0$. Then, $\tau_1 = \emptyset$ and, hence, $\varphi(\tau) = \varphi(\tau_0 \cup \tau_2) = \varphi(\tau_0) \cup \varphi(\tau_2) = \tau_0 \cup \tau_2 = \tau$. Hence, $\varphi(\tau)$ is equal to the simplex τ of K.

Suppose, on the other hand, that $n > 0$. Let $\tau_1 = \{x_j | 1 \leq j \leq n\}$. For each integer j with $1 \leq j \leq n$, let y_j be the unique vertex of K such that $\{x_j, y_j\} \in \mathcal{E}$. From the definition of φ, $\varphi(\tau_1) = \{y_j | 1 \leq j \leq n\}$. It follows that $\varphi(\tau) = \tau_0 \cup \tau_2 \cup \{y_j | 1 \leq j \leq n\}$.

Let j be an integer with $1 \leq j \leq n$. Since $\{x_j, y_j\} \in \mathcal{E}$, $\{x_j, y_j\}$ is an exchangeable pair of vertices of K. Hence, there exists a simple exchange $\varphi_j : K \to K$ of K exchanging x_j and y_j. Since the distinct pairs $\{x_j, y_j\}$, $1 \leq j \leq n$, are in $\mathcal{E}$, they are disjoint. It follows that the composition $\varphi_1 \circ \ldots \circ \varphi_n : K \to K$ is an automorphism ψ of K such that $\psi(\tau_0) = \tau_0$, $\psi(\tau_2) = \tau_2$ and $\psi(x_j) = y_j$, $1 \leq j \leq n$. This implies that $\varphi(\tau) = \psi(\tau)$. Since $\psi : K \to K$ is an automorphism of K and τ is a simplex of K, it follows that $\psi(\tau)$ is a simplex of K; that is to say, $\varphi(\tau)$ is a simplex of K.

This shows that the involution $\varphi : K_0 \to K_0$ extends to a simplicial map $\varphi : K \to K$. Since $\varphi : K_0 \to K_0$ is an involution, its simplicial extension $\varphi : K \to K$ is also an involution and, hence, a simplicial automorphism of K. Hence, $\varphi : K \to K$ is a simplicial automorphism of K such that (i) for each pair $\{x, y\}$ in $\mathcal{E}$, $\varphi_{\mathcal{E}}(x) = y$ and $\varphi_{\mathcal{E}}(y) = x$ and (ii) $\varphi_{\mathcal{E}}(z) = z$ for every vertex z of K which is not an element of some pair in $\mathcal{E}$. Since the stated conditions on $\varphi : K \to K$ determine the restriction $\varphi : K_0 \to K_0$, and since any two simplicial maps which agree on the vertices of their common domain are equal, it follows that $\varphi : K \to K$ is the unique such automorphism of K.

$\square$

Definition 3.30 (Generalized exchange). Let K, $\mathcal{E}$, and $\varphi_{\mathcal{E}} : K \to K$ be as in Proposition 3.29. We call the automorphism $\varphi_{\mathcal{E}} : K \to K$ of K the *generalized exchange of K associated to $\mathcal{E}$.*

If $\mathcal{F}$ and $\mathcal{G}$ are subsets of $\mathcal{E}$, then $\varphi_{\mathcal{F}} \circ \varphi_{\mathcal{G}} = \varphi_{\mathcal{F} \triangle \mathcal{G}}$, where $\mathcal{F} \triangle \mathcal{G}$ denotes symmetric difference. Hence, by Proposition 3.29, we have the following result.

Proposition 3.31. *Let K be a simplicial complex. Let $\mathcal{E}$ be a collection of exchangeable pairs of distinct vertices of K with the property that no two distinct pairs in $\mathcal{E}$ have a common vertex. Then there exists a monomorphism Φ from the Boolean algebra $\mathcal{B}(\mathcal{E})$ of all subsets of $\mathcal{E}$ to $Aut(K)$ such that $\Phi(\mathcal{F}) = \varphi_{\mathcal{F}}$ for every subset $\mathcal{F}$ of $\mathcal{E}$.*

Definition 3.32 (Boolean subgroup). Let K be a simplicial complex. Let $\mathcal{E}$ be a collection of exchangeable pairs of distinct vertices of K with the property that no two distinct pairs in $\mathcal{E}$ have a common vertex. The *Boolean subgroup of $Aut(K)$ corresponding to* $\mathcal{E}$, denoted by $B_{\mathcal{E}}$, is the image $\Phi(\mathcal{B}(\mathcal{E}))$ of the Boolean algebra $\mathcal{B}(\mathcal{E})$ under the monomorphism Φ of Proposition 3.31. In particular, the Boolean subgroup $B_{\mathcal{E}}$ is naturally isomorphic to the Boolean algebra $\mathcal{B}(\mathcal{E})$.

Proposition 3.33. *Let K be a simplicial complex. Let $\mathcal{E}$ be a collection of exchangeable pairs of distinct vertices of K with the property that no two distinct pairs in $\mathcal{E}$ have a common vertex. Let $\varphi \in \mathrm{Aut}(K)$, $\mathcal{F} \subset \mathcal{E}$ and $\mathcal{G} = \varphi(\mathcal{F})$. Then $\mathcal{G}$ is a collection of exchangeable pairs of distinct vertices of K with the property that no two distinct pairs in $\mathcal{G}$ have a common vertex. Moreover, $\varphi \circ \Phi_{\mathcal{F}} \circ \varphi^{-1} = \Phi_{\mathcal{G}}$.*

3.3 Quotient complexes

In this section, we develop a notion of quotient complex which will be used in our study of some simplicial complexes.

Proposition 3.34. *Let K be a simplicial complex on the vertex set V, W be a set, and $\rho : V \to W$ be a map of V onto W. Let L be the collection of all subsets τ of W for which there exists a simplex σ of K such that $\tau = \rho(\sigma)$. Then L is a simplicial complex with vertex set W and $\rho : V \to W$ is a simplicial map from K to L*

Proof. First, we show that L is a simplicial complex. For this, we must show that each singleton subset of W is an element of the collection L and every subset of an element of L is an element of L. Since $\rho : V \to W$ is surjective, each element w of W is the image under ρ of a vertex v of K. Thus, $\{w\} = \rho(\{v\})$. Since K is a simplicial complex and v is a vertex of K, $\{v\}$ is a simplex of K. Hence, by the definition of L, $\{w\}$ is an element of L.

Suppose that τ is an element of L and ϵ is a subset of τ. By the definition of L, there exists a simplex σ of K such that $\tau = \rho(\sigma)$. Let $\delta = \rho^{-1}(\epsilon) \cap \sigma$. Since

δ is contained in the simplex σ of the simplicial complex K, δ is a simplex of K. Since $\tau = \rho(\sigma)$ and ϵ is contained in τ, it follows that $\epsilon = \rho(\delta)$. Hence, by the definition of L, ϵ is an element of L.

This shows that L is a simplicial complex.

Next, we show that $\rho : V \to W$ is a simplicial map from K to L. To this end, let σ be a simplex of K and $\tau = \rho(\sigma)$. By the definition of L, τ is a simplex of L. This proves that $\rho : V \to W$ is a simplicial map from K to L. □

Definition 3.35 (Quotient complex). Let K, V, W, $\rho : V \to W$, and L be as in Proposition 3.34. We say that L *is the quotient complex of* K and $\rho : K \to L$ *is the natural projection* associated to the vertex correspondence $\rho : V \to W$.

Definition 3.36 (Simplicial quotient map). Let K be a simplicial complex with vertex set V, L be a simplicial complex with vertex set W, and $\rho : K \to L$ be a simplicial map. We say that $\rho : K \to L$ is a *simplicial quotient map* if for every subset τ of W, τ is a simplex of L if and only if there exists a simplex σ of K for which $\rho(\sigma) = \tau$.

Proposition 3.37. *Let $\rho : V \to W$ be a simplicial quotient map from a simplicial complex K to a simplicial complex L. Then $\rho : V \to W$ maps V onto W.*

Proof. Let w be an element of W. Then $\{w\}$ is a simplex of L. Since $\rho : K \to L$ is a simplicial quotient map, it follows that there exists a simplex σ of K such that $\{w\} = \rho(\sigma)$; that is to say, $\{w\} = \{\rho(x) \mid x \in \sigma\}$. Thus, there exists a vertex x of σ such that $w = \rho(x)$. Since $x \in \sigma \subset V$, $x \in V$. Hence, there exists an element x of V such that $w = \rho(x)$. This proves that $\rho : V \to W$ maps V onto W. □

Proposition 3.38. *Let $\rho : V \to W$ be a simplicial quotient map from a simplicial complex K to a simplicial complex L and $\alpha : V \to Z$ be a simplicial map from K to a simplicial complex M respectively. Suppose that $\alpha : V \to Z$ is constant on each fiber $\rho^{-1}(w)$, $w \in W$, of $\rho : V \to W$ (i.e. that $\alpha(x) = \alpha(y)$ whenever $\rho(x) = \rho(y)$, $x, y \in V$). Then there exists a unique simplicial map $\beta : V \to Z$ from K to M such that $\alpha = \beta \circ \rho : V \to Z$.*

Proof. By Proposition 3.37, $\rho : V \to W$ maps V onto W. Since $\rho : V \to W$ is surjective and $\alpha : V \to Z$ is constant on the fibers of $\rho : V \to W$, there exists a unique map $\beta : W \to Z$ such that $\alpha = \beta \circ \rho : V \to Z$.

It remains only to show that $\beta : W \to Z$ is a simplicial map from L to M. To this end, suppose that τ is a simplex of L. Since $\rho : K \to L$ is a simplicial quotient map, there exists a simplex σ of K such that $\tau = \rho(\sigma)$. Thus $\beta(\tau) = \beta(\rho(\sigma)) = (\beta \circ \rho)(\sigma) = \alpha(\sigma)$. Since $\alpha : K \to M$ is a simplicial map and σ is a simplex of K, it follows that $\alpha(\sigma)$ is a simplex of M; that is to say, $\beta(\tau)$ is a simplex of M. This proves that $\beta : W \to Z$ is a simplicial map from L to M, which completes the proof. □

Proposition 3.39. *Let $\rho : V \to W$ and $\alpha : V \to Z$ be simplicial quotient maps from a simplicial complex K to a simplicial complex L and a simplicial complex M. Suppose that $\rho : V \to W$ and $\alpha : V \to Z$ have the same fibers (i.e. for each pair of elements, x and y, of V, $\rho(x) = \rho(y)$ if and only if $\alpha(x) = \alpha(y)$). Then there exists a unique simplicial isomorphism $\beta : W \to Z$ from L to M such that $\alpha = \beta \circ \rho : V \to Z$.*

Proof. By Proposition 3.37, $\rho : V \to W$ and $\alpha : V \to Z$ are surjective.

By Proposition 3.38, there exist unique simplicial maps $\beta : W \to Z$ from L to M and $\gamma : Z \to W$ from M to L such that $\alpha = \beta \circ \rho$ and $\rho = \gamma \circ \alpha$.

It remains only to show that $\beta : W \to Z$ and $\gamma : Z \to W$ are inverse maps. To this end, let $\delta = \gamma \circ \beta : W \to W$ and $\epsilon = \beta \circ \gamma : Z \to Z$.

Note that $\delta \circ \rho = (\gamma \circ \beta) \circ \rho = \gamma \circ (\beta \circ \rho) = \gamma \circ \alpha = \rho : V \to W$. Since $\rho : V \to W$ is surjective and $\delta \circ \rho = \rho : V \to W$, it follows that $\delta : W \to W$ is equal to the identity map of W. Likewise, $\epsilon : Z \to Z$ is equal to the identity map of Z.

This proves that $\beta : W \to Z$ and $\gamma : Z \to W$ are inverse maps, completing the proof. □

Proposition 3.39 shows that any two quotient complexes of a given simplicial complex corresponding to simplicial quotient maps with the same fibers are canonically isomorphic. We now construct a canonical model for the isomorphism class of any such quotient complex.

Definition 3.40. Let $\sim$ be an equivalence relation on the vertex set V of an abstract simplicial complex K. Let $\tilde{V}$ be the set of equivalence classes of $\sim$ on V and $\rho : V \to W$ be the associated natural projection which maps each vertex x of K to its equivalence class $[x] = \{y \in V \mid y \sim x\}$. Let $\tilde{K}$ be the quotient complex of K and $\rho : K \to \tilde{K}$ be the natural projection associated to the vertex correspondence $\rho : V \to \tilde{V}$. We say that $\tilde{K}$ *is the quotient of* K *by* $\sim$ and $\rho : K \to \tilde{K}$ *is the natural projection from* K *to* $\tilde{K}$.

We have the following immediate corollary of Proposition 3.39.

Proposition 3.41. *Let $\alpha : V \to Z$ be a simplicial quotient map from a simplicial complex K to a simplicial complex M. Let $\sim$ be the equivalence relation on V defined by the rule $x \sim y$ if and only if $\alpha(x) = \alpha(y)$. Let $\tilde{K}$ be the quotient of K by $\sim$ and $\rho : K \to \tilde{K}$ be the natural projection from K to $\tilde{K}$. Then there exists a unique simplicial isomorphism $\beta : \tilde{V} \to Z$ from $\tilde{K}$ to M such that $\alpha = \beta \circ \rho : V \to Z$.*

Definition 3.42 (Derived complex). Let $\sim$ be the exchange relation on the vertex set V of an abstract simplicial complex K. We denote $\tilde{K}$ by K' and say that K' is the *derived complex of* K.

Proposition 3.43. *Let K be a simplicial complex and $\rho : K \to K'$ be the natural projection from K to the derived complex K' of K. Let $\varphi : K \to K$ be an automorphism of K. Then there exists a unique automorphism $\varphi' : K' \to K'$ such that $\rho \circ \varphi = \varphi' \circ \rho : K \to K'$.*

Definition 3.44 (Derived automorphism). Let K, $\rho : K \to K'$, $\varphi : K \to K$, and $\varphi' : K' \to K'$ be as in Proposition 3.43. We say that the automorphism $\varphi' : K' \to K'$ of K' is the *automorphism of K' derived from $\varphi : K \to K$.*

Proposition 3.45. *Let K be a simplicial complex and K' be its derived complex. Then there is a homomorphism $\rho : \mathrm{Aut}(K) \to \mathrm{Aut}(K')$ defined by the following rule: for each automorphism $\varphi : K \to K$ of K, $\rho(\varphi)$ is equal to the automorphism of K' derived from $\varphi : K \to K$.*

Definition 3.46 (Derivation homomorphism). Let $\rho : \mathrm{Aut}(K) \to \mathrm{Aut}(K')$ be as in Proposition 3.45. We say that $\rho : \mathrm{Aut}(K) \to \mathrm{Aut}(K')$ is the *derivation homomorphism from* $\mathrm{Aut}(K)$ *to* $\mathrm{Aut}(K')$.

Definition 3.47 (Exchange automorphisms group). Let K be a simplicial complex, $\rho : \mathrm{Aut}(K) \to \mathrm{Aut}(K')$ be the derivation homomorphism from $\mathrm{Aut}(K)$ to $\mathrm{Aut}(K')$, and $\mathrm{Aut}(K)$ be the kernel of $\rho : \mathrm{Aut}(K) \to \mathrm{Aut}(K')$. We call $\mathrm{Aut}(K)$ the *group of exchange automorphisms of* K and any element φ of $\mathrm{Aut}(K)$ an *exchange automorphism of* K.

Example 3.48. If x and y are exchangeable vertices of $\mathcal{E}$, then the simple exchange of K exchanging the vertices x and y of K is an exchange automorphism. More generally, let $\mathcal{E}$ be a collection of exchangeable pairs of distinct vertices of K with the property that no two distinct pairs in $\mathcal{E}$ have a common vertex. Then the generalized exchange $\varphi_{\mathcal{E}} : K \to K$ of K associated to $\mathcal{E}$ is an exchange automorphism, and the Boolean subgroup of $\mathrm{Aut}(K)$ corresponding to $\mathcal{E}$, $B_{\mathcal{E}}$, is a subgroup of the group of exchange automorphisms $\mathrm{Aut}(K)$ of K.

Definition 3.49 (Generalized exchange automorphism). Let K, $\mathcal{E}$, and $\varphi_{\mathcal{E}} : K \to K$ be as in Proposition 3.29. We call the automorphism $\varphi_{\mathcal{E}} : K \to K$ of K the *generalized exchange of K associated to $\mathcal{E}$.*

Proposition 3.50. *Let K be a simplicial complex. Let $\mathcal{E}$ be a collection of exchangeable pairs of distinct vertices of K with the property that each pair of distinct vertices of K in $\mathcal{E}$ is an edge of K and no two distinct edges in $\mathcal{E}$ have* *a common vertex. Let $\sim$ be the equivalence relation on the vertex set V of K defined by the rule $x \sim y$ if and only if either $x = y$ or $\{x, y\} \in \mathcal{E}$. Let $\tilde{K}$ be the quotient of K by the equivalence relation $\sim$ on V and $\rho : K \to \tilde{K}$ be the associated natural projection. Let τ be a subset of the vertex set $\tilde{V}$ of $\tilde{K}$. Then τ is a simplex of $\tilde{K}$ if and only if $\rho^{-1}(\tau)$ is a simplex of K.*

Proof. Suppose, on the one hand, that $\rho^{-1}(\tau)$ is a simplex of K. Since $\rho : V \to \tilde{V}$ is surjective, $\rho(\rho^{-1}(\tau)) = \tau$. Since $\rho^{-1}(\tau)$ is a simplex of K and $\rho(\rho^{-1}(\tau)) = \tau$, it follows from the definition of $\tilde{K}$ that τ is a simplex of $\tilde{K}$.

Suppose, on the other hand, that τ is a simplex of $\tilde{K}$. It follows from the definition of $K/\sim$ that there exists a simplex σ of K such that $\rho(\sigma) = \tau$.

Since σ is a simplex of K, there exists a nonnegative integer k and $k+1$ distinct vertices, x_i, $0 \leq i \leq k$, of K such that $\sigma = \{x_i \mid 0 \leq i \leq k\}$.

Let i be an integer with $0 \leq i \leq k$. Let $y_i = x_i$ if x_i is not a vertex of some pair of distinct vertices of K in the collection $\mathcal{E}$. Otherwise, let y_i be the unique vertex of K such that $\{x_i, y_i\}$ is one of the edges of K in the collection $\mathcal{E}$. If $x_i = y_i$, then $\mathrm{St}(x_i, K) = \mathrm{St}(y_i, K)$. Otherwise, $\{x_i, y_i\}$ is an edge of K with an exchangeable pair of vertices and, hence, by Proposition 3.28, $\mathrm{St}(x_i, K) = \mathrm{St}(y_i, K)$. Thus, in any case, $\mathrm{St}(x_i, K) = \mathrm{St}(y_i, K)$.

By our choices of y_i, $0 \leq i \leq k$, it follows from the definition of $\sim$ that $\rho^{-1}(\tau) = \{x_i, y_i \mid 0 \leq i \leq k\}$.

Let i be an integer with $0 \leq i \leq k$. Let $\sigma_i = \sigma \cup \{y_j \mid 0 \leq j \leq i\}$. We shall prove, by induction on i, that σ_i is a simplex of K.

First, consider the case where $i = 0$. Since $x_0 \in \sigma$, σ is a simplex of $\mathrm{St}(x_0, K)$ and, hence, of $\mathrm{St}(y_0, K)$. Since σ is a simplex of $\mathrm{St}(y_0, K)$, $\sigma \cup \{y_0\}$ is a simplex of K. Hence, we may let $\sigma_0 = \sigma \cup y_0\}$.

Now suppose that $0 \leq i < k$. Assume, by induction, that σ_i is a simplex of K. Since $0 < i + 1 \leq k$, x_{i+1} is a vertex of σ and, hence, of σ_i. Thus, σ_i is a simplex of $\mathrm{St}(x_{i+1}, K)$ and, hence, of $\mathrm{St}(y_{i+1}, K)$. Since σ_i is a simplex of $\mathrm{St}(y_{i+1}, K)$, $\sigma_i \cup \{y_{i+1}\}$ is a simplex of K. Since $\sigma_{i+1} = \sigma_i \cup \{y_{i+1}\}$, it follows that σ_{i+1} is a simplex of K.

This proves, by induction, that σ_k is a simplex of K. Since $\sigma_k = \sigma \cup \{y_j \mid 0 \leq j \leq k\}$ and $\sigma = \{x_i \mid 0 \leq i \leq k\}$, it follows that $\sigma_k = \{x_i, y_i \mid 0 \leq i \leq k\}$. In other words, $\sigma_k = \rho^{-1}(\tau)$. It follows that $\rho^{-1}(\tau)$ is a simplex of K, which completes the proof.

□

The following propositions will be useful below for our computations of the automorphism groups of the complex of domains which we study below.

Proposition 3.51. *Let $K, \mathcal{E}, \sim, \tilde{K}, \rho$ be as in Proposition 3.50. Let $\mathrm{Aut}_{\mathcal{E}}(K)$ be the stabilizer of $\mathcal{E}$ in $\mathrm{Aut}(K)$. If $\varphi \in \mathrm{Aut}_{\mathcal{E}}(K)$, then there exists a unique simplicial automorphism $\varphi_* : \tilde{K} \to \tilde{K}$ such that $\varphi_* \circ \rho = \rho \circ \varphi : K \to \tilde{K}$.*

Proof. Let $\varphi \in \mathrm{Aut}_{\mathcal{E}}(K)$; that is to say, suppose that $\varphi(\mathcal{E}) = \{\varphi(e) \mid e \in \mathcal{E}\} = \mathcal{E}$. It follows that the fibers of both simplicial maps $\rho : V \to \tilde{V}$ and $\rho \circ \varphi : V \to \tilde{V}$ from K to $\tilde{K}$ are the equivalence classes of the equivalence relation $\sim$ on the vertex set V of K.

Since the simplicial maps $\rho : V \to \tilde{V}$ and $\rho \circ \varphi : V \to \tilde{V}$ from K to $\tilde{K}$ have the same fibers and $\rho : K \to \tilde{K}$ is a simplicial quotient map, it follows from Proposition 3.38 that there exists a unique simplicial map $\varphi_* : \tilde{K} \to \tilde{K}$ such that $\varphi_* \circ \rho = \rho \circ \varphi : K \to \tilde{K}$.

It remains only to show that $\varphi_* : \tilde{K} \to \tilde{K}$ is a simplicial automorphism of $\tilde{K}$.

To this end, let $\psi = \varphi^{-1} : K \to K$. Since $\varphi(\mathcal{E}) = \mathcal{E}$, it follows that $\psi(\mathcal{E}) = \mathcal{E}$. Hence, by the argument above, there exists a unique simplicial map $\psi_* : \tilde{K} \to \tilde{K}$ such that $\psi_* \circ \rho = \rho \circ \psi : K \to \tilde{K}$.

Note that $(\psi_* \circ \varphi_*) \circ \rho = \psi_* \circ (\varphi_* \circ \rho) = \psi_* \circ (\rho \circ \varphi) = (\psi_* \circ \rho) \circ \varphi = (\rho \circ \psi) \circ \varphi = \rho \circ (\psi \circ \varphi) = \rho \circ id_V = \rho : V \to \tilde{V}$. Since $\rho : V \to \tilde{V}$ is surjective and $(\psi_* \circ \varphi_*) \circ \rho = \rho : V \to \tilde{V}$, it follows that $\psi_* \circ \varphi_* : \tilde{V} \to \tilde{V}$ is the identity map of $\tilde{V}$. Likewise, $\varphi_* \circ \psi_* : \tilde{V} \to \tilde{V}$ is the identity map of $\tilde{V}$.

This proves that $\varphi_* : \tilde{V} \to \tilde{V}$ and $\psi_* : \tilde{V} \to \tilde{V}$ are inverse simplicial maps from $\tilde{K}$ to $\tilde{K}$. It follows that $\varphi_* : \tilde{V} \to \tilde{V}$ is a simplicial automorphism of $\tilde{K}$, completing the proof.

□

Proposition 3.52. *Let $K, \mathcal{E}, \sim, \tilde{K}, \rho$ be as in Proposition 3.50 and Proposition 3.50. Let $\mathrm{Aut}_{\mathcal{E}}(K)$ be the stabilizer of $\mathcal{E}$ in $\mathrm{Aut}(K)$ and $\mathrm{Aut}_{\rho(\mathcal{E})}(\tilde{K})$ be the stabilizer of $\rho(\mathcal{E})$ in $\mathrm{Aut}(\tilde{K})$. Then there exists a unique homomorphism $\eta : \mathrm{Aut}_{\mathcal{E}}(K) \to \mathrm{Aut}_{\rho(\mathcal{E})}(\tilde{K})$ such that for each automorphism $\varphi \in \mathrm{Aut}_{\mathcal{E}}(K)$, $\eta(\varphi)$ is the unique simplicial automorphism $\varphi_* : \tilde{K} \to \tilde{K}$ such that $\varphi_* \circ \rho = \rho \circ \varphi : K \to \tilde{K}$. Moreover, there exists a natural short exact sequence:*

$$1 \to B_{\mathcal{E}} \to \mathrm{Aut}_{\mathcal{E}}(K) \to \mathrm{Aut}_{\rho(\mathcal{E})}(\tilde{K}) \to 1$$

corresponding to inducing automorphisms of $\tilde{K}$ from automorphisms of K which preserve $\mathcal{E}$.

Proof. The existence and uniqueness of such a homomorphism $\eta : \mathrm{Aut}_{\mathcal{E}}(K) \to \mathrm{Aut}_{\rho(\mathcal{E})}(\tilde{K})$ follows from Proposition 3.51. Since $B_{\mathcal{E}}$ is by definition a subgroup of $\mathrm{Aut}_{\mathcal{E}}(K)$, the homomorphism $B_{\mathcal{E}} \to \mathrm{Aut}_{\mathcal{E}}(K)$ is injective. That the kernel of the natural homomorphism $\eta : \mathrm{Aut}_{\mathcal{E}}(K) \to \mathrm{Aut}_{\rho(\mathcal{E})}(\tilde{K})$ is equal to the image of the natural homomorphism $B_{\mathcal{E}} \to \mathrm{Aut}_{\mathcal{E}}(K)$ follows from the definition of the natural projection $\rho : K \to \tilde{K}$ and the definition of $\eta : \mathrm{Aut}_{\mathcal{E}}(K) \to \mathrm{Aut}_{\rho(\mathcal{E})}(\tilde{K})$. □

3.4 Topology

One can associate to any abstract simplicial complex K a topological space called the *geometric realization* of K, and, usually, the topological properties of K refer to those of K. However, there are certain topological properties of K that have a simple definition in terms of K, and we shall use this approach. Thus, we use the following terminology:

We shall say that K is *connected* if for any any two vertices v and w in K, there exists a finite sequence of vertices $v_0 = v, v_1, \ldots, v_n = w$ such that for all $k = 0, 1, n-1$, $\{v_k, v_{k+1}\}$ is an edge of k. We shall say that the sequence $v_0 = v, v_1, v_n = w$ is an edge-path of length n joining v and w.

Note that a simplicial complex is connected if and only if its 1-skeleton is connected.

We shall say that K is *bounded* if there exists an integer n such that any two vertices of K are connected by an edge-path of length $\leq n$.

We shall say that K is *unbounded* if it is not bounded.

We shall say that K is *locally finite* if for any vertex v of K, there are only finitely many edges containing it.

4 Some complexes associated to S

In this section, we shall discuss some abstract simplicial complexes associated to S and on which the extended mapping class group $\Gamma^*(S)$ acts naturally. All of these complexes are finite-dimensional flag complexes.

The simplices of each of the complexes are finite collections of isotopy classes of subspaces of S of a certain type, which can be represented by disjoint subspaces of this type. The extended mapping class group $\Gamma^*(S)$ of S acts naturally on each of these complexes via the natural action of the group of homeomorphisms of S on the relevant subspaces.

For each of these complexes we say that an automorphism of the complex is *geometric* if it is induced by a homeomorphism of S. A question that has been addressed about such complexes is whether every automorphism of such a complex is geometric. The affirmative answer to this question is known to hold for a number of complexes associated to S, see [1], [7], ([12], [20], [22], [21], [24], [25], [35],[38], [40], [47], [**?**], [36] and [37]. The complex of domains plays a special role in this theory, because the answer for that complex is negative, as we shall see later in this chapter (see Theorem 8.8).

4.1 The curve complex $C(S)$

Definition 4.1. The *curve complex of* S, $C(S)$, is the simplicial complex whose n-simplices, for every $n \geq 2$, are collections of $n+1$ distinct isotopy classes of essential disjoint curves on S.

The curve complex was introduced by Harvey in 1978, with the idea that this complex encodes some boundary structure of Teichmüller space, in analogy to Tits buildings which encode a boundary structure of symmetric spaces. This complex turned out to be an extremely interesting object, and it has been studied for itself by Ivanov, Masur, Minsky, Hammenstaedt, Bowditch and others.

Note that a finite collection of vertices of $C(S)$ forms a simplex of $C(S)$ if and only if each pair of vertices in this collection can be represented by disjoint curves on S. In other words, $C(S)$ is a flag complex.

In the case where S is a sphere with at most three holes, $C(S)$ is empty.

In the case where S is a sphere with four holes or a torus with at most one hole, there are infinitely many isotopy classes of curves on S, but no two such curves are disjoint and nonisotopic. Hence, in these cases, $C(S)$ is an infinite set of vertices.

If S is not a sphere with at most four holes or a torus with at most one hole, then $C(S)$ is connected. This result was stated by Harvey in [15], and proofs were given by Harer in [16] and by Masur and Minsky in [41], §2.2. The proof that Masur and Minsky gave in [41] (Lemma 2.1) uses induction on the number of intersection points between curves. In fact, Masur and Minsky gave an upper bound of the distance between two vertices in terms of the intersection number of the curves that represent these vertices. Ivanov gave in [30] another proof of the same fact using Cerf theory.

A maximal simplex in the curve complex $C(S)$ is represented by a family $\mathcal{C}$ of disjoint essential curves on S such that the surface $S_{\mathcal{C}}$ obtained by cutting S along $\mathcal{C}$ is a disjoint union of pairs of pants. Thus, a maximal simplex in $A(S)$ is naturally a pants decomposition.

Proposition 4.2 (The dimension of the curve complex $C(S)$). *If S is a sphere with at most three holes, then $C(S)$ is empty. If S is a closed torus, then $C(S)$ is an infinite set of vertices. Otherwise, all maximal simplices of $C(S)$ have the same number of vertices, which is $3g+b-3$, and* $\dim(C(S)) = 3g+b-4$.

Proof. This follows by a standard Euler characteristic argument similar to that employed in the proof of Proposition 4.6. □

The extended mapping class group acts simplicially on $C(S)$ in a natural manner: if $\gamma \in \Gamma^*(S)$ is the class of a homeomorphism f of S and if σ is a simplex of $C(S)$ which is represented by a collection of curves $C_1, \ldots, C_k$, then $\gamma(\sigma)$ is the simplex represented by the collection of curves $f(C_1), \ldots, f(C_k)$.

For any surface S, the complex $C(S)$ is finite-dimensional. Indeed, there is an upper bound for the number of pairwise disjoint and pairwise non-isotopic essential curves on S. For $g \geq 2$, or $n \geq 3$, the maximal number of non-isotopic essential curves on S is $3g - 3 + b$ (which is equal to the number of essential curves in a pants decomposition of S). Therefore, the dimension of $C(S)$ is equal to $3g-4+b$. Note that this dimension is ≥ 1 provided S is not a sphere with at most four holes or a torus with at most one hole.

The complex $C(S)$ is not locally finite, provided it is connected. The reason is that as soon as a surface contains an essential curve, it contains infinitely many such curves. Thus, if α is an essential curve on S and if $C(S)$ is connected, then there are infinitely many distinct essential curves on the surface S_α obtained from S by cutting it along α, and therefore the vertex representing α in $C(S)$ belongs to infinitely many edges.

From the natural action of the extended mapping class group $\Gamma^*(S)$ on $C(S)$, we obtain a natural homomorphism from $\Gamma^*(S)$ into the group $\mathrm{Aut}(C(S))$ of simplicial automorphisms of $C(S)$.

The basic result on the automorphism group of $C(S)$ is due to N. Ivanov, who proved that for any $g \geq 2$, the natural homomorphism $\Gamma^*(S) \to \mathrm{Aut}(C(S))$ is an isomorphism provided S is not the closed surface of genus 2. In the case of genus 2, Ivanov proved that S the homomorphism is surjective and its kernel is $\mathbb{Z}_2$, generated by the hyperelliptic involution (see [25]).

Korkmaz continued the analysis made by Ivanov and he studied the case of surfaces of genus 0 and 1. He proved in [35] that for such surfaces, any automorphism of $C(S)$ is induced by an element of $\Gamma^*(S)$ if S is not a sphere with ≤ 4 holes or a torus with ≤ 2 holes.

In the cases where S is a torus with one hole or a sphere with four holes, there are automorphisms of $C(S)$ that are not geometric since in each of these cases the curve complex is an infinite countable set of vertices, and therefore its automorphism group is uncountable.

Luo in [38] analyzed a delicate remaining case, which is the case where the surface S is a torus with two holes. He proved that in that case the map $\Gamma^*(S) \to \mathrm{Aut}(C(S))$ is not surjective.

Let us say a few words about that case.

Luo noticed that there is an isomorphism $C(S_{1,2}) \to C(S_{0,5})$ induced by the projection map $\pi : S_{1,2} \to S_{1,2}/\iota$, where ι is a hyperelliptic involution of $S_{1,2}$, and where $S_{0,5}$ is identified with the complement of the singular locus of π in $S_{1,2}/\iota$. Thus, the automorphism group of $C(S_{1,2})$ is isomorphic to the automorphism group of $C(S_{0,5})$. Now it is known that the extended mapping class groups $\Gamma^*(S_{1,2})$ and $\Gamma^*(S_{0,5})$ are not isomorphic. More precisely, $\Gamma^*(S_{1,2})$ is an order-two extension of a subgroup of index 5 in $\Gamma^*(S_{0,5})$. Thus, we have $\Gamma^*(S_{1,2}) \not\simeq \mathrm{Aut}(C(S_{1,2}))$. One can understand the situation as follows. Consider a hyperelliptic involution of the torus with two punctures which exchanges the two punctures. The quotient surface is a sphere with one puncture, and the quotient map is ramified over four points. In this way, the mapping classes of the torus with two punctures correspond to the mapping classes of the sphere with five punctures that preserve one of the punctures and that permute the four others. This gives rise to a subgroup of index five. The extended mapping class group of the torus with two punctures is an extension of that group by the hyperelliptic involution.

The homomorphism $\Gamma^*(S_{1,2}) \to \mathrm{Aut}(C(S_{1,2}))$ is also not injective, since the hyperelliptic involution ι acts trivially on $C(S_{1,2})$. (This was already known, from works of Birman and of Viro, cf. [4] and [51].)

The following theorem summarizes the results on the automorphism group of the complex $C(S)$.

Theorem 4.3 (Ivanov-Korkmaz-Luo). *Consider a surface $S_{g,n}$ whose curve complex $C(S)$ has positive dimension. (Equivalently, the curve complex of $C(S)$ is connected; equivalently, S is not a sphere with at most four holes or a torus with at most one hole). Then, we have the following:*

(1) *For $(g,n) \notin \{(1,2),(2,0)\}$, the natural homomorphism*
$$\Gamma^*(S_{g,n}) \to \mathrm{Aut}(C(S_{g,n}))$$
is an isomorphism.

(2) *The homomorphism $\Gamma^*(S_{2,0}) \to \mathrm{Aut}(C(S_{2,0}))$ is surjective and its kernel is of order two, generated by the hyperelliptic involution.*

(3) *The homomorphism $\Gamma^*(S_{1,2}) \to \mathrm{Aut}(C(S_{1,2}))$ is neither surjective nor injective. The kernel of this homomorphism is of order two, generated by the hyperelliptic involution, and its image is a subgroup of index 5 in $\mathrm{Aut}(C(S_{1,2}))$. The image consists in the simplicial automorphisms of $C(S_{1,2})$ that preserve the set of vertices represented by nonseparating curves.*

Luo, in his paper [38], gave a proof of Thorem 4.3 that includes all the cases and which is different from the proofs by Ivanov and by Korkmaz. Luo's proof uses induction, and it is in the spirit of Grothendieck's reconstruction principle (see Chapter 17 of Volume II of this Handbook [39]).

We note finally that for any domain X on S, we have a natural simplicial embedding
$$C(X) \hookrightarrow C(S).$$

4.2 The arc complex $A(S)$

Definition 4.4. The *arc complex of* S, $A(S)$, is the simplicial complex whose n-simplices are collections of $n+1$ pairwise distinct isotopy classes of disjoint essential arcs on S.

Since homeomorphisms and isotopies of S take homotopic arcs to homotopic arcs and disjoint arcs to disjoint arcs, the extended mapping class group of S acts simplicially on $A(S)$.

Let us first consider a few cases of surfaces of low genus and small number of components.

It is easy to see that $A(S)$ is empty if either $b=0$ or $g=0$, $b=1$, and that it is reduced to a single vertex if $g=0$, $b=2$

For $g=0, b=3$, $A(S)$ is a finite 2-dimensional simplicial complex having six vertices, nine edges and four 2-cells, see Figure 7.

In all the other cases, $A(S)$ is a locally infinite connected complex with infinitely many vertices.

A maximal simplex in the arc complex $A(S)$ is represented by a family $\mathcal{A}$ of disjoint essential arcs on S such that the surface $S_{\mathcal{A}}$ obtained by cutting S along $\mathcal{A}$ is a disjoint union of hexagons. In other words, a maximal simplex in $A(S)$ is naturally an ideal triangulation in the following sense:

Definition 4.5 (Ideal triangulation). An *ideal triangulation* of S is a system of disjoint and pairwise non-isotopic arcs in S that is maximal with respect to inclusion.

The complement on S of an ideal triangulation is a collection of hexagons, where a hexagon is a disk with six distinct points on its boundary, dividing this boundary into six arcs called the *distinguished edges.* Three non-consecutive edges arise from three arcs on the surface, and the other edges are segments in the boundary of S. We shall call such a hexagon an *ideal hexagon.* The names *ideal triangulation* and *ideal hexagon* stem from the fact that if we pinch each boundary component of S to a point, obtaining, as a quotient, a closed surface with distinguished points arising from the boundary components of S, then each ideal hexagon becomes, in the quotient surface, a triangle whose vertices are at the set of distinguished points, and the ideal triangulation of S becomes a decomposition into triangles having all of their vertices at the distinguished points; that is, an ideal triangulation in the usual sense.

We shall study a graph called the *ideal triangulation graph* in Section 4.8below.

Proposition 4.6 (The dimension of the arc complex $A(S)$). *If S is a closed surface or a sphere with one hole, then $A(S)$ is empty. If S is a sphere with two holes, then $A(S)$ is a singleton. In all the other cases, all maximal simplices of*

$A(S)$ have the same number of vertices, which is $6g+3b-6$, and $\dim(A(S)) = 6g+3b-7$.

Proof. This follows by a standard standard Euler characteristic argument. $\square$

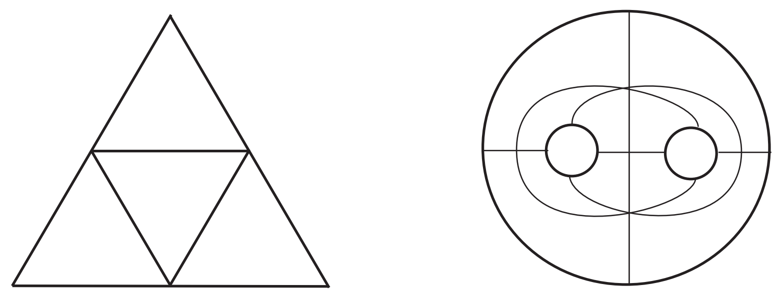

Figure 7. The finite simplicial complex on the left-hand side represents the arc complex of the sphere with three holes. The six vertices of this complex are the isotopy classes of the arcs represented in the right-hand side.

Irmak and McCarthy gave a complete description of the automorphism group of the arc complex. They proved the following:

Theorem 4.7 (Irmak-McCarthy [23]). *Let $S_{g,n}$ be a surface with nonempty boundary and with negative Euler characteristic. Then, the natural homomorphism*

$$\rho : \Gamma^*(S) \to \mathrm{Aut}(A(S))$$

is surjective and it is an isomorphism provided $(g,n) \notin \{(1,1),(0,3)\}$. In the excluded cases, the kernel of ρ is the centre of $\Gamma^*(S)$. In other words, we have the following:

(1) *if S is a pair of pants, the kernel of ρ is $\mathbb{Z}_2$, generated by the isotopy class of any orientation-reversing involution of S that preserves each boundary component of S;*

(2) *if S is a torus with one hole, the kernel of ρ is $\mathbb{Z}_2$, generated by the hyperelliptic involution of S.*

We note that the proof of this result, given in [23], does not make use of the corresponding result for the curve complex (Theorem 4.3 above). We also note that in the same paper, Irmak and McCarthy obtained a stronger result, namely, they proved that any injective simplicial self-map of $A(S)$ is induced by a homeomorphism of S.

We end this section by mentioning a few natural maps between arc complexes and curve complexes of surfaces.

There is an operation of *doubling* a surface $S = S_{g,b}$ with nonempty boundary along one or several of its boundary components. It is defined as follows. We choose a subset ∂_0 of the boundary ∂S of S, and we assume δ_0 is a union of k boundary components. The *double* of S along ∂_0 is a surface $S^d_{\partial_0}$ of genus $2g+k-1$ having $b-k$ boundary components, equipped with a system ∂_0' of k curves, having the property that the surface $S^d_{\partial_0}$ cut along ∂_0 consists of two copies of S, such that the image of the union of boundary curves ∂_0 by the two natural inclusions of S in $S^d_{\partial_0}$ is the union of curves ∂_0', and such that there is an orientation-reversing involution of $S^d_{\partial_0}$ that fixes pointwise the set $|\partial_0'|$ and exchanges the two copies of S in $S^d_{\partial_0}$.

Now given a surface S with nonempty boundary $\partial = \partial S$, and given a collection $\partial_0 \subset \partial$ of k components of ∂, we denote by $A(S, \partial_0)$ the subcomplex of $A(S)$ induced by the vertices represented by arcs on S having its two endpoints on ∂_0. The union of any arc in S whose two endpoints are on ∂_0 with its image in the double $S^d_{\partial_0}$ by the natural involution is a curve in $S^d_{\partial_0}$ of of S along ∂_0. This gives a natural simplicial embedding

$$A(S_{g,b}, \partial_0) \hookrightarrow C(S_{2g+k-1,b-k}).$$

In particular, if we take ∂_0 to be the union of all the boundary components of $S = S_{g,b}$, then the resulting surface $S^d_\partial = S_{2g+b-1,0}$ is called the *double* of $S = S_{g,b}$. The union of any arc on $S_{g,b}$ with its image by the natural involution of S^d is a curve on $S_{g+b,0}$, and this association defines a natual simplicial embedding

$$A(S_{g,b}) \hookrightarrow C(S_{2g+b-1,0}).$$

If X is a domain on S such that a nonempty set of boundary components of X are boundary components of S, then we have a natural simplicial embedding

$$A(X, \partial X \cap \partial S) \hookrightarrow A(S).$$

This map can be useful for studying arc complexes of surfaces of infinite type. Let S_∞ be a surface with boundary which has infinite type, and suppose that S_∞ admits an exhaustion by subsurfaces of finite type with boundary,

$$S_0 \subset S_1 \subset S_2 \subset \ldots,$$

such that a nonempty subset of the boundary components of S_0 are boundary components of S_∞, such that for every $i \geq 0$, S_i is a domain on S_{i+1} and $S_\infty = \cup_{i=1}^\infty S_i$. Then, we have a sequence of natural embeddings

$$A(S_0, \partial S_0 \cap \partial S_1) \hookrightarrow A(S_1, \partial S_1 \cap \partial S_2) \hookrightarrow A(S_2, \partial S_2 \cap \partial S_3) \ldots$$

4.3 The arc and curve complex $AC(S)$

We now introduce an abstract simplicial complex in which the curve complex and the arc complex naturally embed.

Definition 4.8 (The arc and curve complex). The *arc and curve complex*, $AC(S)$, of S is the simplicial complex whose k-simplices, for each $k \geq 0$, are collections of $k+1$ distinct isotopy classes of one-dimensional submanifolds which can be either essential simple closed curves or essential arcs in S, and such that this collection can be represented by disjoint curves or arcs on the surface.

This complex was studied by Hatcher in [18], who proved that this complex is contractible.

Note that if $b = 0$, then there are no arcs on $S = S_{g,b}$, and in that case $AC(S_{g,0}) = C(S_{g,0})$. When talking about the arc and curve complex, we shall assume that $b \geq 1$.

Proposition 4.9 (Maximal simplices in the arc and curve complex [36]). *A maximal simplex of $AC(S)$ that has maximal dimension consists of arcs, i.e. it is an ideal triangulation of S. The dimension of such a simplex is $6g+3n-7$. The dimension of a maximal simplex Δ in $AC(S)$ that has minimal dimension is $3g+2n-4$. There exist maximal simplices in $AC(S)$ of all dimensions between $3g+2n-4$ and $6g+3n-7$.*

This proposition immediately gives the following rigidity result:

Theorem 4.10. *Let $S = S_{g,n}$ and $S' = S_{h,p}$ be two surfaces of types (g,n) and (h,p) respectively, and assume that the two corresponding arc and curve complexes $AC(S)$ and $AC(S')$ are homeomorphic. Then, S is homeomorphic to S'.*

Proof. From Proposition 4.9, if $AC(S)$ and $AC(S')$ are homeomorphic, we have $6g+3n-7 = 6h+3p-7$ and $3g+2n-4 = 3g+2n-4$. The two equations imply $g = h$ and $n = p$; that is, the surfaces are homeomorphic. $\square$

Each element of the extended mapping class group $\mathrm{Mod}^*(S)$ naturally acts in a simplicial way on the complex $AC(S)$, and its is clear that the resulting map from $\mathrm{Mod}^*(S)$ to the simplicial automorphism group $\mathrm{Aut}(AC(S))$ of $AC(S)$ is a homomorphism.

There are natural simplicial maps from the curve complex $C(S)$ and from the arc complex $A(S)$ into the arc and curve complex, which extend the natural inclusions at the level of the vertices. These maps are injectve, and the acomplexes $C(S)$ and $A(S)$ are naturally subcomplexes of the arc and curve complex.

We have the following:

Theorem 4.11 (Korkmaz-Papadopoulos [36]). *If the surface $S_{g,n}$ is not a sphere with one, two or three punctures nor a torus with one puncture, then the natural homomorphism* $\mathrm{Mod}^*(S_{g,b}) \to \mathrm{Aut}(AC(S_{g,b}))$ *is an isomorphism.*

This result says in particular that there are no automorphisms of the arc and curve complex that sends a vertex represented by an arc (respectively a curve) to a vertex represented by a curve (respectively an arc). In fact, this can be seen directly by counting the dimensions of maximal simplices containing vertices, and this is used in the proof of Theorem 4.11 given [36]. Two different proofs of 4.11 are given in [36]; one proof uses the induced action of an automorphism of $AC(S)$ on the arc complex, and another one uses the induced action on the curve complex.

Let us say a few words on the cases that are excluded by the hypothesis of Theorem 4.11.

If S is a sphere with one puncture, then $AC(S)$ is empty.

If S is a sphere with two punctures, $AC(S) = A(S)$ is a single point, and if S is a sphere with three punctures, $AC(S) = A(S)$ is a finite complex (see Figure 7 above). In both cases, by Theorem 4.7, the natural homomorphism from $\mathrm{Mod}^*(S_{g,n})$ to $\mathrm{Aut}(AC(S_{g,n}))$ is surjective and its kernel is $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, which is the center of $\mathrm{Mod}^*(S_{g,n})$.

Finally, in the case where S is a torus with one puncture, the natural homomorphism $\mathrm{Aut}(AC(S_{g,n})) \to \mathrm{Aut}(A(S_{g,n}))$ is an isomorphism, which implies that we have an isomorphism $\mathrm{Mod}^*(S_{g,n})/\mathbb{Z}_2 \simeq \mathrm{Aut}(AC(S))$.

Similar to the simplicial map $A(S_{g,b}) \to C(S_{2g+b-1,0})$ defined in §4.2, there is a natural injective simplicial map $AC(S_{g,b}) \to C(S_{2g+b-1,0})$ obtained by doubling. In fact, taking doubles along subsets of the set of boundary components (see the definition in §4.2), gives a sequence of simplicial injections $AC(S_{g,b}) \to AC(S_{2g+k-1,b-k})$, defined for $1 \leq k \leq b$.

4.4 The pants decomposition graph $P_1(S)$

A *elementary move* between two pants decompositions on S is a transformation in which a single curve C is modified (that is to say, the two pants decompositions involved in that move contain the same set of curves except for that curve C), such that C and the curve C' obtained from C by the move have the smallest possible intersection number. Thus, $i(C, C') = 1$ or 2, depending on whether C is on the boundary of one or of two pairs of pants (and the same situation holds at the same time in the two pants decompositions that are involved in the move). The two types of elementary moves are represented in Figure 8.

We regard a elementary move as an operation which is well defined up to isotopy, so we can talk of two isotopy classes of pants decompositions that are obtained from each other by a elementary move.

We shall say that the elementary move is *performed* on the curve that is transformed in the pair of pants decomposition. Note that the number of possible elementary moves performed on a given curve is always infinite.

Definition 4.12 (Pants decomposition graph). The *pants decomposition graph* $P_1(S)$ is the one-dimensional simplicial complex whose vertices are isotopy classes of pants decompositions and where two vertices are joined by an edge if and only if the two pants decompositions that represent them (up to homotopy) differ by an elementary move.

If S is a sphere with at most two punctures or a closed torus, $P_1(S)$ is empty. If S is a pair of pants, then $P_1(S)$ consists of one vertex. In all the other cases, $P_1(S)$ is locally infinite: each vertex is contained in infinitely many edges.

The pants decomposition graph was introduced by Hatcher and Thurston in the appendix to their paper [19]. Hatcher and Thurston proved that $P_1(S)$ is connected, that is, any two isotopy classes of pants decompositions on a given surface can be obtained from each other by a finite sequence of elementary moves.

D. Margalit proved the following rigidity result:

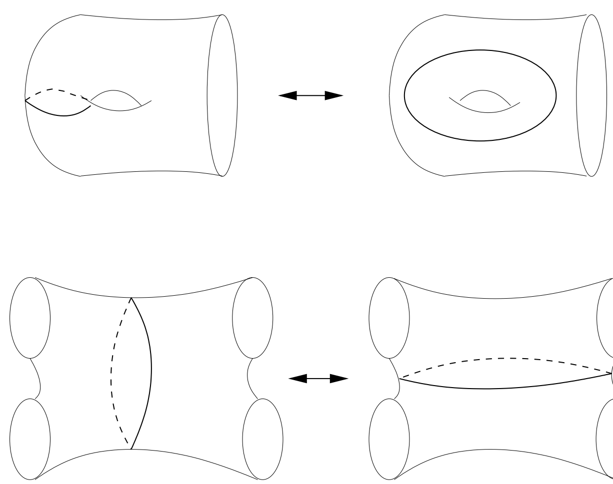

Figure 8. The two types of elementary moves between pants decompositions.

Theorem 4.13 (cf. [40]). *Let $S_{g,n}$ be a surface of negative Euler characteristic. If $(g,n) \notin \{(0,3),(1,1),(1,2),(2,0),(0,4)\}$, then the homomorphism*

$$\Gamma^*(S_{g,n}) \to \mathrm{Aut}(P_1(S_{g,n}))$$

is an isomorphism.

Furthermore, in the excluded cases, we have the following:

(1) *The homomorphism $\Gamma^*(S_{0,3}) \to \mathrm{Aut}(P_1(S_{0,3}))$ is not injective. (This is because $P_1(S_{0,3})$ is reduced to a point, and therefore $\mathrm{Aut}(P_1(S))$ is trivial, whereas $\Gamma^*(S_{0,3})$ is not trivial: it is an order-two extension of the permutation group on three elements.)*

(2) *The homomorphism $\Gamma^*(S_{0,4}) \to \mathrm{Aut}(P_1(S_{0,4}))$ is surjective, and its kernel kernel is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated by two hyperelliptic involutions.*

(3) *In the case where $(g,n) = (1,1),(1,2)$ or $(2,0)$, the homomorphism $\Gamma^*(S_{g,n}) \to \mathrm{Aut}(P_1(S_{g,n}))$ is surjective, and its kernel is $\mathbb{Z}_2$, generated by a hyperelliptic involution.*

Note that a pants decomposition of $S_{g,n}$ can be regarded as a maximal simplex in the curve complex $C(S_{g,n})$ of $S_{g,n}$. The graph $P_1(S_{g,n})$ can be regarded as a subcomplex of the dual complex to the curve complex $C(S_{g,n})$. The proof by Margalit of Theorem 4.13 does not use this fact but nevertheless it uses the result of Ivanov, Korkmaz and Luo on the automorphisms of the curve complex (Theorem 4.3 above).

The pants graph has other important features. In the paper [8], J. Brock proved that this graph, endowed with its natural simplicial metric, is quasi-isometric to the Teichmüller space of S endowed with its Weil-Petersson metric.

4.5 The ideal triangulation graph $T(S)$

In this section, S is a surface with nonempty boundary. The *ideal triangulation graph* of S, $T(S)$, is the simplicial graph whose vertices are the isotopy classes of ideal triangulations of S and in which an edge connects two vertices whenever these vertices differ by an elementary move. The elementary moves are described in Figure 9. Thus, in an elementary move, we replace some (homotopy class of) edge of a triangulation by a different one and we keep the other (homotopy classes of) edges unchanged. The (homotopy class of) edge that is transformed by the move is said to be *exchanged* by the move, and the move is said to be *performed* on that edge.

The ideal triangulation graph has been studied by several authors, in particular Harer [16] and Hatcher [18]. The rigidity result for the automorphism group of this graph was obtained by Korkmaz and Papadopoulos in [37] (see Theorem 4.15 below).

Let Δ be an ideal triangulation on S. An arc on S which is an element of the system of arcs defining Δ will be called an *edge* of Δ. There is an important distinction between exchangeable and non-exchangeable edges of Δ. This notion is defined as follows.

An edge e of Δ is said to be *exchangeable* if an elementary move can be performed on e, giving rise to a new ideal triangulation. The edge e is said to be *non-exchangeable* if no elementary move can be performed on e. Non-exchangeable edges on S are those that are on the boundary of a unique hexagon in the dual triangulation. The configuration is represented in Figure 10.

The proof of the following rigidity result follows easily from the distinction made between exchangeable and non-exchangeable edges.

Theorem 4.14 ([37]). *Let $T(S_{g,n})$ and $T(S_{h,m})$ be the two ideal triangulation graphs associated of two surfaces $S_{g,n}$ and $S_{h,m}$ respectively. Then, $T(S_{g,n})$ and $T(S_{h,m})$ are homeomorphic if and only if the surfaces $S_{g,n}$ and $S_{h,m}$ are homeomorphic.*

Proof. The non-trivial direction is the "only if" direction, and it follows easily from the following valency considerations in the ideal trianguation graph.

The *valency* of a vertex in $T(S_{g,n})$ is the number of edges abutting (locally) at that point. An ideal triangulation that represents a vertex of maximal valency in the ideal triangulation graph is an ideal triangulation that does not contain any non-exchangeable edge. There exist such triangulations on any surface. It is also easy to see that an ideal triangulation representing a vertex of minimal valency contains a configuration of the form represented in Figure 11, in which all the boundary curves of S are involved. Such a triangulation also exists on any surface, provided the surface has at least two boundary components. From this, it easily follows by an Euler characteristic

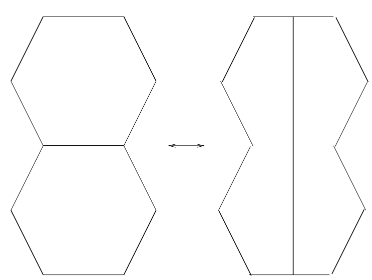

Figure 9. An elementary move on an ideal triangulation: A pair of adjacent hexagons is replaced by a different pair of adjacent hexagons. The segments in bold lines represent the arcs that are edges of the triangulation, an the other segments are contained in the boundary of the surface.

argument that for any g and n, the maximum valency at a vertex of $T(S_{g,n})$ is $6g+3n-6$ and the minimal valency is $6g+3n-6-(n-1)=6g+2n-5$. Thus, if the two graphs $T(S_{g,n})$ and $T(S_{h,m})$ are homeomorphic, we have $6g+3n-6=6h+3m-6$ and $6g+2n-5=6h+2m-5$. The two equations imply that $g=h$ and $n=m$; that is, the surfaces are homeomorphic. (A special easy argument is needed in case one of the surfaces has only one boundary component.) □

The extended mapping class group $\Gamma^*(S)$ acts naturally on $T(S)$ by simplicial automorphisms, and we have the following.

Theorem 4.15 (Korkmaz-Papadopoulos [37]). *Let S be a connected orientable surface with at least one puncture. If S is not a sphere with at most three punctures or a torus with one puncture, then the natural homomorphism* $\Gamma^*(S) \to \mathrm{Aut}(T(S))$ *is an isomorphism.*

The proof of Theorem 4.15 given in [37] is based on the analogous theorem for the arc complex obtained by Irmak and McCarthy (Theorem 4.7). It is shown that any automorphism of the ideal triangulation graph induces an automorphism of the arc complex, and this is used in the proof of 4.15.

We note that the graph $T(S)$ is the one-skeleton of the simplicial complex dual to the arc complex $A(S)$, and therefore any automorphism of $A(S)$ induces an automorphism of the graph $T(S)$. This fact is not used in the proof of Theorem 4.15. We also note that $T(S)$ is a strict subcomplex of the dual complex of $A(S)$, and a priori its automorphism group could be larger than the automorphism group of $A(S)$. Theorem 4.15 shows that this is not the case.

The surfaces that admit ideal triangulations and that are excluded by the hypothesis of Theorem 4.15 are the sphere with two or three punctures and the torus with one puncture. These cases are also analyzed in the paper [37], and the results are as follows:

In the case where $S=S_{0,2}$ is the sphere with two punctures, $T(S_{0,2})$ consists of a single vertex, hence its automorphism group is trivial, and the natural homomorphism $\Gamma^*(S_{0,2}) \to \mathrm{Aut}(T(S_{0,2})$ is surjective and not injective.

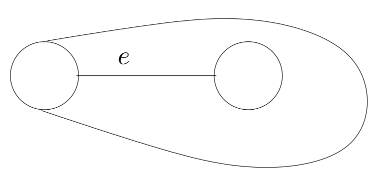

Figure 10. The edge e is a non-exchangeable edge in an ideal triangulation: it is on the boundary of a unique ideal hexagon.

In the case where $S = S_{0,3}$ is a sphere with three punctures, the graph $T(S_{0,3})$ is finite, and it is homeomorphic to a tripod, whose central vertex is represented by the unique (up to isotopy) ideal triangulation of $S_{0,3}$ in which every edge is exchangeable. The automorphism group $\mathrm{Aut}(T(S_{0,3}))$ is isomorphic to the permutation group on three elements, the mapping class group of $\Gamma(S_{0,3})$ is also isomorphic to the permutation group on three elements (the punctures of $S_{0,3}$), and the natural homomorphism $\Gamma(S_{0,3}) \to \mathrm{Aut}(T(S_{0,3}))$ is an isomorphism. Hence, the natural homomorphism $\Gamma^*(S_{0,3}) \to \mathrm{Aut}(T(S_{0,3}))$ is surjective and its kernel is the center of $\Gamma^*(S_{0,3})$, a cyclic group of order two.

In the case where $S = S_{1,1}$ is a torus with one puncture, then its the ideal triangulation graph is a regular infinite tree in which every vertex has valency three. The automorphism group of such a tree is uncountable. Thus, the natural homomorphism $\Gamma^*(S_{1,1}) \to \mathrm{Aut}(T(S_{1,1}))$ is highly non-surjective.

As we did in the preceding sections, we can double the surface $S = S_{g,b}$ along its boundary components and obtain a closed surface $S_{g+b,0}$. There is a natural simplicial embedding

$$T(S_{g,b}) \hookrightarrow P_1(S_{g+b,0})$$

obtained by taking the doubles of ideal triangulations on $S_{g,b}$, an operation that gives pairs of pants decompositions on $S_{g+b,0}$, and noting that the double of an elementary move between ideal triangulations on $S_{g,b}$ is an elementary move between the two corresponding pairs of pants decompositions of $S_{g+b,0}$.

4.6 The Schmutz graph of nonseparating curves $G(S)$

In [50], Paul Schmutz Schaller introduced and studied a new one-dimensional simplicial complex $G(S)$ associated to S. There are two different definitions, depending on whether the genus of S is 0 or ≥ 1.

Definition 4.16 (The Schmutz graph). Let $S = S_{g,n}$ be a surface of negative Euler characteristic which is not a pair of pants. Then:

(1) If $g \geq 1$, the vertex set of $G(S)$ is the set of isotopy classes of nonseparating simple closed curves on S, and two vertices are related by an edge whenever their geometric intersection number is 1.

(2) If $g = 0$, the vertex set of $G(S)$ is the set of isotopy classes of simple closed curves on S which separate S into two components one of which is a pair of pants. (Note that two of the boundary components of this pair of pants are boundary components of S, and therefore such a vertex does not exist if $b \leq 1$.) In this case, two vertices are related by an edge whenever their geometric intersection is equal to two.

Schmutz Schaller proved that $G(S)$ is connected and that the automorphism group of this complex is equal to the mapping class group modulo its centre. More precisely, he proved the following:

Theorem 4.17 (Schmutz Schaller [50]). *Let $S = S_{g,n}$ be a surface of negative Euler characteristic which is not a pair of pants. Then, if $(g, n) \notin \{(0, 4), (1, 1), (1, 2), (2, 0)\}$, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{G}((S))$$

is an isomorphism.

Furthermore, in the exceptional cases, the situation is as follows:

(1) *for $(g, n) \in \{(1, 1), (1, 2), (2, 0)\}$, the homomorphism is surjective, and its kernel is $\mathbb{Z}_2$, generated by the hyperelliptic involution of S;*

(2) *for $(g, n) = (0, 4)$, the homomorphism is surjective, and its kernel is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated by two hyperelliptic involutions.*

4.7 The complex of nonseparating curves $N(S)$

In this section, S is a compact, connected, orientable surface of genus $g \geq 2$ with $p \geq 0$ boundary components.

Definition 4.18 (The complex of nonseparating curves). The complex $N(S)$ of nonseparating curves of S is the simplicial complex whose k-simplices, for every $k \geq 0$, are the collections of $k+1$ isotopy classes of nonseparating curves that can be represented by disjoint and pairwise non-isotopic curves.

Note that $N(S)$ admits a canonical simplicial injection as the subcomplex of the curve complex $C(S)$ induced by the set of vertices that are isotopy classes of nonseparating simple closed curves.

E. Irmak proved the following:

Theorem 4.19 (Irmak [22]). *If S is not the closed surface of genus 2, then the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(N(S))$$

is an isomorphism. In the case where S is the closed surface of genus 2, the automorphism group of $N(S)$ is $\Gamma^*(S)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the hyperelliptic involution of S.

Notice that although the vertex set of the complex $N(S)$ of nonseparating curves is the same as the vertex set of the Schmutz graph $G(S)$, the one-skeleton of $N(S)$ is not simplicially equivalent to the Schmutz graph. However, the proof of Theorem 4.19 by Irmak uses the corresponding theorem by Schmutz Schaller (Theorem 4.17).

4.8 The cut system graph $HT_1(S)$

A *cut system* on the surface S is (the isotopy class of) a system of curves such that S cut along this system is a sphere with holes. Note that each of the curves defining a cut system is necessarily nonseparating, and that the cardinality of such a system is equal to the genus of the surface S (by one of the definitions of the genus). In particular, if the genus of S is 0, then there is no cut system on S. Thus, for the rest of this section, we suppose that the genus of S is ≥ 1.

To simplify notation, we shall often identify a cut system and the set of homotopy classes of curves in that system.

Hatcher and Thurston introduced the following notion of *elementary move* between cut systems.

A *elementary move* is the operation of replacing the (homotopy class of) a curve in a cut system by a new (homotopy class of) curve such that the result is again a cut system, and such that the geometric intersection number between the old and the new (homotopy classes of) curve is equal to one.

Definition 4.20 (The cut-system graph). The *cut-system graph,* $HT_1(S)$, of S is the simplicial graph whose vertex set is the set of cut systems on S and whose edges are the pairs of cut systems that are related by an elementary move.

The cut system graph is also called the *Hatcher-Thurston graph,* hence the notation $HT_1(S)$. Note that the index 1 refers to the fact that the Hatcher-Thurston graph is the one-skeleton of a thicker CW-complex called the cut-system complex (or the Hatcher-Thurston complex) and denoted by $HT(S)$, and which we shall not deal with here.

In the case where the genus of S is 1, a cut system on S is reduced to a single nonseparating curve, and the cut-system graph coincides with the Schmutz graph $G(S)$ defined in §4.6 above.

The automorphism group of the cut system graph was studied by E. Irmak and M. Korkmaz, who proved in [24] that the group $\mathrm{Aut}(HT_1(S))$ of simplicial automorphisms of $HT_1(S)$ is the extended mapping class group modulo its centre. More precisely, they obtained the following.

Theorem 4.21 (Irmak and Korkmaz [24]). *Let $S = S_{g,b}$ be a compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components. If S is not a torus with at most two holes or a closed surface of genus 2, then the natural map*

$$\Gamma^*(S) \to \mathrm{Aut}(HT_1(S))$$

is an isomorphism. In the excluded cases, this map is surjective and its kernel is $\mathbb{Z}/2$, the centre of $\Gamma^*(S)$.

Irmak and Korkmaz proved Theorem 4.21 by passing through the Schmutz complex. A beautiful ingredient in their proof is the encoding of nonseparating simple closed curves in S by vertices and edges in the cut system graph. More precisely, a pair (v, e), where v is a vertex of $HT_1(S)$ and e an edge containing v, determines in a natural way a homotopy class of a nonseparating curve, namely, the homotopy class of curves in a cut system representing v that is transformed by the elementary move representing the edge e. (Recall that every curve in a cut-system is nonseparating.) Irmak and Korkmaz use this fact to associate to each automorphism f of $HT_1(S)$ an automorphism $\widetilde{f}$ of the Schmutz graph. More precisely, they proceed as follows. Start with an automorphism f of $HT_1(S)$. Take an (isotopy class of) nonseparating curve C_1 on S. Complete it into a cut system $\mathcal{C} = \{C_1, \ldots, C_g\}$. Perform an elementary move on C_1 in the cut system $\mathcal{C}$, replacing C_1 by some curve D. Then, the collection $\mathcal{C}' = \{D, \ldots, C_g\}$ (that is, the system obtained from $\mathcal{C}$ by replacing C with D) is also a cut system on S, and, as a vertex of $HT_1(S)$, this cut system is connected to the vertex $\mathcal{C}$ by an edge. Now since $f : HT_1(S) \to HT_1(S)$ is simplicial, the vertices $f(\mathcal{C})$ and $f(\mathcal{C}')$ are also connected by an edge. Irmak and Korkmaz define $\widetilde{f}(C_1)$ as the unique (homotopy class of) nonseparating curve that is in $f(\mathcal{C})$ and that is not in $f(\mathcal{C}')$. They then prove that the resulting map $\widetilde{f}$ is independent of all the choices involved. The map $\widetilde{f}$ is then showed to be an automorphism of $HT_1(S)$ that sends any pair of isotopy classes of nonseparating curves whose geometric intersection number is equal to one to a pair satisfying the same property. From this, Irmak and Korkmaz obtain a homomorphism from $\mathrm{Aut}(HT_1(S))$ to the Schmutz complex $G(S)$, and they finally prove that this map is an isomorphism.

Irmak states in her paper [22] p. 84, that the Isomorphism Theorem 4.21 can also be deduced from her result on the automorphism group of the complex of nonseparating curves (Theorem 4.19 above), using the same methods of proof.

We note that the automorphism rigidity theorem stated in the paper [24] concerns the Hatcher-Thurston CW complex, and not the graph, but the proof given in that paper works equally for the Hatcher-Thurston graph.

4.9 The complex of separating curves $CS(S)$

In this section $S = S_{g,n}$ is a compact, connected, orientable surface of negative Euler characteristic, of genus g with n boundary components.

Definition 4.22 (The complex of separating curves). The complex $CS(S)$ of separating curves of S is the flag simplicial complex whose k-simplices, for every $k \geq 0$, are the collections of $k+1$ isotopy classes of curves that can be represented by disjoint and pairwise non-isotopic separating curves.

Note that E is canonically isomorphic to the full subcomplex of the curve complex $C(S)$ spanned by all vertices that are isotopy classes of separating curves.

Theorem 4.23 (Brendle & Margalit [7], Kida [34]). *Suppose that the genus of S is ≥ 1, and that S is not a torus with at most two holes or a surface of genus two with at most one hole. Then, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(CS(S))$$

is an isomorphism.

Theorem 4.23 was obtained by Brendle and Margalit [7] for closed surfaces. The generalization as stated is due to Kida [34].

4.10 The Torelli complex $TC(S)$

In this section $S = S_{g,n}$ is a compact, connected, orientable surface of negative Euler characteristic, of genus g with n boundary components.

A *bounding pair* in S is a pair of nonseparating curves whose union separates S (see Figure 12).

There is a notion of a *Dehn twist along a bounding pair*. This is defined as the product of a positive Dehn twist along one of the two curves in the bounding pair, and a negative Dehn twist along the other curve.

Dehn twists along bounding pairs play an important role in the study of the Torelli group, in particular because of a theorem of D. Johnson asserting that the Torelli group of any closed surface of genus ≥ 3 is generated by a finite collection of Dehn twists along bounding pairs, see [33]. Before Johnson obtained that result, Birman and Powell had proved that the Torelli group is generated by the infinite collection of all Dehn twists along separating curves and bounding pairs, cf. [3] and [49]

Definition 4.24 (The Torelli complex). The Torelli complex of S, denoted by $TC(S)$, is the flag simplicial complex whose vertices can be of the following types:

(1) an isotopy class of a separating curve on S;

(2) an isotopy class of a bounding pair on S.

For $k \geq 2$, a collection of k vertices is a $(k-1)$-simplex of $TC(S)$ if and only if these vertices can be represented by curves or bounding pairs that are mutually non-isotopic and disjoint.

In the case of surfaces of genus zero, the Torelli complex coincides with the curve complex, since any closed curve on such a surface is separating.

Brendle and Margalit obtained in [7] the following theorem which was conjectured by Farb:

Theorem 4.25 (Brendle-Margalit [7]). *For any closed surface S of genus $g \geq 4$ the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(TC(S))$$

is an isomorphism.

In their paper [12], Farb and Ivanov had given an outline of the proof of the same result, but with an additional structure on the vertices and on the two-simplices of that complex, and for the case where the genus of S is ≥ 5.

Kida obtained a more general version of that theorem, and we finally have the following:

Theorem 4.26 (Brendle & Margalit [7], Kida [34]). *Let $S = S_{g,n}$ be a connected compact surface of genus $g \geq 1$ with $n \geq 0$ boundary components, such that S is not a torus with at most two punctures or a surface of genus two with at most one puncture. Then, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(TC(S))$$

is an isomorphism.

5 The complex of domains and its subcomplexes

5.1 The complex of domains $D(S)$

Definition 5.1. The *complex of domains of* S, $D(S)$, is the simplicial complex whose k-simplices, for all $k \geq 0$, are the collections of $k+1$ distinct isotopy classes of disjoint domains on S.

Clearly, there is a natural simplicial embedding $C(S) \to D(S)$ obtained *via* the association to each curve on S a regular neighborhood of that curve, and considering that regular neighborhood as a domain on S. We shall describe below injections of other simplicial complexes into $D(S)$. There is no natural injection from the arc complex into the complex of domains, but we shall describe a subcomplex of the arc complex, namely, the complex of boundary graphs, which is naturally injected in the complex of domains (see §5.3 below).

From Proposition 2.8, for every domain X on S, we have a natural simplicial embedding

$$D(X) \hookrightarrow D(S).$$

We shall study in detail the complex of domains. In particular, we shall describe its automorphism group in §8 below.

Let us first briefly discuss the complex of domains associated to some surfaces of low genus and small nomber of boundary components.

If $S = S_{0,3}$ is a sphere with at most three holes, then S has no essential curves. Since a domain on a surface has at least an essential curve, $D(S_{0,3})$ is empty.

Proposition 5.2. *If S is a sphere with four holes, then $D(S) \simeq C(S) \times \Delta_2$ where Δ_2 is a triangle.*

Proof. Suppose that $S = S_{0,4}$ is a spheres with four holes. We recall that in that case $C(S)$ is an infinite vertex set. Let X be a domain on S. Then X must have at least one essential boundary component on S.

Suppose, on the one hand, that X has at least two essential boundary components C and D on S. Then, since any two non-homotopic essential curves on $S_{0,4}$ have a nonempty intersection, there exists an annular domain A on S such that C and D are the two boundary components of A. Moreover, there are exactly two codomains, P and Q, of A on S, both of which are biperipheral pairs of pants on S. We may assume that P has C as its unique essential boundary component on S and Q has D as its unique essential boundary component on S. Since C and D are both boundary components of X it follows that X is equal to A.

Suppose, on the other hand, that X has exactly one essential boundary component C on S. Then X must be one of the two biperipheral pairs of pants on S which have C as their unique essential boundary components on S. It follows that every domain on S is represented by either an annulus on S or a biperipheral pair of pants on S.

□

This description of $D(S)$ exhibits this simplicial complex as a bundle over the infinite vertex set $C(S)$ with fiber a triangle Δ_2. There is a natural section of $D(S)$ corresponding to the biperipheral annuli on S.

Proposition 5.3. *If S is a closed torus, then $D(S) \simeq C(S)$ is an infinite set of vertices.*

Proof. Suppose that $S = S_{1,0}$ is a closed torus. Then each domain on S is an annulus, and the natural map $\eta : C(S) \to D(S)$ is an isomorphism.

□

Proposition 5.4. *If S is a torus with one hole, then $D(S) \simeq C(S) \times \Delta_1$ where Δ_1 is an edge.*

Proof. Suppose that $S = S_{1,1}$ is a torus with one hole. In this case, each domain on S is either an annulus or a monoperipheral pair of pants, and two domains on S are disjoint and nonisotopic if and only if one is a monoperipheral pair of pants and the other is an annulus in its complement. Thus, we have the following

□

In the case where S is a torus with one hole, the component of $D(S)$ corresponding to the component $\{[\alpha]\} \times \Delta_1$ of $C(S) \times \Delta_1$ is the edge of $D(S)$ whose vertices correspond to a regular neighborhood of α on S and to a monoperipheral pair of pants in its complement. This description of $D(S)$ exhibits $D(S)$ as a bundle over $C(S)$ with fiber an edge Δ_1. There are two natural sections of $D(S)$, one corresponding to the annuli on S, the other corresponding to the monoperipheral pairs of pants on S. This bundle is therefore trivializable with a natural trivialization.

Proposition 5.5. *If S is not a sphere with at most four holes or a torus with at most one hole , then $D(S)$ is connected.*

Proof. Let v be a vertex representing a domain C on S. Then, C has at least one essential boundary component. Let A be an annular domain representing this boundary component, and let w be the vertex in $D(S)$ represented by A. Since C and A are isotopic to disjoint surfaces, the vertices v and w are joined by an edge. Thus, any vertex in $D(S)$ can be joined by an edge to a vertex in the natural image of $C(S)$ in $D(S)$. Since $C(S)$ is connected, this implies that $D(S)$ is connected. □

We now study maximal simplices in the complex of domains.

One difference between the complex $D(S)$ and complexes such as $A(S)$ or $C(S)$ is that in $A(S)$ and $D(S)$, the maximal simplices are the top-dimensional simplices, whereas not all of the maximal simplices of $D(S)$ are top-dimensional simplices. In fact, in $D(S)$, there are maximal simplices of all dimensions between 1 and the dimension of the top-dimensional simplices.

We shall describe maximal simplices in $D(S)$.

For these descriptions, it is helpful to distinguish between various types of vertices of $D(S)$, corresponding to some special domains. Domains of particular interest include annuli, nonannular domains, pairs of pants, peripheral pairs of pants, monoperipheral pairs of pants and biperipheral pairs of pants.

The following proposition gives a relation between pants decompositions and maximal simplices.

Proposition 5.6. *Suppose that S is not a sphere with at most three holes or a closed torus. Let $C = \{C_i \mid 1 \leq i \leq n\}$ be a maximal system of curves on S. Let $X = \{X_i \mid 1 \leq i \leq n\}$ be a collection of disjoint annuli on S such that*

X_i is a regular neighborhood of C_i on S, $1 \leq i \leq n$. Let $Y = \{Y_j, 1 \leq j \leq k\}$ be the collection of components of the closure of the complement of X in S. Then:

(1) *$n = 3g - 3 + b$;*

(2) *$k = 2g - 2 + b$;*

(3) *the subsurfaces X_i, $1 \leq i \leq n$ are annular domains on S;*

(4) *the subsurfaces Y_j, $1 \leq j \leq k$, are pairs of pants on S;*

(5) *the vertices x_i, $1 \leq i \leq n$ and y_j, $1 \leq j \leq k$ represented by, respectively, X_i, $1 \leq i \leq n$ and Y_j, $1 \leq j \leq k$ are the vertices of a simplex of $D(S)$ which has exactly $5g - 5 + 2b$ vertices and which is top-dimensional.*

It is easy to construct maximal simplices in $D(S)$ that are not top-dimensional. To describe the general maximal simplices in $D(S)$, we introduce the notion of tiling of a surface.

A *tiling* $\mathcal{F}$ *of* S is a system of domains on S which is maximal with respect to inclusion. An element of such a tiling $\mathcal{F}$ is called a *tile* of $\mathcal{F}$. The maximal simplices of $D(S)$ are the unions of vertices representing a tiling of $\mathcal{F}$.

A *tie of a tiling of* S is a codomain of a tiling of S.

Suppose first that S is a closed torus. If $\mathcal{F}$ is a tiling of S, then $\mathcal{F}$ has a unique tile and a unique tie, which are both annuli and which are glued along their two boundary components.

If $\mathcal{F}$ is a collection of disjoint domains on a closed torus S, then the following are equivalent:

(1) $\mathcal{F}$ is a tiling of S;

(2) $\mathcal{F}$ is a system of domains on S;

(3) $|\mathcal{F}|$ has exactly one codomain.

Most of the propositions in the rest of this section are easy to prove, and the proofs are left to the reader.

Proposition 5.7. *Suppose now that S is not a closed torus, let $\mathcal{F}$ be a tiling of S and let T be a tie of $\mathcal{F}$. Then T is an annulus on S with essential boundary components C and D such that there exists a unique pair of domains of $\mathcal{F}$, N and A, such that C is an essential boundary component of N, D is an essential boundary component of A, N is not an annulus, A is an annulus, and A is isotopic to T.*

Proposition 5.8. *Suppose that S is not a closed torus. Let $\mathcal{F}$ be a tiling of S, let X be a tile of $\mathcal{F}$ and C be an essential boundary component of X. Then there exists a unique tie T of $\mathcal{F}$ such that C is an essential boundary component of T.*

Proposition 5.9. *Suppose that S is not a closed torus and let $\mathcal{F}$ be a collection of disjoint domains on S. Then $\mathcal{F}$ is a tiling of S if and only if every codomain of $|\mathcal{F}|$ is an annulus joining an annular domain of $|\mathcal{F}|$ to a nonannular codomain of $\mathcal{F}$.*

Proposition 5.10 (Tilings and maximal simplices of $D(S)$). *Suppose that S is not a sphere with at most three holes or a closed torus and let $C = \{C_i \mid 1 \leq i \leq n\}$ be a system of curves on S. Let $X = \{X_i \mid 1 \leq i \leq n\}$ be a collection of disjoint annuli on S such that for all $1 \leq i \leq n$-, X_i is a regular neighborhood of C_i on S. Let $Y = \{Y_j, 1 \leq j \leq k\}$ be the collection of components of the closure of the complement of X in S. Let $\mathcal{F} = \{X_i, Y_j \mid 1 \leq i \leq n, 1 \leq j \leq k\}$. Then:*

1. *the subsurfaces X_i, $1 \leq i \leq n$ are annular domains on S;*
2. *the subsurfaces Y_j, $1 \leq j \leq k$, are thick domains on S;*
3. *the collection $\mathcal{F}$ is a tiling of S;*
4. *the components of the closure of $R_i \setminus X_i$, $1 \leq i \leq n$, are the ties of $\mathcal{F}$;*
5. *for $1 \leq i \leq n$ and $1 \leq j \leq k$, the vertices x_i, and y_j represented by, respectively, X_i and Y_j, are the vertices of a simplex of $D(S)$which has Δ_C has exactly $n + k$ verticesand which is a maximal simplex of $D(S)$.*

Definition 5.11 (The canonical maximal simplex of $D(S)$ associated to a system of curves). Suppose that S is not a sphere with at most three holes or a closed torus and let C be a system of curves on S. The simplex Δ_C provided by Proposition 5.10 is called the *canonical maximal simplex of $D(S)$ associated to C*.

Proposition 5.12. *Suppose that S is not a sphere with at most three holes or a closed torus and let Δ be a maximal simplex of $D(S)$. Then there exists a system of curves C on S such that $\Delta = \Delta_C$, where Δ_C is the canonical maximal simplex of $D(S)$ associated to C.*

Proof. Each maximal simplex Δ contains a nonempty set of vertices which are represented by annular domains. We take C to be the system of curves that represent the homotopy classes the union of these annular domains. □

Proposition 5.13. *Suppose that S is not a sphere with at most three holes or a closed torus, let Δ be a simplex of $D(S)$ and let C be a system of curves on S such that $\Delta = \Delta_C$ is the canonical maximal simplex of $D(S)$ associated to C. Then the following are equivalent:*

1. *Δ is a top-dimensional simplex of $D(S)$;*
2. *C is a pants decomposition of S.*

Proposition 5.14 (The dimension of the complex of domains $D(S)$). *If S is a sphere with at most three holes, then $D(S)$ is empty. If S is a closed torus, then $D(S)$ is an infinite set of vertices. Otherwise,* $\dim(D(S)) = 5g + 2b - 6$.

Proof. A sphere with at most three holes has no essential curves and, hence, no domains. Therefore, if S is a sphere with at most three holes, $D(S)$ is empty.

If S is a closed torus, then the natural map $\eta : C(S) \to D(S)$ is an isomorphism and, hence, $D(S)$ is an infinite set of vertices.

Assume that S is not a sphere with at most three holes or a closed torus. Then, it follows from Proposition 5.6 that a top-dimensional simplex of $D(S)$ has $5g + 2b - 5$ vertices. Hence, $\dim(D(S)) = 5g + 2b - 6$. □

We shall now construct a natural tiling of S associated to any system of domains on S. We assume that S is not a closed torus.

Let $\mathcal{F}$ be a system of domains on S.

Let $\mathcal{A}$ be the collection of annular domains in $\mathcal{F}$.

Let $\mathcal{P}$ be the collection of nonannular domains in $\mathcal{F}$.

Let $\mathcal{D}$ be the collection of annular codomains of $\mathcal{F}$

Let $\mathcal{R}$ be the collection of nonannular codomains of $\mathcal{F}$

Let $D \in \mathcal{D}$. Since D is an annular domain on S and annular domains do not have any peripheral boundary components, there exists a unique subset $\{F, G\}$ of $\mathcal{F}$ such that D joins F to G.

Suppose, on the one hand, that $F = G$. In this case, we say that D *is a coannulus of $\mathcal{F}$ attached to the domain F in $\mathcal{F}$.*

Suppose that D is a coannulus of $\mathcal{F}$ attached to the domain F in $\mathcal{F}$. Then $F \cup D$ is a domain on S with genus one greater than that of F, the same number of peripheral boundary components as F, and two less essential boundary components than F. In particular, if F is an annulus, then $F \cup D$ is a closed torus and, hence, S is a closed torus.

Since S is not a closed torus, it follows that each such coannulus of $\mathcal{F}$ joins a nonannular domain F in $\mathcal{F}$ to itself (i.e. a domain F in $\mathcal{P}$ to itself).

Suppose, on the other hand, that $F \neq G$. In this case, we say that D *is a coannulus of $\mathcal{F}$ attached to the distinct domains F and G in $\mathcal{F}$.* Note that in this case, it is possible that either F or G is an annular domain on S.

Suppose that D is a coannulus of $\mathcal{F}$ attached to the distinct domains F and G in $\mathcal{F}$. Then $F \cup D \cup G$ is a domain on S with genus equal to the sum of the genera of F and G, with the same peripheral boundary components as $F \cup G$, and two less essential boundary components than $F \cup G$.

Let $\mathcal{Q}$ be a collection of domains on S which is obtained from $\mathcal{R}$ by replacing each domain R in $\mathcal{R}$ by a domain Q which is obtained from R by shrinking R on S. In particular, Q is contained in the interior of R and Q is isotopic to R

on S. Hence, Q and R represent the same vertex of $D(S)$. Moreover, since R is not an annulus on S, Q is not an annulus on S.

Note that $\mathcal{Q}$ has the same number of elements as $\mathcal{R}$, that $\mathcal{F}$ and $\mathcal{Q}$ are disjoint collections of domains on S, and that $\mathcal{F} \cup \mathcal{Q}$ is a collection of disjoint domains on S.

Let $\mathcal{E}$ be the collection of codomains of $\mathcal{F} \cup \mathcal{Q}$.

Suppose that $E \in \mathcal{E}$. Note that there exists a unique pair of distinct domains on S, $\{F, Q\}$, such that $F \in \mathcal{F}$ and $Q \in \mathcal{Q}$ such that E joins F to Q on S. Hence, E joins a domain F in $\mathcal{F} \cup \mathcal{Q}$ to a nonannular domain Q in $\mathcal{F} \cup \mathcal{Q}$. Note that it is possible that F is also a nonannular domain on S.

It follows from Proposition 2.23 that $\mathcal{F} \cup \mathcal{Q}$ is a system of domains on S.

Note that $\mathcal{D}$ and $\mathcal{E}$ are disjoint sets, $\mathcal{D} \cup \mathcal{E}$ is a collection of disjoint annuli on S, and $\mathcal{D} \cup \mathcal{E}$ is the collection of codomains of the system of domains, $\mathcal{F} \cup \mathcal{Q}$.

Let $\mathcal{D}_{\text{nonann}}$ be the subcollection of $\mathcal{D}$ consisting of all annuli D in $\mathcal{D}$ which join a nonannular domain F in $\mathcal{F}$ to a nonannular domain G in $\mathcal{F}$.

Let $\mathcal{B}$ be a collection of domains on S which is obtained from $\mathcal{D}_{\text{nonann}}$ by replacing each domain D in $\mathcal{D}_{\text{nonann}}$ by a domain B which is obtained from D by shrinking D on S. In particular, B is contained in the interior of D and B is isotopic to D on S. Hence, B and D represent the same vertex of $D(S)$. Moreover, since D is an annulus on S, B is an annulus on S.

Let $\mathcal{E}_{\text{nonann}}$ be the subcollection of $\mathcal{E}$ consisting of all annuli E in $\mathcal{E}$ which join a nonannular domain F in $\mathcal{F}$ to a nonannular domain Q in $\mathcal{Q}$.

Let $\mathcal{C}$ be a collection of domains on S which is obtained from $\mathcal{E}_{\text{nonann}}$ by replacing each domain E in $\mathcal{E}_{\text{nonann}}$ by a domain C which is obtained from E by shrinking E on S. In particular, C is contained in the interior of E and C is isotopic to E on S. Hence, C and E represent the same vertex of $D(S)$. Moreover, since E is an annulus on S, C is an annulus on S.

Note that $\mathcal{F}$, $\mathcal{Q}$, $\mathcal{B}$, and $\mathcal{C}$ are disjoint collections of domains on S, and $\mathcal{F} \cup \mathcal{B} \cup \mathcal{C}$ is a collection of disjoint domains on S.

Let $\mathcal{G} = \mathcal{F} \cup \mathcal{B} \cup \mathcal{C}$. Note that $\mathcal{G}$ contains $\mathcal{F}$ and $\mathcal{G} = \mathcal{P} \cup \mathcal{Q} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$.

Let $\mathcal{T}$ be the collection of codomains of $\mathcal{G}$ on S.

Suppose that $T \in \mathcal{T}$. Then, T is an annulus joining a nonannular domain X in $\mathcal{G}$ (i.e. a domain X in $\mathcal{P} \cup \mathcal{Q}$) to an annulus Y in $\mathcal{G}$ (i.e. a domain Y in $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$).

Hence, from the above construction, we have the following result.

Proposition 5.15. *Let $\mathcal{F}$ be a system of domains on S. Let $\mathcal{A}$ be the collection of annular domains in $\mathcal{F}$, let $\mathcal{P}$ be the collection of nonannular domains in $\mathcal{F}$, let $\mathcal{D}$ be the collection of annular codomains of $\mathcal{F}$ on S and let $\mathcal{R}$ be the collection of nonannular codomains of $\mathcal{F}$ on S. Finally, let $\mathcal{Q}$, $\mathcal{B}$, and $\mathcal{C}$ be the collections of domains that are constructed as above and let $\mathcal{G} = \mathcal{P} \cup \mathcal{Q} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$. Then:*

(1) *$\mathcal{G}$ is a tiling of S containing $\mathcal{F}$;*

(2) $\mathcal{G}$ *is well-defined up to isotopies on* S *which fix the support* $|\mathcal{F}|$ *of* $\mathcal{F}$ *pointwise;*

(3) *for each domain* $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$*, there exists a domain* $F \in \mathcal{F}$ *and an essential boundary component* ∂ *of* F *on* S *such that* G *is isotopic to a regular neighborhood of* ∂ *on* S*;*

(4) *for each domain* $F \in \mathcal{Q}$ *and each essential boundary component* ∂ *of* F *on* S*, there exists a unique domain* $G \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ *such that* G *is isotopic to a regular neighborhood of* ∂ *on* S*;*

(5) *the number of elements of* $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$ *is equal to the number of isotopy classes of essential boundary components of domains of* S *in* $\mathcal{F}$*;*

(6) *for each domain* $Q \in \mathcal{Q}$*, there exists a unique domain* $R \in \mathcal{R}$ *such that* Q *is contained in the interior of* R *and* Q *is isotopic to* R *on* S*;*

(7) *for each domain* $R \in \mathcal{R}$*, there exists a unique domain* $Q \in \mathcal{Q}$ *such that* Q *is contained in the interior of* R *and* Q *is isotopic to* R *on* S*;*

(8) *the number of elements of* $\mathcal{G}$ *is equal to the sum of the number of nonannular domains in* $\mathcal{F}$*, the number of nonannular codomains of* $\mathcal{F}$ *on* S*, and the number of isotopy classes of essential boundary components of domains of* S *in* $\mathcal{F}$*.*

Proposition 5.16. *Let* σ *be a simplex of* $D(S)$*, let* $\mathcal{F}$ *be a system of domains on* S *whose elements represent the vertices of* σ *and let* $\mathcal{G}$ *be the tiling of* S *that is associated to* $\mathcal{F}$ *given by Proposition 5.15. Then, the simplex* τ *of* $D(S)$ *whose vertices are represented by the domains in* $\mathcal{G}$ *is the unique maximal simplex of* $D(S)$ *which contains the simplex* σ *and which has the least number of vertices among all maximal simplices of* $D(S)$ *containing* σ*.*

Definition 5.17. Let σ be a simplex of $D(S)$. The simplex τ of $D(S)$ that is provided by Proposition 5.16 is called the *canonical maximal simplex* of $D(S)$ containing σ.

Proposition 5.18. *Suppose that* S *is not a closed torus. Let* X *be a domain on* S*. Let* $Y_1, \ldots, Y_k$ *be the* k *codomains of* X *on* S*. Let* $x, y_1, \ldots, y_k$ *be the vertices of* $D(S)$ *represented by* $X, Y_1, \ldots, Y_k$*. Then* $\{x, y_1, \ldots, y_k\}$ *is a* k*-simplex of* $D(S)$*.*

Proof. Let $\{\partial_j \mid 1 \leq j \leq n\}$ be the collection of all essential boundary components of X on S. Let $\{A_j \mid 1 \leq j \leq n\}$ be a collection of disjoint annuli on X such that $A_j \cap \partial X = \partial_j$, $1 \leq j \leq n$. Let Z be the closure of the complement of $|\mathcal{A}|$ in X. Note that Z is a domain on S which is isotopic to X on S. In particular, Z represents the vertex x of $D(S)$.

Note that $\{Z, Y_i \mid 1 \leq i \leq k\}$ is a collection of disjoint domains on S and that $\{A_i \mid 1 \leq i \leq n\}$ is the collection of codomains of $\{Z, Y_i \mid 1 \leq i \leq k\}$.

Suppose that $\{Z, Y_j \mid 1 \leq j \leq k\}$ is not a system of domains on S. It follows from Proposition 2.23 that there exists an annular codomain A_i of $\{Z, Y_i \mid 1 \leq i \leq k\}$ which joins the annulus Z to an annular codomain Y_j of X on S, where $1 \leq i \leq n$ and $1 \leq j \leq k$.

Since X is a domain on S which is isotopic to the annular domain Z on S, it follows that X is an annular domain on S. Hence, Y_j is an annular codomain of the annular domain X on S. It follows that S is a closed torus. This is a contradiction.

Suppose that $\{Z, Y_i \mid 1 \leq i \leq k\}$ is a system of domains on S.

It follows that $\{[Z], [Y_i] \mid 1 \leq i \leq k\}$ is a k-simplex of $D(S)$; that is to say, $\{x, y_1, \ldots, y_k\}$ is a k-simplex of $D(S)$. $\square$

In the next subsections, we describe several subcomplexes of the complex of domains. For all these complexes, a finite collection of vertices forms a simplex if and only if each pair of vertices in this collection can be represented by disjoint domains on S. In other words, all these complexes are flag complexes.

5.2 The truncated complex of domains $D^2(S)$

Definition 5.19. The *truncated complex of domains of* S, $D^2(S)$, is the induced subcomplex of $D(S)$ corresponding to those vertices of $D(S)$ that are not represented by biperipheral pairs of pants.

Note that $D^2(S) = D(S)$ when $b \leq 1$. In particular, $D^2(S) = D(S)$ for any closed surface S.

A *biperipheral curve on* S is a curve on S which is a boundary component of a biperipheral pair of pants.

There is a unique projection

$$\pi : D(S) \to D^2(S)$$

which sends each vertex of $D^2(S)$ to itself and sends each remaining vertex of $D(S)$ to the vertex of $D^2(S)$ represented by a regular neighborhood of the unique essential boundary component of any biperipheral pair of pants representing this vertex.

For each vertex x of $D^2(S)$ which is not represented by a regular neighborhood of a biperipheral curve on S, the fiber $\pi^{-1}(x)$ of $\pi : D(S) \to D^2(S)$ above x is equal to $\{x\}$.

Suppose that x is a vertex of $D^2(S)$ which is represented by a regular neighborhood of a biperipheral curve γ on S.

In the case where S is a sphere with four holes, the fiber $\pi^{-1}(x)$ of $\pi : D(S) \to D^2(S)$ above x is the triangle of $D(S)$ induced by the vertices of $D(S)$

corresponding to a regular neighborhood of γ on S and the two biperipheral pairs of pants on S of which γ is a boundary component.

Suppose that S is not a sphere with four holes. Then the fiber $\pi^{-1}(x)$ of $\pi : D(S) \to D^2(S)$ above x is the edge of $D(S)$ induced by the vertices of $D(S)$ corresponding to a regular neighborhood of γ on S and the unique biperipheral pair of pants on S of which γ is a boundary component. For most of what concerns us here (in particular, for the rigidity result we prove in §8.2), these edge fibers "duplicate information". Passing from $D(S)$ to $D^2(S)$ or, what is essentially the same, applying the natural projection, amounts to removing this "duplication of information".

In the same way as for the case of the complex of domains $D(S)$, there is a natural inclusion

$$C(S) \hookrightarrow D^2(S)$$

which maps the vertex of $C(S)$ represented by a curve α on S to the vertex of $D^2(S)$ represented by a regular neighborhood of α on S.

We now briefly discuss the truncated complex of domains of a few surfaces of low genus and small number of boundary components.

If S is a sphere with at most three holes, then $D^2(S)$ is empty.

If S is a spheres with four holes, then, from Proposition 5.2 and the discussion that precedes it, $D^2(S) \simeq C(S)$ and, hence, $D^2(S)$ is an infinite set of vertices.

If S is a closed torus, then, since S has no holes, the natural map $\eta : C(S) \to D^2(S)$ is an isomorphism and $D(S) \simeq D^2(S)$ is an infinite set of vertices.

If S is a torus with one hole, then, $D^2(S) = D(S) \simeq C(S) \times \Delta_1$ where Δ_1 is an edge.

Proposition 5.20. *If S is a surface of positive genus, then* $\dim(D^2(S)) = \dim(D(S)) = 5g + 2b - 6$.

Proof. If $b \leq 1$, then $D^2(S) = D(S)$. Suppose $b \geq 2$. If $g = 1$, then, S is a torus with b holes, and we can find a pair of pants decomposition of S with no biperipheral curves by using the decomposition pictured in Figure 13. If $g \geq 2$, then there exists a torus with $b + 1$ holes embedded in S, and we can find a pair of pants decomposition of S with no biperipheral curves by using again the decomposition pictured in Figure 13. The tiling associated to such a pants decomposition defines at the same time a top-dimensional simplex of S and a top-dimensional simplex of $D^2(S)$. This proves the result. ☐

Proposition 5.21. *If S is a sphere with at least four holes, then* $\dim(D^2(S)) = \dim(D(S)) - 2 = 5g + 2b - 8$.

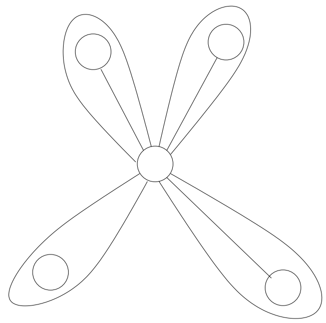

Figure 11. On a surface $S_{g,n}$ with $n \geq 2$, an ideal triangulation containing such a configuration involving all the boundary components of $S_{g,n}$ represents a vertex of $T(S_{g,n})$ that has minimal valency.

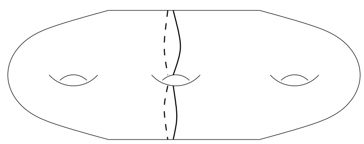

Figure 12. A bounding pair.

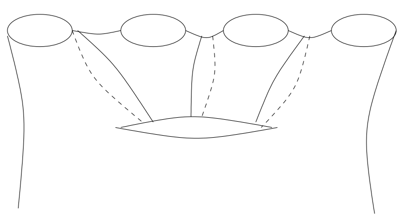

Figure 13. The case where S has genus $g \geq 1$ and $b \geq 2$ boundary components. Either S is a torus with b holes, or there is a torus with $b+1$ holes which is embedded in S. We can complete the system of curves represented in this picture to a pants decomposition of S in which no curve is biperipheral.

Proof. Suppose S is a sphere with at least four holes and let $\mathcal{F}$ be a tiling defining a top-dimensional simplex of $D^2(S)$. Let C be a boundary curve of a tile which is essential in S. Then, C is a separating curve on S. Let S_1 and S_2 be the two components of $S \setminus C$. Each of these components is a sphere with at least two holes. Since the tiling $\mathcal{F}$ defines a top-dimensional simplex of $D^2(S)$, each of the components S_1 and S_2 contains a tile which is a biperipheral annulus. The tiling $\mathcal{F}$ can be extended to a tiling $\mathcal{F}'$ of S defining a top-dimensional simplex of $D(S)$, by adding two biperipheral pants to the elements of $\mathcal{F}$. This gives $\dim(D^2(S)) = \dim(D(S)) - 2$. □

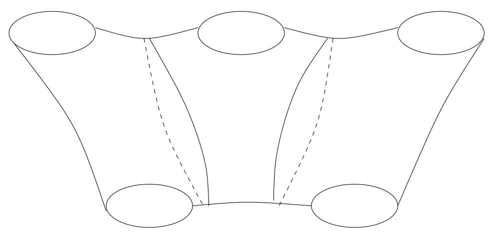

Figure 14. In any pants decomposition of a sphere with $b \geq 4$ holes, there are necessarily two biperipheral pairs of pants.

Proposition 5.22. *The natural projection $\rho : D(S) \to D^2(S)$ is a simplicial quotient map.*

Proof. The map ρ, from the vertex set of $D(S)$ to the vertex set of $D^2(S)$, is surjective, and all what is needed is to show that a set of vertices of $D^2(S)$ is a simplex if and only if there exists a simplex σ of $D(S)$ such that $\tau = \rho(\sigma)$.

If S is a sphere with four holes, then the simplicial complex $D^2(S)$ is reduced to its set of vertices, which are all annular vertices. Any simplex τ of $D^2(S)$ is a vertex of $D^2(S)$, and its inverse image $\sigma = \rho^{-1}(\tau)$ is a triangle of $D(S)$, whose elements are that annular vertex together with the two associated biperipheral pairs of pants. We have $\tau = \rho(\sigma)$.

If S is not a sphere with four holes, then for any simplex τ of $D^2(S)$, its inverse image $\sigma = \rho^{-1}(\tau)$ consists of the union of the vertices of τ considered as vertices in $D(S)$ together with a unique biperipheral pair of pants for each biperipheral vertex of τ. Consider a system of domains $\mathcal{F}$ on S representing the simplex τ of $D^2(S)$. We can complete $\mathcal{F}$ to a system of domains $\mathcal{F}'$ by adding to each biperipheral annulus in $\mathcal{F}$ a corresponding biperipheral pair of pants. The system of domains $\mathcal{F}'$ represents the vertex set σ, and therefore, σ is a simplex of $D(S)$. We have $\tau = \rho(\sigma)$. This completes the proof.

□

We end this section by describing some maximal simplices in $D^2(S)$.

Proposition 5.23 (Pants decompositions). *Suppose that S is not a sphere with at most three holes or a closed torus. Let $C = \{C_i \mid 1 \leq i \leq n\}$ be a maximal system of curves on S. Let $X = \{X_i \mid 1 \leq i \leq n\}$ be a collection of disjoint annuli on S such that X_i is a regular neighborhood of C_i on S, $1 \leq i \leq n$. Let $Y = \{Y_j \mid 1 \leq j \leq p\}$ be the collection of components of the closure of the complement of X in S which are not biperipheral pairs of pants on S. Then:*

1. *$n = 3g - 3 + b$;*
2. *$p = 2g - 2 + b$;*
3. *the subsurfaces X_i, $1 \leq i \leq n$ are annular domains on S;*
4. *the subsurfaces Y_j, $1 \leq j \leq k$, are pairs of pants on S;*
5. *the vertices x_i, $1 \leq i \leq n$ and y_j, $1 \leq j \leq k$ represented by, respectively, X_i, $1 \leq i \leq n$ and Y_j, $1 \leq j \leq k$ are the vertices of a simplex, Δ'_C, of $D(S)$;*
6. *Δ'_C has exactly $n + p$ vertices;*
7. *if $g = 0$, then Δ'_C has exactly $5g - 7 + 2b$ vertices;*
8. *if $g > 0$, then Δ'_C has exactly $5g - 5 + 2b$ vertices;*
9. *Δ'_C is a top-dimensional simplex;*

Definition 5.24 (The canonical maximal simplex of $D^2(S)$ associated to C). Let C be a system of curves on S, let Δ_C be the canonical maximal simplex of $D(S)$ associated to C and let Δ'_C be the simplex of $D^2(S)$ whose vertices are the vertices of Δ_C which are not represented by biperipheral pairs of pants on S. Then, Δ'_C is called the canonical maximal simplex of $D^2(S)$ associated to C.

5.3 The complex of boundary graphs $B(S)$

Definition 5.25. The *complex of boundary graphs of* S, $B(S)$, is the simplicial complex whose n-simplices are collections of $n + 1$ distinct isotopy classes of disjoint essential boundary graphs on S.

Note that $B(S)$ is empty if $b = 0$. If S is either a disk (i.e. a sphere with one hole), then $B(S)$ is also empty. Note also that $B(S)$ is a nonempty finite set of vertices if $g = 0$, $2 \leq b \leq 3$.

If S is an annulus, then S has a unique isotopy class of an essential arc, hence $B(S)$ has a unique vertex and no higher dimensional simplices.

If S is a torus with one hole, then $B(S)$ is, like $A(S)$, an infinite vertex set.

If S is a surface of positive genus with two holes, then $B(S)$ is disconnected and has higher dimensional simplices. Indeed, $B(S)$ contains infinitely many components with exactly one vertex corresponding to essential arcs on S which are contained in biperipheral pairs of pants on S. The boundary graph of any such arc necessarily intersects any other boundary graph. These are the only "isolated" vertices of $B(S)$. All other vertices correspond to arcs which are contained in domains on S which are monoperipheral pairs of pants on S. Since any such domain on S is disjoint from at least one other such domain on S, these vertices are contained in at least one edge of $B(S)$.

When S is a pair of pants, then $B(S)$ has exactly six vertices, represented by the boundary graphs that are asssociated to the six isotopy classes of essential arcs in S (see Figure 7), and no higher dimensional simplices.

Note that the boundary graph of an arc α is not only determined by that arc but, in turn, it determines the arc. Indeed, α is the closure in S of the complement of $G_\alpha \cap \partial S$ in G_α.

Let α and β be arcs on S. Let $\varphi : S \times [0,1] \to S$ be an isotopy from G_α to G_β. Note that the boundary components of the boundary graphs $\varphi_t(G_\alpha)$ remain constant throughout the isotopy. Hence, isotopic boundary graphs have the same boundary components. Since ∂S is invariant under any isotopy, it follows that α is isotopic to β if and only if G_α is isotopic to G_β. Hence, there exists a natural bijection $B_0(S) \to A_0(S)$. If G_α is disjoint from G_β, then α is disjoint from β. Hence, this bijection extends to a natural simplicial inclusion

$$i : B(S) \to A(S).$$

This identifies $B(S)$ with a subcomplex of $A(S)$ having the same vertex set as $A(S)$. Note, however, that the boundary graphs G_α and G_β of disjoint arcs α and β are disjoint only when the boundary components of S joined by α are distinct from those joined by β. Hence, in general, the image subcomplex of $A(S)$ has fewer simplices than $A(S)$. More precisely, the image subcomplex of $A(S)$ is the subcomplex consisting of those simplices σ of $A(S)$ which have the property that each pair of distinct vertices of σ are represented by disjoint arcs α and β on S such that the boundary components of S joined by α are distinct from those joined by β.

Assume now that $b > 0$ and either $g > 0$ or $b > 3$. We already noted that $A(S)$ is connected with infinitely many vertices. Since $B_0(S) \simeq A_0(S)$, $B(S)$ has infinitely many vertices.

Suppose that S has exactly one boundary component. It follows that $g > 0$. Note that the boundary graphs of any two arcs on X must intersect, since they both contain the nonempty boundary of S. Hence, no two vertices of $B(S)$ are joined by an edge of $B(S)$. Thus, $B(S)$, unlike $A(S)$, is an infinite set of vertices and in particular it is disconnected.

Suppose that S has exactly two boundary components. Again, it follows that $g > 0$. Let α be an arc on S joining the two boundary components ∂_1 and ∂_2 of S. Note that the boundary graph G_β of any arc β on S must intersect G_α. After all, at least one of the two boundary components of S is contained in both G_α and G_β. It follows that the vertex of $B(S)$ represented by α is not connected by an edge of $B(S)$ to any other vertex of $B(S)$. It follows that $B(S)$, unlike $A(S)$ has at least two connected components. Actually, since $g > 0$, it can be shown that there are infinitely many distinct vertices of $B(S)$ joining the two boundary components of S. It follows that $B(S)$, unlike $A(S)$, has infinitely many connected components. This shows, in particular, that the natural inclusion map $B(S) \to A(S)$ need not be a homotopy equivalence.

Proposition 5.26. *If $g \geq 1$ and $b \geq 3$, or if $g = 0$ and $b \geq 4$, then $B(S)$ is connected.*

The dimension of the complex of boundary graphs is given below (Proposition 5.35).

If S is a surface with nonempty boundary and if $\partial_0 \subset \partial S$ is a union of components of ∂, then we denote by $B(S, \partial_0)$ the subcomplex of $B(S)$ induced by the vertices represented by boundary graphs S that are associated to arcs whose endpoints are on ∂_0. We have a natural simplicial embedding

$$B(S, \partial_0) \hookrightarrow B(S).$$

In particular, for any domain X on S having a nonmpty collection of boundary components that are boundary components of S, we have a natural simplicial embedding

$$B(X, \partial X \cap \partial S) \hookrightarrow B(S).$$

5.4 The complex of peripheral pairs of pants $P_\partial(S)$

Definition 5.27. The *complex of peripheral pairs of pants on S*, $P_\partial(S)$ is the subcomplex of $D(S)$ induced by the set of vertices of $D(S)$ that are represented by peripheral pairs of pants on S.

If S is a sphere with at most three holes ora torus with at most one hole, then $P_\partial(S)$ is empty.

If S is a sphere with four holes, then, each peripheral pair of pants P on S is a biperipheral pair of pants, and we have a simplicial isomorphism $P_\partial(S) \simeq C(S) \times \Delta_1$, where Δ_1 is an edge.

Now we describe a natural map

$$\eta : B(S) \to P_\partial(S).$$

Suppose that S is not a sphere with at most three holes.

Let α be an arc on S. Let P_α be a regular neighborhood on S of the boundary graph G_α of α such that P_α is an essential surface on S. Since either $g \geq 1$ or $b \geq 4$, P_α is an essential peripheral pair of pants on S. It follows that there is a natural map from the vertex set of $B(S)$ to the vertex set of $P_\partial(S)$ which maps the vertex of $B(S)$ represented by G_α to the vertex of $P_\partial(S)$ represented by P_α.

Note that the vertices of any simplex of $B(S)$ can be represented by arcs with disjoint boundary graphs. Furthermore, we may choose disjoint essential regular neighborhoods of these disjoint boundary graphs. It follows that the natural map $B_0(S) \to P_0(S)$ extends to a natural simplicial map $\eta : B(S) \to P_\partial(S)$.

Proposition 5.28. *Suppose that S is not a sphere with at most three holes. Then the natural simplicial map $\eta : B(S) \to P_\partial(S)$ is surjective.*

We cannot always identify $B(S)$ with the complex of peripheral pairs of pants $P_\partial(S)$ on S as the natural map $\eta : B(S) \to P_\partial(S)$ need not be injective. Indeed, $\eta : B(S) \to P_\partial(S)$ is injective precisely when $b = 1$; in which case, $B(S)$ is either empty, when $g = 0$, or an infinite vertex set, when $g \geq 1$.

The following proposition describes precisely the failure of $\eta : B(S) \to P_\partial(S)$ to be, in general, injective.

Proposition 5.29. *Suppose that S is not a sphere with at most three holes. Let α be an essential arc on S, P_α be an essential regular neighborhood of the boundary graph G_α of α on S, u be the vertex of $B(S)$ represented by G_α, z be the vertex of $P_\partial(S)$ represented by P_α, and $\eta^{-1}(z)$ be the fiber of η over the vertex z of $P_\partial(S)$. Then:*

(1) *if P_α has exactly one boundary component on ∂S, then $\eta^{-1}(w)$ is the single vertex, u, of $B(S)$;*

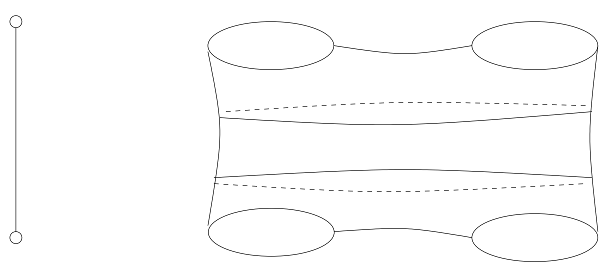

Figure 15. A component of $P_\partial(S_{0,4})$ and domains representing its two vertices

(2) *if P_α has exactly two boundary components on ∂S, then $\eta^{-1}(w)$ is a set of diameter 2 in $B(S)$ consisting of three distinct vertices, u, v, and w, of $B(S)$, corresponding to arcs of S contained in P_α.*

Proof. This follows easily from the classification of isotopy classes of arcs in pairs of pants. See Figure 7. $\square$

Corollary 5.30. *Suppose that S is not a sphere with at most three holes. Let $\eta : B(S) \to P_\partial(S)$ be the natural simplicial map. If σ is a k-simplex of $B(S)$, then $\eta(\sigma)$ is a k-simplex of $P_\partial(S)$.*

Proof. Since $\eta : B(S) \to P_\partial(S)$ is simplicial, $\eta(\sigma)$ is an l-simplex of $B(S)$ for some nonnegative integer $l \leq k$. Suppose that $l < k$ and, hence, that there exists a pair of distinct vertices u and v of σ such that $\eta(u) = \eta(v)$. Let $z = \eta(u)$. Since S is not a sphere with two or three holes and the distinct vertices u and v of $B(S)$ are both in the fiber of η over z, it follows from Proposition 5.29 that z is represented by a biperipheral pair of pants P on S and u and v are represented by the boundary graphs of arcs α and β on S contained in P. Note that each endpoint of α and β lies on one of the two peripheral boundary component of P.

Since P is a domain on S and G_α and G_β represent distinct vertices of the simplex σ of $B(S)$, we may assume that G_α and G_β are disjoint. It follows that α joins one of the two peripheral boundary components of P to itself, and β joins the other of these two peripheral boundary components of S to itself. It follows from the classification of isotopy classes of arcs in pairs of pants that α and β intersect. This is a contradiction. Hence, $l = k$; that is to say, $\eta(\sigma)$ is a k-simplex of $P_\partial(S)$. $\square$

Proposition 5.31. *Let ∂_i be a boundary component of S and σ be an essential curve on S. Then there exists an arc α of type $\{i, i\}$ on S such that σ is an essential boundary component of a regular neighborhood P_α of G_α on S.*

Proof. Since S is connected, there exists an embedded path J in S such that $J \cap \partial_i$ is one endpoint of J and $J \cap \sigma$ is the other endpoint of J. Let $G = \partial_i \cup J \cup \sigma$. Let N be an essential regular neighborhood of the graph G on S. Then N is an essential peripheral pair of pants on S with one of its essential boundary components isotopic to σ on S. By isotoping N on S, we may assume that σ is an essential boundary component of N. Let α be an essential arc in N joining ∂_i to itself. Then N is a regular neighborhood of G_α. $\square$

Suppose now that S is a sphere with four holes.

The structure of $B(S)$ can be obtained from studying the natural map $\eta : B(S) \to P_\partial(S)$. Let e be an edge of $P_\partial(S)$ and P and Q be disjoint biperipheral pairs of pants representing the vertices x and y of e. Since S is not a sphere with two or three holes and x is represented by the biperipheral pair of pants P on S, it follows from Propositions 5.28 and 5.29 that $\eta^{-1}(x)$ is equal to the set of three vertices of $B(S)$ corresponding to the boundary graphs on S contained in P. Note that no two of these three vertices of $B(S)$ are joined by an edge of $B(S)$. Likewise, $\eta^{-1}(y)$ is equal to the set of three vertices of $B(S)$ corresponding to the boundary graphs on S contained in Q and no two of these three vertices of $B(S)$ are joined by an edge of $B(S)$.

Note, furthermore, that all the vertices of $\eta^{-1}(x)$ are joined by edges to all the vertices of $\eta^{-1}(y)$.

Let $G(X,Y)$ be a simplicial graph whose vertex set is the disjoint union of two nonempty sets, X and Y, and whose edges are the pairs $\{u,v\}$, $u \in X, v \in Y$. We say that $G(X,Y)$ is the complete bipartite graph on X and Y.

It follows that $\eta^{-1}(e)$ is the complete bipartite graph $G(\eta^{-1}(x), \eta^{-1}(y))$ on $\eta^{-1}(x)$ and $\eta^{-1}(y)$. Moreover, each edge of $G(\eta^{-1}(x), \eta^{-1}(y))$ is mapped by $\eta : B(S) \to P_\partial(S)$ onto the edge $e = \{x,y\}$ of $P_\partial(S)$.

Let γ be an essential curve on S and A be a regular neighborhood of γ on S. Let P and Q be the two codomains of A on S, both of which are biperipheral pairs of pants on S. Let x and y be the vertices of $P_\partial(S)$ represented by P and Q, and $G_\gamma = G(\eta^{-1}(x), \eta^{-1}(y))$.

From these considerations, we deduce the following description of $B(S)$.

Proposition 5.32. *Let $G(3,3)$ be a complete bipartite graph on two sets of cardinality 3. $C(S)$ is an infinite set of vertices and $B(S) \simeq G(3,3) \times C(S)$, where the component of $B(S)$ corresponding to the component $G(3,3) \times \{\gamma\}$ of $G(3,3) \times C(S)$ is equal to G_γ.*

Let P be a peripheral pair of pants on S. If P is monoperipheral and ∂_i is the unique boundary component of S which is a boundary component of P, then there exists an essential arc α on P joining ∂_i to itself. If P is biperipheral and ∂_i and ∂_j are the unique boundary components of S which are boundary components of P, then there exists an essential arc α on P joining ∂_i to ∂_j.

There is a natural inclusion

$$i : P_\partial(S) \to B(S)$$

which maps the vertex of $P_\partial(S)$ corresponding to a peripheral pair of pants P on S to the vertex of $B(S)$ corresponding to the boundary graph G_α of an essential arc α on P joining the peripheral boundary components of P as above.

Proposition 5.33. *The composition $\eta \circ i : P_\partial(S) \to P_\partial(S)$ of the natural inclusion $i : P_\partial(S) \to B(S)$ with the natural map $\eta : B(S) \to P_\partial(S)$ is equal to the identity of $P_\partial(S)$.*

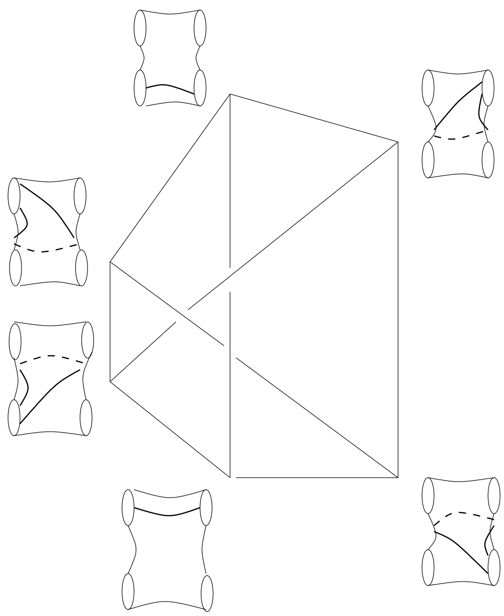

Figure 16. A component of $B(S_{0,4})$ and boundary graphs representing its six vertices

Proposition 5.34 (The dimension of the complex of peripheral pairs of pants). *If S is a sphere with at most three holes or a closed surface, then $P_\partial(S)$ is empty. If S is not a sphere with at most three holes or a closed surface and if $g = 0$, then* $\dim(P_\partial(S)) = b-3$. *If $g \geq 1$, then* $\dim(P_\partial(S)) = b-1$.

Proof. We shall give the argument in the cases where S is not a sphere with at most three holes or a surface with at most one hole. In all of these cases, the calculation of $\dim(P_\partial(S))$ follows from the calculation of $\dim(D^2(S))$. In genus zero, this calculation shows, on the one hand, that each maximal collection of disjoint peripheral pairs of pants on S exhausts the boundary of S and has at least two biperipheral pairs of pants and, on the other hand, that there exists a maximal collection of disjoint peripheral pairs of pants on S with only two biperipheral pairs of pants. In positive genus, this calculation shows, on the one hand, that each maximal collection of disjoint peripheral pairs of pants on S exhausts the boundary of S and, on the other hand, that there exists a maximal collection of disjoint peripheral pairs of pants on S with no biperipheral pairs of pants. The calculation follows. □

Proposition 5.35 (The dimension of the complex of boundary graphs). *If S is a closed surface or a sphere with one hole, then $B(S)$ is empty. If S is a sphere with two holes, then $B(S)$ is a singleton set. If S is a sphere with three holes, then $B(S)$ is a set of six vertices. If S is not a closed surface or a sphere with at most three holes, then* $\dim(B(S)) = \dim(P_\partial(S))$. *Hence, if $g = 0$, then* $\dim(B(S)) = b - 3$*; and, if $g \geq 1$, then* $\dim(B(S)) = b - 1$.

Proof. We shall give the argument when S is not a closed surface or a sphere with at most three holes. Since S is not a sphere with two or three holes, it follows from Proposition 5.28 that $\dim(P_\partial(S)) \leq \dim(B(S))$. Likewise, it follows from Corollary 5.30 that $\dim(B(S)) \leq \dim(P_\partial(S))$. Hence, $\dim(B(S)) = \dim(P_\partial(S))$. The result follows then from Proposition prop:dimPpartial. □

5.5 Other subcomplexes of $D(S)$

The *complex of elementary domains $E(S)$ of S* is the subcomplex of $D(S)$ induced by the set of vertices of $D(S)$ which are represented by essential annuli and pairs of pants on S.

Proposition 5.36 (The dimension of the complex of elementary domains $E(S)$). *If S is a sphere with at most three holes, then $E(S)$ is empty. If S is a closed torus, then the natural map $\eta : C(S) \to E(S)$ is an isomorphism and, hence, $E(S)$ is an infinite set of vertices. Otherwise,* $\dim(E(S)) = \dim(D(S)) = 5g + 2b - 6$.

Proof. Since $E(S)$ is a subcomplex of $D(S)$, we have $\dim(E(S)) \leq \dim(D(S)) = 5g + 2b - 6$. If S is neither a sphere with at most three hole nor a closed torus, we let C be a pants decomposition of S, and Δ_C the canonical maximal simplex of $D(S)$ associaed to C. The vertices of Δ_C are in $E(S)$, hence, Δ_C is also a simplex of $E(S)$. Its dimension is $5g + 2b - 6$. It follows that $\dim(E(S)) \geq 5g + 2b - 6$. This shows that $\dim(E(S)) = \dim(D(S)) = 5g + 2b - 6$.

□

Definition 5.37. The *complex of annular domains $R(S)$ of S* is the subcomplex of $D(S)$ induced by the set of vertices of $D(S)$ which are represented by essential annuli on S.

There is a natural simplicial isomorphism

$$C(S) \simeq R(S)$$

which maps the vertex of $C(S)$ represented by a curve α on S to the vertex of $R(S)$ represented bya regular neighborhood of α on S.

We shall identify $C(S)$ with the complex of annuli $R(S)$ on S via this natural isomorphism.

Definition 5.38. The *complex of pairs of pants on S, $R(S)$* is the subcomplex of $D(S)$ induced by the set of vertices of $D(S)$ which are represented by pairs of pants on S.

Proposition 5.39 (The dimension of the complex of pairs of pants). *If S is a sphere with at most three holes or a closed torus, then $P(S)$ is empty. Otherwise,* $\dim(P(S)) = 2g + b - 3$.

Definition 5.40. The *complex of thick domains $TD(S)$ of S* is the induced subcomplex of $D(S)$ corresponding to those vertices of $D(S)$ which are represented by domains on S which are not annuli (i.e. which have negative Euler characteristic).

Proposition 5.41 (The complex of thick domains $TD(S)$). *If S is a sphere with at most three holes or a closed torus, then $TD(S)$ is empty. Otherwise,* $\dim(TD(S)) = 2g + b - 3$.

6 Topology of S recognized by $D(S)$ and $D^2(S)$

The aim of this section is to show on a series of specific cases how topological information on the surface S can be recognized by combinatorial information in the simplicial complexes $D(S)$ and $D^2(S)$.

The results of §6.4 and §6.5 below, entitled *Recognizing annular vertices in* $D^2(S)$ and *Recognizing biperipheral edges in* $D(S)$, will be used in the proofs of the rigidity results on the automorphisms od $D^2(S)$ and $D(S)$ that we give in §7 and 8.

We start with the following:

6.1 Recognizing elementary vertices in $D(S)$

We say that a vertex of $D(S)$ is *elementary* if it is represented by elementary domains on S.

The significance of this notion appears in the following characterizations of elementary vertices of $D(S)$ and various subtypes of elementary vertices of $D(S)$. In particular, these characterizations imply that any automorphism of $D(S)$ preserves the subcomplex of $D(S)$ induced by the set of elementary vertices of $D(S)$.

Proposition 6.1. *Let x be a vertex of $D(S)$. If* $\mathrm{Lk}(\mathrm{Lk}(x)) = \{x\}$*, then x is elementary.*

Proof. Let X be a domain on S representing x. Suppose that $\mathrm{Lk}(\mathrm{Lk}(x)) = \{x\}$ and x is not elementary. By Proposition 2.18, there exists a pair of curves α and β on S such that $i(\alpha, \beta) \neq 0$ and α and β are contained in the interior of X. Let W be a regular neighborhood of α in the interior of X and w be the vertex of $D(S)$ represented by W. Since $i(\alpha, \beta) \neq 0$, W is not isotopic to any domain disjoint from X. Hence, $w \notin \mathrm{Lk}(x)$. On the other hand, since $W \subset X$, $w \in \mathrm{St}(\mathrm{Lk}(x))$. Since $w \notin \mathrm{Lk}(x)$ and $w \in \mathrm{St}(\mathrm{Lk}(x))$, $w \in \mathrm{Lk}(\mathrm{Lk}(x))$. Hence, $w = x$; that is to say, the annulus W is isotopic to X on S. Since X is not an annulus, this is a contradiction. Hence, x is elementary. This proves the proposition. $\square$

Proposition 6.2 (Recognizing elementary vertices in $D(S)$). *Suppose that S is not a closed torus and let x be a vertex of $D(S)$. Then the following are equivalent:*

(1) *x is elementary.*

(2) *There exists a simplex Δ in $D(S)$ such that* $\mathrm{Lk}(\Delta) = \{x\}$.

(3) $\mathrm{Lk}(\mathrm{Lk}(x)) = \{x\}$.

Proof. Proposition 6.1 shows that (3) implies (1) (without the hypothesis that is not a closed torus).

Proposition 3.16 gives (2) $\Rightarrow$ (3).

It remains only to prove that (1) implies (2).

Suppose that x is elementary and let X be a domain on S representing x. Since x is elementary, X is either an annulus or a pair of pants.

If X is an annulus, let Δ be the simplex of $D(S)$ whose vertices are the vertices of $D(S)$ which are represented by codomains of X on S.

If X is a pair of pants, let Δ be the simplex of $D(S)$ whose vertices are the vertices of $D(S)$ which are represented by codomains of X on S together with the vertices of $D(S)$ which are represented by regular neighborhoods of essential boundary components of X on S.

Since X is isotopic to a domain on S which is disjoint from all the codomains of X on S and regular neighborhoods of all the essential boundary components of X on S, it follows that $x \in \mathrm{St}(\Delta)$.

On the other hand, by Proposition 5.18, $x \notin \Delta$, hence, $x \in \mathrm{Lk}(\Delta)$.

Suppose, conversely, that $w \in \mathrm{Lk}(\Delta)$ and let W be a domain on S representing w. Since $w \in \mathrm{Lk}(\Delta)$, W is isotopic to a domain on S which is disjoint from all the codomains of X on S. Hence, without loss of generality, W is contained in X.

Suppose that X is an annulus. Then W is isotopic to X on S.

Suppose that X is a pair of pants. Suppose that W is not isotopic to X on S. Then W is isotopic to a regular neighborhood of an essential boundary component of X on S. This implies that $w \in \Delta$. Since $w \in \mathrm{Lk}(\Delta)$, this is a contradiction. Hence, W is isotopic to X on S.

In any case, therefore, W is isotopic to X on S; that is to say, $w = x$ and, hence, $\mathrm{Lk}(\Delta) \subset \{x\}$.

$\square$

The following is an immediate corollary of Proposition 6.2.

Corollary 6.3. *Every simplicial automorphism of $D(S)$ preserves the subcomplex of $D(S)$ induced by the set of elementary vertices of $D(S)$.*

6.2 Recognizing nonseparating annuli in $D(S)$

Proposition 6.4 (Recognizing nonseparating annuli in $D(S)$). *Suppose that S is not a torus with at most one hole and let $x \in D(S)$. Then the following are equivalent:*

(1) *There exists a nonseparating curve α whose essential regular neighborhoods on S represent x.*

(2) *There exists a vertex y of $D(S)$ such that* $\mathrm{Lk}(y) = \{x\}$.

Proposition 6.4 is vacuously true when S is a sphere with at most three holes and false when S is a torus with at most one hole.

Proof. Suppose that x is represented by an essential regular neighborhood X of a nonseparating curve α on S. Since α is nonseparating, the complement Y of the interior of X in S is a domain on S. Let y be the vertex of $D(S)$ represented by Y. Since α is nonseparating and S is not a torus, Y is not isotopic to X on S; that is to say, $x \neq y$.

Let M be a regular neighborhood of α contained in the interior of X. Since M is isotopic to X on S, M also represents x. On the other hand, since M is disjoint from Y, $\{x, y\}$ is a simplex of $D(S)$ (and x and y are distinct vertices). Since x is in this simplex but not equal to y, $x \in \mathrm{Lk}(y)$.

Suppose that $z \in \mathrm{Lk}(y)$. Then we may choose an essential surface Z in S representing z and disjoint from Y. Since Z is disjoint from Y, Z is an essential surface contained in the interior of the annulus X. It follows that Z is isotopic to X on S; that is to say, $z = x$.

This proves that (1) implies (2).

Suppose that y is a vertex of $D(S)$ such that $\mathrm{Lk}(y) = \{x\}$. Then $x \neq y$ and $\mathrm{St}(y) = \{x, y\}$.

Since $x \in \mathrm{Lk}(y)$, we may choose disjoint nonisotopic domains, X and Y, on S representing, respectively, x and y.

Let C be an essential boundary component of X. Since X is disjoint from Y, there exists a regular neighborhood Z of C such that Z is disjoint from Y. Note that Z is a domain on S. Let z be the vertex of $D(S)$ represented by Z. Since Y and Z are disjoint, it follows that $z \in \mathrm{St}(y)$; that is to say, either $z = x$ or $z = y$.

Suppose that $z = y$. In this case, it follows that Y is the regular neighborhood of some essential curve E on S. Let M be a regular neighborhood of E contained in the interior of Y. Note that M is isotopic to Y on S. Hence, M also represents the vertex y of $D(S)$.

Suppose that E is separating. Then the complement of the interior of Y in S has exactly two components, U and V. Then U, V, and M are disjoint nonisotopic domains on S. It follows that $\mathrm{Lk}(y)$ has at least two elements. Indeed the vertices of $D(S)$ represented by U and V are distinct elements of $\mathrm{Lk}(y)$. Since $\mathrm{Lk}(y) = \{x\}$, this is a contradiction.

It follows that E is nonseparating. In particular, $g \geq 1$. Hence, there exists a one-holed torus T on S with E contained in its interior. Since Y is a regular neighborhood of E on S, we may assume that Y is contained in the interior of T.

Let G be the boundary of T. Since $(g, n) \neq (1, 0), (1, 1)$, it follows that G is an essential curve on S. Note that there exists a regular neighborhood H of G disjoint from Y.

Note furthermore that the complement of the interior of Y in T is a pair of pants P on S.

Now H and P are nonisotopic domains in the complement of M. Moreover, neither H nor P is isotopic to M. Again, it follows that $\mathrm{Lk}(y)$ has at least two elements. Indeed the vertices of $D(S)$ represented by H and P are distinct elements of $\mathrm{Lk}(y)$. Since $\mathrm{Lk}(y) = \{x\}$, this is a contradiction.

It follows that $z \neq y$ and, hence, $z = x$. Then X is isotopic to the regular neighborhood Z of the essential curve C on S. This implies that X is the regular neighborhood of some essential curve α on S. Let M be a regular neighborhood of α contained in the interior of X. Note that M is isotopic to X on S. Hence, M also represents the vertex x of $D(S)$.

Suppose that α is separating. Then the complement of the interior of X in S has exactly two components. Since Y is connected and disjoint from X, Y is contained in the interior of one of these components, U. Let V be the other component and v be the vertex of $D(S)$ represented by V.

Since Y and V are disjoint domains on S, $v \in \mathrm{St}(y)$.

Since V and M are nonisotopic domains on S, $v \neq x$. Hence, $v = y$. In other words, Y is isotopic to V on S. Hence, there exists a homeomorphism $h : S \to S$ such that h is isotopic to the identity map of S and $h(Y) = V$.

Let β be an essential boundary component of the domain Y on S. Since Y is contained in the complement of α, α and β are disjoint essential curves on S.

Suppose that β is not isotopic to α. Then there exists an essential curve γ such that $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) \neq 0$. This implies that γ is isotopic to a curve in U. Since h is isotopic to the identity map of S, γ is isotopic to $h(\gamma)$. Hence, $i(\alpha, h(\gamma)) = i(\alpha, \gamma) \neq 0$. On the other hand, since $\alpha \subset Y$ and $h(\gamma) \subset V$ and $Y \cap V = \emptyset$, $i(\alpha, h(\gamma)) = 0$. This is a contradiction. Hence, β is isotopic to α.

Suppose that Y is an annulus. Then X and Y are regular neighborhoods of isotopic curves α and β on S. Hence, X and Y are isotopic; that is to say, $x = y$. This is a contradiction. Hence, Y is not an annulus.

Since Y is a domain on S and Y is not an annulus, there exists an essential pair of pants P contained in Y having β as one of its boundary components.

Suppose that the remaining two boundary components of P are peripheral on S, then no isotopy can carry Y into the complement of U. This is a contradiction. Hence, there exists an essential boundary component ϵ of P on S distinct from β.

Note that ϵ and α are disjoint.

Suppose that ϵ is isotopic to α. Then ϵ is isotopic to β. Hence, ϵ and β are isotopic boundary components of an essential pair of pants P on S. This implies that β is nonseparating. This is contradiction. Hence, ϵ is not isotopic to α.

Replacing β with ϵ and repeating the above argument, we arrive, again, at a contradiction.

Hence, α is nonseparating.

This proves that (2) implies (1).

$\square$

Corollary 6.5. *Suppose that S is not a torus with at most one hole. Then every simplicial automorphism of $D(S)$ preserves the subcomplex of $D(S)$ induced by the set of vertices of $D(S)$ which are represented by regular neighborhoods of nonseparating curves on S.*

6.3 Recognizing elementary vertices in $D^2(S)$

The proof of the following proposition is the same as that given for Proposition 6.1 and therefore we omit it.

Proposition 6.6. *Let $x \in D^2(S)$. If $\mathrm{Lk}(\mathrm{Lk}(x)) = \{x\}$, then x is elementary.*

Proposition 6.7 (Recognizing elementary vertices in $D^2(S)$). *Let $x \in D^2(S)$. Suppose that S is neither a sphere with four holes nor a closed torus. Then the following are equivalent:*

(1) *x is elementary.*

(2) *There exists a simplex Δ in $D^2(S)$ such that $\mathrm{Lk}(\Delta) = \{x\}$.*

(3) $\mathrm{Lk}(\mathrm{Lk}(x)) = \{x\}$.

Proof. Form Proposition 3.16, (2) implies (3). From Proposition 6.6, (3) implies (1). It remains only to prove that (1) implies (2).

Suppose that x is elementary. Let X be a domain on S representing x. Since x is elementary, X is either an annulus or a pair of pants.

If X is an annulus, let Δ be the simplex of $D^2(S)$ whose vertices are the vertices of $D^2(S)$ that are represented by codomains of X on S that are not biperipheral pairs of pants on S.

If X is a pair of pants, let Δ be the simplex of $D^2(S)$ whose vertices are the vertices of $D^2(S)$ that are represented by codomains of X on S that are not biperipheral pairs of pants on S together with the vertices of $D^2(S)$ that are represented by regular neighborhoods of essential boundary components of X on S.

Since X is isotopic to a domain on S which is disjoint from all the codomains of X on S and regular neighborhoods of all the essential boundary components of X on S, it follows that $x \in \mathrm{St}(\Delta)$.

On the other hand, by Proposition 5.18, $x \notin \Delta$. Hence, $x \in \mathrm{Lk}(\Delta)$.

Suppose, conversely, that $w \in \mathrm{Lk}(\Delta)$. Let W be a domain on S representing w.

Suppose that X is an annulus.

Since $w \in \mathrm{Lk}(\Delta)$, W is isotopic to a domain on S which is disjoint from all the codomains of X on S which are not biperipheral pairs of pants on S.

Hence, without loss of generality, W is contained in the union U of X with all the codomains of X on S which are biperipheral pairs of pants on S.

Suppose, on the one hand, that no codomain of X on S is a biperipheral pair of pants on S. It follows that $U = X$ and, hence, $W \subset X$. Since X is an annular domain on S and W is a domain on S contained in X, it follows that W is isotopic to X on S. Thus, $w = x$.

Suppose, on the other hand, that X has a codomain Y on S which is a biperipheral pair of pants on S. It follows that X is a separating annulus on S and hence, has exactly one codomain Z other than Y.

Since S is not a sphere with four holes, Z is not a biperipheral pair of pants.

It follows that $U = X \cup Y$. Since X is an annular domain on S and $X \cap Y$ is one of the essential boundary components of X, it follows that U is isotopic to Y on S. Hence, W is isotopic to a domain on S contained in Y. Since W represents a vertex of $D^2(S)$ and Y is a biperipheral pair of pants on S, it follows that W is not isotopic on S to Y. Since Y is a pair of pants and W is a domain on S contained in Y, it follows that W is isotopic on S to a regular neighborhood of the unique essential boundary component of Y. This implies that W is isotopic to X on S. Thus, $w = x$.

This shows that $w = x$, if X is an annulus.

Suppose that X is a nonbiperipheral pair of pants.

Let B be the union of all the essential boundary components of X on S that are boundary components of codomains of X on S which are biperipheral pairs of pants. Since $w \in \mathrm{Lk}(\Delta)$, W is isotopic to a domain on S which is disjoint from B and all the codomains of X on S which are not biperipheral pairs of pants on S. It follows that W is contained in either $X \setminus B$ or $Y \setminus B$ for some biperipheral codomain Y of X on S.

Suppose that W is contained in $Y \setminus B$. Then W is a domain on S contained in Y. Since W represents a vertex of $D^2(S)$ and Y is a biperipheral pair of pants on S, it follows that W is not isotopic on S to Y. Since Y is a pair of pants and W is a domain on S contained in Y, it follows that W is isotopic on S to a regular neighborhood of the unique essential boundary component of Y. Since $w \in \mathrm{Lk}(\Delta)$ and hence, w is not in Δ, this is a contradiction.

It follows that W is contained in $X \setminus B$. Then W is a domain on S contained in X.

Suppose that W is not isotopic to X on S. Since X is a pair of pants and W is a domain on S contained in X, it follows that W is isotopic on S to a regular neighborhood of an essential boundary component of X. Since w is not in Δ, this is a contradiction.

Hence, W is isotopic to X on S. Thus, $w = x$.

□

Proposition 6.8 (Distinguishing nonseparating annuli from regular neighborhoods of biperipheral curves in $D^2(S)$). *Suppose that the genus of S is positive* *and S is not a torus with at most one hole. Let α be an essential curve on S which is either nonseparating or biperipheral and let x be the vertex of $D(S)$ represented by an essential regular neighborhood of α on S. Then the following are equivalent:*

(1) *α is nonseparating.*

(2) *There exists a top-dimensional simplex of $D^2(S)$ having x as one of its vertices.*

Proof. Suppose that α is nonseparating. There exists a pants decomposition C of S with no biperipheral pairs of pants and such that α is one of the curves in C. By Proposition 5.6, the corresponding simplex Δ_C of $D(S)$ is a top dimensional simplex of $D(S)$. On the other hand, having no vertices corresponding to biperipheral pairs of pants, Δ_C is a simplex of $D^2(S)$. Hence, it certainly is a top-dimensional simplex of $D^2(S)$.

This proves that (1) implies (2).

Suppose that there exists a top-dimensional simplex Δ of $D^2(S)$ having x as one of its vertices.

By Proposition 5.20, $\dim(D^2(S)) = \dim(D(S))$. Hence, Δ is a top-dimensional simplex of $D(S)$. It follows that Δ is a maximal simplex of $D(S)$.

By Proposition 5.12, there exists a system of curves C of S such that $\Delta = \Delta_C$.

By Proposition 5.13, C is a pants decomposition. Since x is a vertex of Δ_C, x corresponds to either a component of C or to a pair of pants of C. Since x corresponds to α, we may assume, by isotoping α, that α is a component of C.

Suppose that α is biperipheral. Then the biperipheral pair of pants, Y, corresponding to α must be a pair of pants of C. Hence, the corresponding vertex y of $D(S)$ is an element of Δ_C. Since Δ_C is a simplex of $D^2(C)$, it follows that y is a vertex of $D^2(C)$. Since no vertex of $D^2(C)$ corresponds to a biperipheral pair of pants, this is a contradiction. Hence, α is nonseparating.

This proves that (2) implies (1).

□

Proposition 6.9. *Suppose that S is neither a sphere with at most three holes nor a torus with at most one hole and let $[X]$ be a vertex of $D^2(S)$. Suppose that X is not a regular neighborhood of a nonseparating curve on S nor a regular neighborhood of a biperipheral curve on S. Then the following are equivalent:*

(1) *Either X is a regular neighborhood of a nonbiperipheral separating curve, or S is a torus with two holes and X is a nonperipheral pair of pants on S, or S is a sphere with five holes and X is a monoperipheral pair of pants on S.*

(2) *There exists an edge e of $D^2(S)$ such that* $\mathrm{Lk}(e) = \{[X]\}$.

Proof. Suppose that X is a regular neighborhood of a nonbiperipheral separating curve. Let e be $\{[Y],[Z]\}$ where Y and Z are the distinct codomains of X. Since X is nonbiperipheral, neither Y nor Z are biperipheral pairs of pants. Hence, e is an edge of $D^2(S)$. Since Y and Z are the codomains of X, it follows that $\mathrm{Lk}(e) = \{x\}$.

Suppose that S is a torus with two holes and X is a nonperipheral pair of pants on S. Note that X has exactly two codomains, U and Z, where U is a biperipheral pair of pants and Z is a regular neighborhood of a nonseparating curve on S. Let Y be a regular neighborhood of the unique essential boundary component of U such that Y and Z are disjoint.

Let e be $\{[Y],[Z]\}$. Since Y and Z are not biperipheral pairs of pants on S, $e \subset D^2(S)$. Since Y is a regular neighborhood of a biperipheral curve on S and Z is a regular neighborhood of a nonseparating curve on S, Y and Z are nonisotopic on S. Hence, since Y and Z are disjoint domains on S, e is an edge of $D^2(S)$.

Since X is a pair of pants, X is isotopic to neither Y nor Z. Note that X is isotopic to a domain on S which is disjoint from $Y \cup Z$. Hence, $[X] \in \mathrm{Lk}(e)$.

Let $w \in \mathrm{Lk}(e)$. Then there exists a domain W representing w which is disjoint from Y and Z and not isotopic to either Y or Z. Since W is disjoint from Y and Z, W is contained in either U or X.

Suppose that W is contained in U. Since W is a domain on S and U is a biperipheral pair of pants on S, it follows that either W is isotopic to U or W is isotopic to a regular neighborhood of the unique essential boundary component of U. Since W represents a vertex of $D^2(S)$, W is not isotopic to U. Hence, W is isotopic to a regular neighborhood of the unique essential boundary component of U. It follows that W is isotopic to either Y or Z. This is a contradiction. Hence, W is not contained in U.

It follows that W is contained in X. Since W is a domain on S and X is a pair of pants on S, then W is isotopic to either X or a regular neighborhood of an essential boundary component of X.

Suppose that W is isotopic to a regular neighborhood of an essential boundary component of X. Then W is isotopic to either Y or Z. This is a contradiction. Hence, W is isotopic to X on S; that is to say, $w = [X]$. Hence, $\mathrm{Lk}(e) = \{[X]\}$.

Suppose that S is a sphere with five holes and X is a monoperipheral pair of pants. Let e be $\{[Y],[Z]\}$ where Y and Z are disjoint regular neighborhoods of the two essential boundary components of X. Since S is not a torus with one hole, Y and Z are nonisotopic. Hence, e is an edge of $D^2(S)$.

Note that $X \cup Y \cup Z$ is a domain on S isotopic to X with exactly two codomains, U and V, both of which are biperipheral pairs of pants on S.

Moreover, the essential boundary components of U and V are isotopic to the essential boundary components of X.

Since Y and Z are disjoint regular neighborhoods of essential boundary components of X, X is isotopic to a surface disjoint from Y and Z. Since X is a pair of pants and Y and Z are annuli, X is not isotopic to either Y or Z. Hence, $[X] \in \mathrm{Lk}(e)$.

Let $w \in \mathrm{Lk}(e)$. Then there exists a domain W representing w which is disjoint from Y and Z and not isotopic to either Y or Z. Since W is disjoint from Y and Z, W is contained in either X, U, or V. Since W represents a vertex of $D^2(S)$, W is not a biperipheral pair of pants. Suppose that W is contained in U. Then, since W is not a biperipheral pair of pants, W must be isotopic to a regular neighborhood of the essential boundary component of U. Hence, W is isotopic to either Y or Z. This is a contradiction. Hence, W is not contained in U. Likewise, W is not contained in V. It follows that W is contained in X. Since X is a monoperipheral pair of pants on S, W is isotopic to either X or a regular neighborhood of an essential boundary component of X. The latter possibility would imply that W is isotopic to either Y or Z, leading to a contradiction. Hence, W is isotopic to X; that is to say, $w = [X]$.

Hence, $\mathrm{Lk}(e) = \{x\}$.

This establishes, in any case, that (1) implies (2).

Suppose that there exists an edge e of $D^2(S)$ such that $\mathrm{Lk}(e) = \{[X]\}$. Let $e = \{y, z\}$ and $x = [X]$. Note that $\{x, y, z\}$ is a triangle of $D^2(S)$. Since $\mathrm{Lk}(e) = \{x\}$, $\{x, y, z\}$ is a maximal simplex of $D^2(S)$.

Suppose that $\{y, z\}$ is an edge of $D^2(S)$ such that $\mathrm{Lk}(e) = \{[X]\}$.

Since $[X]$ is the link of a simplex of $D^2(S)$, it follows by Proposition 6.6, that X is elementary.

Since e is an edge of $D^2(S)$, S is neither a sphere with four holes nor a closed torus.

Let $\Delta = \{x, y, z\}$. Then Δ is a triangle of $D^2(S)$ and a maximal simplex of $D^2(S)$. Since $D(S)$ is finite dimensional, there exists a maximal simplex $\overline{\Delta}$ of $D(S)$ containing Δ. It follows by Proposition 5.12 that there exists a system of curves C on S such that $\overline{\Delta} = \Delta_C$.

Let $\mathcal{F} = \{Y_i \mid 1 \leq i \leq m\}$ be a tiling representing Δ_C and $\{A_j \mid 1 \leq j \leq n\}$ be the set of ties of $\mathcal{F}$.

Let B be the set of vertices of $\overline{\Delta}$ which are not vertices of Δ.

Since Δ is a maximal simplex of $D^2(S)$, the vertices in B are represented by biperipheral pairs of pants on S. It follows that the vertices of B are nonannular vertices of the tiling $\{Y_i \mid 1 \leq i \leq m\}$. Hence, there exists a map $f : B \to \Delta_0$ defined by the rule $[Y_i] \mapsto [Y_k]$, where Y_k is the unique annular tile tied to the nonannular tile Y_i by an annular codomain of the tiling $\{Y_i \mid 1 \leq i \leq m\}$.

Suppose that the map $B \to \Delta_0$ is not injective. Then there exist two disjoint biperipheral pairs of pants Y_i and Y_j and an annular tile Y_k such that

Y_i is tied by an annular codomain A_p of $\{Y_i \mid 1 \leq i \leq m\}$ to Y_k and Y_K is tied by an annular codomain A_q of $\{Y_i \mid 1 \leq i \leq m\}$ to Y_j. It follows that $S = Y_i \cup A_p \cup Y_k \cup A_q \cup Y_j$ and hence, S is a sphere with four holes. This is a contradiction. Hence, the map $B \to \Delta_0$ is injective.

Let r be the number of elements of B. Since Δ is a triangle and the map $B \to \Delta_0$ is injective, it follows that $r \leq 3$.

Let X, Y, and Z be the tiles of $\{Y_i \mid 1 \leq i \leq m\}$ which represent the vertices x, y, and z of $\overline{\Delta}$

Suppose that $r = 3$. In other words, suppose that the map $B \to \Delta$ is bijective. Since x, y, and z are vertices of Δ and the map $B \to \Delta$ is bijective, there exist distinct tiles U, V, and W of $\{Y_i \mid 1 \leq i \leq m\}$ and distinct annular codomains, A, B, and C of $\{Y_i \mid 1 \leq i \leq m\}$ such that A joins U to X. B joins V to Y, and C joins W to Z. Let $R = U \cup A \cup X \cup V \cup B \cup Y \cup W \cup C \cup Z$ and T be the closure of the complement of R in S. Note that T is a domain on S with at least three essential boundary components. Hence, T is neither an annulus nor a biperipheral pair of pants. Since T is not a biperipheral pair of pants on S, T represents a vertex t of $D^2(S)$. Note that T is isotopic to a domain on S which is disjoint from $X \cup Y \cup Z$. Since T is not an annulus, T is not isotopic on S to X, Y, or Z. Hence, it follows that $t \neq x$ and $t \in \mathrm{Lk}(\{y, z\}) = \{x\}$. This is a contradiction. Hence, $r < 3$.

Suppose that $r = 0$. In other words, suppose that $\overline{\Delta} = \Delta$. Then Δ is a maximal simplex of $D(S)$. Hence, $\mathcal{F} = \{X, Y, Z\}$. It follows that $x \in \mathrm{Lk}(\{y, z\}, D(S))$.

Suppose, conversely, that $w \in \mathrm{Lk}(\{y, z\}, D(S)))$. It follows that W is represented by a domain contained in the complement of $Y \cup Z$ in S.

Since $\{X, Y, Z\}$ is a tiling of S with three tiles, the complement of $Y \cup Z$ in S is equal to the union of X with the ties of $\mathcal{F}$. By Proposition 5.7, each tie of $\mathcal{F}$ either joins X to Y, X to Z, or Y to Z. The unique codomain X' of $Y \cup Z$ on S which contains X is equal to the union of X with those ties of $\mathcal{F}$ which join X to either Y or Z. Any remaining codomains of $Y \cup Z$ on S are ties of $\mathcal{F}$ joining Y to Z.

Note that X' is isotopic to X on S.

Since W is a domain on S contained in the complement of $Y \cup Z$ in S, either W is contained in X or W is contained in an annular codomain of $X \cup Y \cup Z$ joining Y to Z.

Suppose that W is contained in a tie A of $\mathcal{F}$ joining Y to Z. By Proposition 5.7, since $\mathcal{F} = \{X, Y, Z\}$ and A is a tie of $\mathcal{F}$ joining Y to Z, either Y or Z is an annulus isotopic to A on S. Since W is a domain on S contained in the annulus A on S, W is isotopic to A on S. It follows that W is isotopic to either Y or Z on S. Since $w \in \mathrm{Lk}(\{y, z\}, D(S)))$, this is a contradiction. Hence, W is not contained in a tie of $\mathcal{F}$ joining Y to Z.

It follows that W is contained in X'. Since X' is isotopic on S to X, we may assume that W is contained in X. Since X is elementary and W is a domain on S contained in X, W is isotopic to either X or a regular neighborhood of an essential boundary component of X on S.

Suppose that W is isotopic to a regular neighborhood of an essential boundary component C of X on S. There exists a unique tie A of $\mathcal{F}$ having C as an essential boundary component. Note that A is an annulus joining X to either Y or Z. We may assume that A joins X to Y. By Proposition 5.7, since $\mathcal{F} = \{X, Y, Z\}$ and A is a tile of $\mathcal{F}$ joining X to Y, either X or Y is an annulus isotopic to A. Since C is an essential curve on S contained in the annulus A and W is a regular neighborhood of C on S, A is isotopic to W on S. Hence, W is isotopic to either X or Y on S.

Suppose that W is isotopic to Y on S. Then $w = y$. Since $w \in \mathrm{Lk}(\{y, z\}, D(S)))$, this is a contradiction.

Hence, W is isotopic to X; that is to say, $w = x$.

Suppose that $r = 1$. In other words, suppose that $\overline{\Delta} = \{x, y, z, p\}$ where p is represented by a biperipheral pair of pants on S.

Let P be the tile of $\mathcal{F}$ representing p. Then $\mathcal{F} = \{X, Y, Z, P\}$.

Let C be the unique essential boundary component of P. Let A be the unique tie of $\mathcal{F}$ containing C. Note that A is an annulus on S joining the nonannular tile P to an annular tile Q of $\mathcal{F}$ which is isotopic to A on S. It follows that Q is a regular neighborhood of a biperipheral curve on S. Since X is not a regular neighborhood of a biperipheral curve on S, it follows that Q is equal to either Y or Z. We may assume that Q is equal to Y.

It follows that Y is an annular tile of $\mathcal{F}$ and $P \cup A \cup Y$ is a domain P' on S which is isotopic to P on S.

Let D be the unique essential boundary component of P' on S and let B be the unique tie of $\mathcal{F}$ containing D.

Note that B is an annulus on S joining the annular tile Y to a nonannular tile R and that R is equal to either X or Z.

Suppose that R is equal to Z. Then Z is not an annulus.

Since X is a domain on S, X has an essential boundary component E on S. Let F be the unique tie of $\mathcal{F}$ containing E. Note that F is an annulus on S joining X to either Y or Z or P. On the other hand, no such tie can join X to either P or Y. Hence, F joins X to Z. Since Z is not an annulus and R is a tie of $\mathcal{F}$ joining Z to X, it follows that X is an annulus isotopic to F on S.

Since X is an annular domain on S and E is an essential boundary component of X on S, there exists an essential boundary component G of X on S which is not equal to E.

Let H be the unique tie of $\mathcal{F}$ containing G. As for the codomain F, it follows that H joins X to Z.

Since $F \cup X \cup H$ is an annulus on S joining two distinct boundary components of the domain Z of S and the interior of $F \cup X \cup H$ is disjoint from Z, it follows that X is a regular neighborhood of a nonseparating curve on S. This is a contradiction.

Hence, R is equal to X. Then X is not an annulus.

Since Z is a domain on S, Z has an essential boundary component E on S. Let F be the unique tie of $\mathcal{F}$ containing E. Note that F is an annulus on S joining Z to either X or Y or P. On the other hand, no such tie can join Z to either P or Y. Hence, F joins Z to X. Since X is not an annulus and R is a tie of $\mathcal{F}$ joining X to Z, it follows that Z is an annulus isotopic to F on S.

Since Z is an annular domain on S and E is an essential boundary component of Z on S, there exists an essential boundary component G of Z on S which is not equal to E.

Let H be the unique tie of $\mathcal{F}$ containing G. As for the tie F, it follows that H joins Z to X.

It follows that $S = P \cup A \cup Y \cup B \cup X \cup F \cup Z \cup H$. Since X is elementary with at least 3 boundary components, one on each of the ties B, F, and H of $\mathcal{F}$, it follows that X is a pair of pants. Hence, S is a torus with two holes and X is a nonperipheral pair of pants on S.

Suppose that $r = 2$. In other words, suppose that $\overline{\Delta} = \{x, y, z, p, q\}$ where p and q are represented by disjoint biperipheral pairs of pants on S.

Let P and Q be the tiles of $\{Y_i \mid 1 \leq i \leq m\}$ representing p and q.

Then $\mathcal{F} = \{X, Y, Z, P, Q\}$.

Let C and D be the unique essential boundary components of P and Q and let A be the unique tie of $\mathcal{F}$ containing C.

Note that A is an annulus on S joining the nonannular tile P to an annular tile R of $\mathcal{F}$ which is isotopic to A on S.

It follows that R is a regular neighborhood of a biperipheral curve on S.

Since X is not a regular neighborhood of a biperipheral curve on S, it follows that R is equal to either Y or Z. We may assume that R is equal to Y.

Let B be the unique tie of $\mathcal{F}$ containing D. Then B is an annulus on S joining the nonannular tile Q to an annular tile T of $\mathcal{F}$ which is isotopic to B on S. It follows that T is a regular neighborhood of a biperipheral curve on S.

Since X is not a regular neighborhood of a biperipheral curve on S, it follows that T is equal to either Y or Z.

Suppose that T is equal to Y.

Then $S = P \cup A \cup Y \cup B \cup Q$. Since $\{X, Y, Z, P, Q\}$ is a tiling of S, this is a contradiction. Hence, T is equal to Z.

By arguments used in the proof for the case where $r = 1$, it follows that there exist distinct ties F and G of $\mathcal{F}$ joining Y to X and Z to X and, hence, $S = P \cup A \cup Y \cup F \cup X \cup G \cup Z \cup B \cup Q$.

Since Y is an annulus and F is a tie of $\mathcal{F}$ joining Y to X, X is not an annulus.

Since X is elementary it follows that X is a pair of pants.

Since $S = P \cup A \cup Y \cup F \cup X \cup G \cup Z \cup B \cup Q$, this implies that S is a sphere with five holes and X is a monoperipheral pair of pants on S. $\square$

6.4 Recognizing annular vertices in $D^2(S)$

Proposition 6.10. *Suppose that S is not a torus with one hole and let $x \in D^2_0(S)$. Then the following are equivalent:*

(1) *x is an annular vertex of $D^2(S)$.*

(2) *For each vertex y of $D^2(S)$ which is not equal to x,* $\mathrm{St}(x)$ *is not contained in* $\mathrm{St}(y)$.

Proof. Since $D^2(S)$ is a flag complex, requiring property (2) of a vertex x of $D^2(S)$ is equivalent to requiring that for each vertex y of $D^2(S)$ which is not equal to x, there exists a vertex z of $D^2(S)$ such that $\{x, z\}$ is a simplex of $D^2(S)$ and $\{y, z\}$ is not a simplex of $D^2(S)$.

We begin by proving that (1) implies (2). To this end, let x be an annular vertex of $D^2(S)$ and y be a vertex of $D^2(S)$ such that $y \neq x$.

We shall deduce (2) by contradiction. To this end, suppose that:

> (*) For every vertex z of $D^2(S)$ such that $\{x, z\}$ is a simplex of $D^2(S)$, $\{y, z\}$ is a simplex of $D^2(S)$.

Since x is an annular vertex of $D^2(S)$, there exists an essential curve α on S such that x is represented by a regular neighborhood of α on S.

Choose a maximal system $\mathcal{C}$ of curves on S containing α.

Let R be a regular neighborhood of the support $|\mathcal{C}|$ of $\mathcal{C}$ on S. For each curve β in the system $\mathcal{C}$, let R_β be the unique component of R which contains β and $x_\beta = [R_\beta] \in D^2(S)$.

Let $\beta \in \mathcal{C}$. Since $x_\alpha = x$, it follows that $\{x, x_\beta\}$ is a simplex of $D^2(S)$ and, hence, by condition (*), $\{y, x_\beta\}$ is a simplex of $D^2(S)$.

In particular, $\{y, x\} = \{y, x_\alpha\}$ is a simplex of $D^2(S)$. Since $y \neq x$, this implies that $\{y, x\}$ is an edge of $D^2(S)$. Hence, S is neither a sphere with at most four holes nor a closed torus.

In particular, the Euler characteristic of S is negative. Hence, the maximal system $\mathcal{C}$ of curves on S is a pants decomposition of S. Let $\mathcal{P}$ be the collection of pairs of pants of $\mathcal{C}$.

Let Y be a domain on S representing y.

Since $\{y, x_\beta\}$ is a simplex of $D^2(S)$ for every curve β in the pants decomposition $\mathcal{C}$ of S, it follows that Y is a domain on S which is isotopic on S either to R_β for some β in $\mathcal{C}$ or to P for some pair of pants P in $\mathcal{P}$.

Suppose that Y is isotopic on S to R_β for some β in $\mathcal{C}$. Then $x_\beta = y \neq x = x_\alpha$ and, hence, $\beta \neq \alpha$.

Since α and β are disjoint nonisotopic essential curves on S, it follows from Proposition 2.3 that there exists a curve γ on S such that $i(\alpha, \gamma) = 0$ and $i(\beta, \gamma) \neq 0$.

Let Z be a regular neighborhood of γ on S and $z = [Z] \in D^2(S)$. Since $i(\alpha, \gamma) = 0$, $\{x, z\}$ is a simplex of $D^2(S)$. Hence, by condition (*), $\{y, z\}$ is a simplex of $D^2(S)$. Since $\{y, z\}$ is a simplex of $D(S)$ and z is an annular vertex, it follows that Y is isotopic on S to a domain which is disjoint from Z. Hence, $i(\beta, \gamma) = 0$, which is a contradiction.

Thus, Y is isotopic on S to some pair of pants P in $\mathcal{P}$.

Since P represents the vertex y of $D^2(S)$, P is not a biperipheral pair of pants on S. Since S is not a torus with one hole and P is a domain on S which is a nonbiperipheral pair of pants on S, it follows that there exists a pair of distinct nonisotopic essential boundary components, ϵ and η, of P.

Since each essential boundary component of a pair of pants of a pants decomposition of S is isotopic to one of the curves of the pants decomposition, there exist distinct curves, β and δ of $\mathcal{C}$ such that ϵ and η are isotopic on S to β and δ.

Since $\beta \neq \delta$, we may assume that $\alpha \neq \beta$. As before, it follows from Proposition 2.3 that there exists a curve γ on S such that $i(\alpha, \gamma) = 0$ and $i(\beta, \gamma) \neq 0$.

Let Z be a regular neighborhood of γ on S and $z = [Z] \in D^2(S)$. Since $i(\alpha, \gamma) = 0$, $\{x, z\}$ is a simplex of $D^2(S)$. Hence, by condition (*), $\{y, z\}$ is a simplex of $D^2(S)$. Since $\{y, z\}$ is a simplex of $D(S)$ and z is an annular vertex, it follows that Y is isotopic on S to a domain which is disjoint from Z. Since β is isotopic on S to ϵ and $\epsilon \subset Y$, it follows that $i(\beta, \gamma) = i(\epsilon, \gamma) = 0$, which is a contradiction.

This shows that (1) implies (2).

We shall now show that (2) implies (1). To this end, suppose that the vertex x of $D^2(S)$ is not an annular vertex of $D^2(S)$.

Let X be a domain on S representing x. Since x is not an annular vertex of $D^2(S)$, X is not an annulus.

Since X is a domain on S, X has an essential boundary component on S. Let Y be a regular neighborhood of an essential boundary component of X on S and $y = [Y] \in D^2(S)$. Then Y is isotopic on S to a domain on S which is disjoint from X. Since X is not an annulus and Y is an annulus, Y is not isotopic to X on S. It follows that y is not equal to x and $\{x, y\}$ is an edge of $D^2(S)$.

Suppose that z is a vertex of $D^2(S)$ such that $\{x, z\}$ is a simplex of $D^2(S)$; that is to say, suppose that either $x = z$ or $\{x, z\}$ is an edge of $D^2(S)$.

Suppose, on the one hand, that $z = x$. Then $\{y, z\}$ is equal to the simplex $\{x, y\}$ of $D^2(S)$.

Suppose, on the other hand, that $\{x, z\}$ is an edge of $D^2(S)$. Then z is represented by a domain Z on S which is disjoint from the domain X on S. Since Y is a regular neighborhood of an essential boundary component of X, it follows that Y is isotopic to a domain on S which is disjoint from Z. This implies that if Z is isotopic to Y, then $\{y, z\}$ is equal to the simplex $\{y\}$ of $D^2(S)$, whereas, if Z is not isotopic to Y on S, then $\{y, z\}$ is an edge of $D^2(S)$. In any case, $\{y, z\}$ is a simplex of $D^2(S)$.

This shows that (2) implies (1).

□

Corollary 6.11. *Suppose that S is not a torus with one hole. Then every simplicial automorphism of $D^2(S)$ restricts to a simplicial automorphism of the subcomplex $\iota(C(S))$ of $D^2(S)$ induced from the set of annular vertices of $D^2(S)$.*

Proof. Let $\varphi \in \mathrm{Aut}(D^2(S))$.

Suppose that x is an annular vertex of $D^2(S)$ (i.e. a vertex of $\iota(C(S))$). By Proposition 6.10, for each vertex y of $D^2(S)$ which is not equal to x, there exists a vertex z of $D^2(S)$ such that $\{x, z\}$ is a simplex of $D^2(S)$ and $\{y, z\}$ is not a simplex of $D^2(S)$.

Let $u = \varphi(x)$. Since $\varphi \in \mathrm{Aut}(D^2(S))$, it follows that for each vertex v of $D^2(S)$ which is not equal to u, there exists a vertex w of $D^2(S)$ such that $\{u, w\}$ is a simplex of $D^2(S)$ and $\{v, w\}$ is not a simplex of $D^2(S)$. Hence, by Proposition 6.10, u is a vertex of $\iota(C(S))$. This shows that φ maps the zero skeleton of $\iota(C(S))$ into the zero skeleton of $\iota(C(S))$.

Note that a simplex σ of $D^2(S)$ is a simplex of $\iota(C(S))$ if and only if each of its vertices is a vertex of $\iota(C(S))$. It follows that φ restricts to a simplicial map $\mu : \iota(C(S)) \to \iota(C(S))$. Likewise, the simplicial automorphism $\varphi^{-1} : D^2(S) \to D^2(S)$ restricts to a simplicial map $\lambda : \iota(C(S)) \to \iota(C(S))$.

Note that the restrictions $\mu : \iota(C(S)) \to \iota(C(S))$ and $\lambda : \iota(C(S)) \to \iota(C(S))$ of $\varphi : D^2(S) \to D^2(S)$ and $\varphi^{-1} : D^2(S) \to D^2(S)$ are inverse simpicial maps. Hence, $\mu : \iota(C(S)) \to \iota(C(S))$ is a simplicial isomorphism. This proves that $\varphi : D^2(S) \to D^2(S)$ restricts to a simplicial automorphism $\varphi : \iota(C(S)) \to \iota(C(S))$. □

6.5 Recognizing biperipheral edges in $D(S)$

In this section, we give a characterization biperipheral edges of $D(S)$. This will be used in the proof of the rigidity result on the automorphism group of the complex of domains, proved in 8 below.

Each biperipheral pair of pants on S has a unique biperipheral boundary component. It follows that there is a natural map from the set of vertices of $D(S)$ corresponding to biperipheral pairs of pants on S to the set of vertices of $D(S)$ corresponding to biperipheral curves on S. This map is a bijection if and only if S is not a sphere with four holes.

Definition 6.12. Suppose that X and Y are domains on S such that X is a biperipheral pair of pants on S and Y is isotopic to a regular neighborhood of the unique essential boundary component of X on S. Then we say that

$\{X, Y\}$ *is a biperipheral pair of domains on* S and sometimes, a *biperipheral pair* , and the edge $\{[X], [Y]\}$ of $D(S)$ is a *biperipheral edge of* $D(S)$.

Suppose that $\{X, Y\}$ is a biperipheral pair of domains on S. We may assume that X is a biperiphal pair of pants on S. Since Y is isotopic to a regular neighborhood of an essential boundary component of X on S, Y is isotopic to a domain Y_1 on S which is disjoint from X. Since X is not an annulus and Y_1 is an annulus, X and Y_1 are not isotopic on S. It follows that $\{X, Y_1\}$ is a system of domains on S and, hence, $\{[X], [Y]\} = \{[X], [Y_1]\}$ is, indeed, an edge of $D(S)$. Since $\{X, Y_1\}$ is both a biperipheral pair of domains on S and a system of domains on S, we say that $\{X, Y_1\}$ *is a biperipheral system of domains on* S.

Proposition 6.13 (vertices with nested stars in $D(S)$). *Let x and y be distinct vertices of $D(S)$. Then the following are equivalent:*

(1) $\mathrm{St}(x, D(S)) \subset \mathrm{St}(y, D(S))$.

(2) *There exist disjoint domains, X and Y, on S representing x and y which belong to one of the following cases:*

((a) *X is not an annulus and Y is an annulus on S which is joined to X by exactly one annular codomain of $X \cup Y$ on S.*

((b) *X is not an annulus and Y is an annulus on S which is joined to X by exactly two annular codomains of $X \cup Y$ on S.*

((c) *Y is a biperipheral pair of pants on S which is joined to X by exactly one annular codomain of $X \cup Y$ on S.*

((d) *Y is a monoperipheral pair of pants on S which is joined to X by exactly two annular codomains of $X \cup Y$ on S.*

((e) *Y is a nonperipheral pair of pants on S which is joined to X by exactly three annular codomains of $X \cup Y$ on S.*

Proof. Suppose that $\mathrm{St}(x, D(S)) \subset \mathrm{St}(y, D(S))$.

Since $x \in \mathrm{St}(x, D(S))$, it follows that $x \in \mathrm{St}(y, D(S))$. Since $x \neq y$, this implies that $\{x, y\}$ is an edge of $D(S)$; that is to say, x and y are represented by disjoint nonisotopic domains X and Y on S.

Suppose that Y is not elementary. By Proposition 2.18, there exist curves α and β on S such that $i(\alpha, \beta) \neq 0$ and α and β are contained in the interior of Y. Let W be a regular neighborhood of α on S such that W is contained in the interior of Y. Since W is contained in Y and X is disjoint from Y, X is disjoint from W. This implies that $\{x, w\}$ is a simplex of $D(S)$, where w is the vertex of $D(S)$ represented by W. It follows that w is a vertex of $\mathrm{St}(x, D(S))$

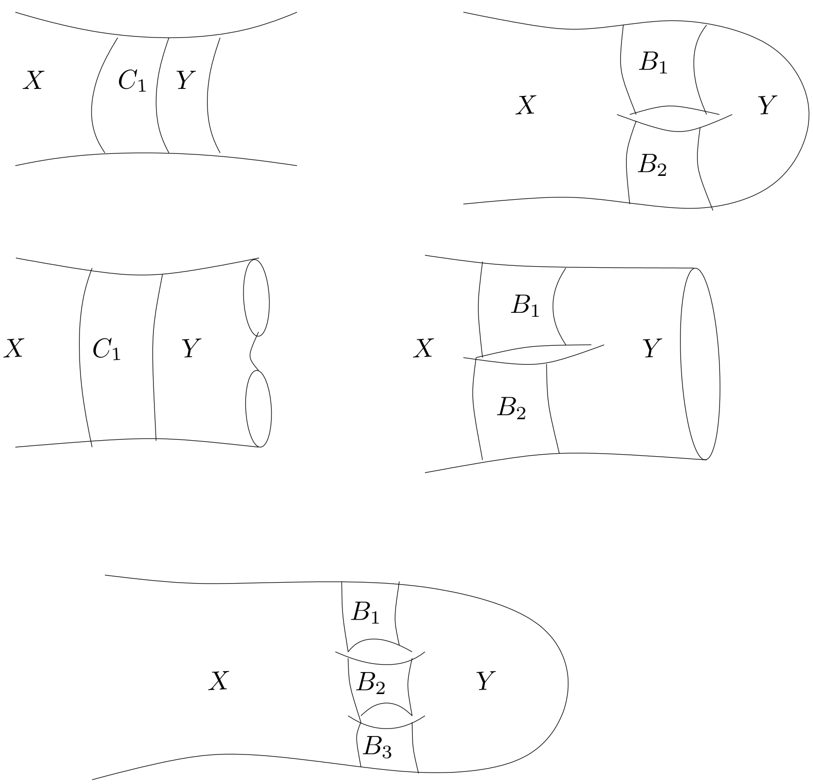

Figure 17. Ordered pairs of disjoint domains (X, Y) representing ordered pairs of vertices (x, y) that satisfy the equation $\mathrm{St}(x, D(S)) \subset \mathrm{St}(y, D(S))$ with $x \neq y$. See Proposition 6.13.

and, hence, w is a vertex of $\mathrm{St}(y, D(S))$; that is to say, $\{y, w\}$ is a simplex of $D(S)$. Since $\{y, w\}$ is a simplex of $D(S)$, either $y = w$ or $\{y, w\}$ is an edge of $D(S)$. Since Y is not an annulus on S and W is an annulus on S, Y is not isotopic to W on S; that is to say, $y \neq w$. Hence, $\{y, w\}$ is an edge of $D(S)$. It follows that W is isotopic on S to a domain on S which is disjoint from Y. Since α is contained in W, it follows that α is isotopic on S to a curve α_1 which is disjoint from Y and, hence, from β. Since α is isotopic on S to α_1 and α_1 and β are disjoint, it follows that $i(\alpha, \beta) = i(\alpha_1, \beta) = 0$, which is a contradiciton. Hence, Y is elementary.

Suppose that there exists an essential boundary component α of Y which is not isotopic to any essential boundary component of X. Since X and Y are disjoint, it follows from Proposition 2.3 that there exists an essential curve γ on S such that $i(\alpha, \gamma) \neq 0$ and $i(\beta, \gamma) = 0$ for every essential boundary component β of X.

We may assume that the collection $\mathcal{C}$ of curves on S consisting of α, γ, and the essential boundary components of X on S is in minimal position. It follows from the above constraints on geometric intersection numbers, that γ is disjoint from X.

Hence, there exists a regular neighborhood Z of γ on S such that Z is disjoint from X. Since Z is disjoint from X, Z represents a vertex z of $\mathrm{St}(x, D(S))$ and, hence, of $\mathrm{St}(y, D(S))$. Thus, $\{y, z\}$ is a simplex of $D(S)$.

Since $\{y, z\}$ is a simplex of $D(S)$, either $y = z$ or $\{y, z\}$ is an edge of $D(S)$.

Suppose that $y = z$; that is to say, suppose that Y is isotopic to Z on S. Since Z is an annulus on S, it follows that Y is an annulus on S. Thus Y is isotopic to a regular neighborhood of its essential boundary component α. Since Z is a regular neighborhood of γ and Y is isotopic to Z, it follows that α is isotopic to γ. Hence, $i(\alpha, \gamma) = i(\alpha, \alpha) = 0$ which is a contradiction.

□

Proposition 6.14 (vertices with the same star in $D(S)$). *Let x and y be distinct vertices of $D(S)$. Then the following are equivalent:*

(1) $\mathrm{St}(x, D(S)) = \mathrm{St}(y, D(S))$.

(2) *There exist disjoint domains, X and Y, on S representing x and y which belong to one of the following cases:*

- ((a) *X is a biperipheral pair of pants on S, Y is an annulus on S, and $X \cup Y$ has exactly two codomains, exactly one of which is an annulus joining X to Y.*
- ((b) *Case (2a) with the roles of X and Y interchanged.*
- ((c) *S is a sphere with four holes, X and Y are biperipheral pairs of pants on S, and $X \cup Y$ has exactly one codomain, an annulus joining X to Y.*

((d) *S is a torus with two holes, X and Y are monoperipheral pairs of pants on S, and $X \cup Y$ has exactly two codomains, both of which are annuli joining X to Y.*

((e) *S is a closed surface of genus two, X and Y are pairs of pants on S, and $X \cup Y$ has exactly three codomains, all of which are annuli joining X to Y.*

((f) *S is a torus with one hole, X is a monoperipheral pair of pants on S, Y is an annulus on S, and $X \cup Y$ has exactly two codomains, both of which are annuli joining X to Y.*

((g) *Case (2f) with the roles of X and Y interchanged.*

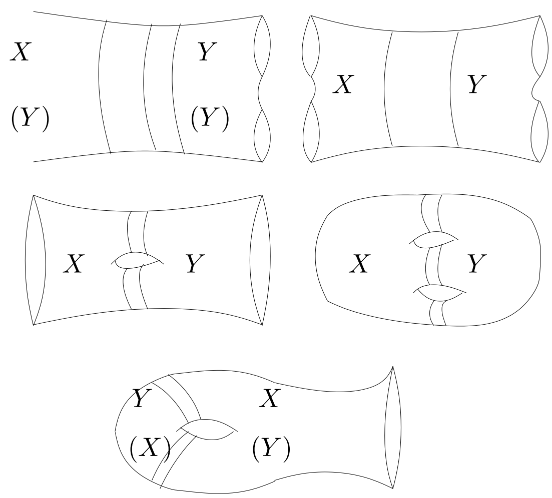

Figure 18. The seven topological types of ordered pairs of disjoint domains (X, Y) representing ordered pairs of vertices (x, y) that satisfy the equation $\mathrm{St}(x, D(S)) = \mathrm{St}(y, D(S))$ with $x \neq y$. See Proposition 6.14.

Proof. We begin by proving that (1) implies (2). To this end, suppose that $\mathrm{St}(x, D(S)) = \mathrm{St}(y, D(S))$. Since $x \neq y$ and $\mathrm{St}(x, D(S)) \subset \mathrm{St}(y, D(S))$, it follows from Proposition 6.13 that x and y are represented by disjoint nonisotopic domains X and Y satisfying one of the five cases, (2a), (2b), (2c), (2d), or (2e), of Proposition 6.13.

Note that for such domains, X and Y, the ordered triple (S, X, Y) is uniquely determined up to isotopies on S. Since $x \neq y$ and $\mathrm{St}(y, D(S)) \subset \mathrm{St}(x, D(S))$, it follows that (Y, X) satisfies one of the five cases obtained by interchanging the roles of X and Y in Proposition 6.13.

Hence, X and Y are each either an annulus or a pair of pants. Moreover, if either is a pair of pants, all of its essential boundary components are joined to essential boundary components of the other by annular codomains of $X \cup Y$ and if either is an annulus, one or both of its essential boundary components is joined to the other by annular codomains of $X \cup Y$. Since X is not isotopic to Y on S and X and Y are joined by at least one annular codomain of $X \cup Y$ on S, it follows that either X is a pair of pants on S or Y is a pair of pants on S.

If X and Y are both pairs of pants, it follows that they have the same number n of essential boundary components, with $1 \leq n \leq 3$. Hence, X and Y satisfy Case (2c), when $n = 1$; Case (2d), when $n = 2$; and Case (2e), when $n = 3$.

If X is a pair of pants and Y is an annulus, it follows that X is either biperipheral when X is joined to Y by exactly one annular codomain of $X \cup Y$ on S or monoperipheral when X is joined to Y by exactly two annular codomains of $X \cup Y$ on S. Hence, X and Y satisfy Case (2a), when X is biperipheral; and Case (2f), when X is monoperipheral.

Likewise, if X is an annulus and Y is a pair of pants, then X and Y satisfy Case (2b) or Case (2g), according as Y is either biperipheral or monoperipheral.

This proves that (1) implies (2).

We now prove that (2) implies (1).

To this end, suppose that X and Y are disjoint domains as in (2). We must prove that $\mathrm{St}(x, D(S)) = \mathrm{St}(y, D(S))$. Since $D(S)$ is a flag complex, it follows from Proposition 3.9 that it suffices to prove that $\mathrm{St}_0(x, D(S)) = \mathrm{St}_0(y, D(S))$.

We shall give the arguments for Cases (2a) and (2c). The argument for Case (2b) is similar to that for Case (2a). The argument for each of Cases (2d), (2e), (2f), and (2g) is similar to that for Case (2c).

First, consider Case (2a). Let X and Y be as in this case.

Suppose, on the one hand, that w is a vertex of $\mathrm{St}(x, D(S))$. In other words, suppose that $\{x, w\}$ is a simplex of $D(S)$. Then either $w = x$ or $\{x, w\}$ is an edge of $D(S)$. If $w = x$, then $\{y, w\}$ is the simplex $\{x, y\}$ of $D(S)$. Suppose that $\{x, w\}$ is an edge of $D(S)$. Then w is represented by a domain W on S which is disjoint from X. Since W is a domain on S disjoint from X and Y is an annulus on S which is joined to X along the unique essential boundary component of X by the unique annular codomain of $X \cup Y$ on S, it follows that Y is isotopic on S to a domain which is disjoint from W. It follows, in any case, that $\{y, w\}$ is a simplex of $D(S)$; that is to say, w is a vertex of $\mathrm{St}(y, D(S))$.

Suppose, on the other hand, that w is a vertex of $\mathrm{St}(x, D(S))$. In other words, suppose that $\{y, w\}$ is a simplex of $D(S)$. Then either $w = y$ or $\{y, w\}$ is an edge of $D(S)$. If $w = y$, then $\{x, w\}$ is the simplex $\{x, y\}$ of $D(S)$. Suppose that $\{y, w\}$ is an edge of $D(S)$. Then w is represented by a domain

W on S which is disjoint from Y. Since W is a domain on S disjoint from Y and Y is an annulus on S which is joined to X along the unique essential boundary component of X by the unique annular codomain of $X \cup Y$ on S, it follows that W is isotopic on S to a domain on S which is contained in either X or the complement of X. If W is contained in the complement of X, then $\{x, w\}$ is a simplex of $D(S)$. Suppose that W is contained in X. Since W is a domain on S which is contained in the biperipheral pair of pants X on S, either W is isotopic to X on S or W is isotopic to a regular neighborhood of the unique essential boundary component of X on S and, hence, to Y. Hence, $\{x, w\}$ is equal to either the simplex $\{x\}$ of $D(S)$ or the simplex $\{x, y\}$ of $D(S)$. It follows, in any case, that $\{x, w\}$ is a simplex of $D(S)$; that is to say, w is a vertex of $\mathrm{St}(x, D(S))$.

This proves that $\mathrm{St}_0(x, D(S)) = \mathrm{St}_0(y, D(S))$. This completes the argument for Case (2a).

Now consider Case (2c). Let X and Y be as in this case. Let Z be the unique codomain of $X \cup Y$ on S. Note that Z is an annulus on S joining the unique essential boundary component of X on S to the unique essential boundary component of Y on S. Hence, $S = X \cup Z \cup Y$.

Since X and Y are biperipheral pairs of pants on S, it follows that $X \cup Z$ and $Y \cup Z$ are biperipheral pairs of pants on S which are isotopic to X and Y on S and Z is isotopic on S to regular neighborhoods on S of the unique essential boundary components of each of X and Y.

Suppose, on the one hand, that w is a vertex of $\mathrm{St}(x, D(S))$. In other words, suppose that $\{x, w\}$ is a simplex of $D(S)$. Then either $w = x$ or $\{x, w\}$ is an edge of $D(S)$. If $w = x$, then $\{y, w\}$ is the simplex $\{x, y\}$ of $D(S)$. Suppose that $\{x, w\}$ is an edge of $D(S)$. Then w is represented by a domain W on S which is disjoint from X. Since W is a domain on S disjoint from X, it follows that W is a domain on S contained in $Y \cup Z$. Since $Y \cup Z$ is isotopic on S to Y, we may assume that W is contained in Y. Since Y is a biperipheral pair of pants on S, it follows that W is isotopic on S to either Y or the unique essential boundary component of Y on S and, hence, to Z. In other words, either $w = y$ or $w = z$. Hence, $\{y, w\}$ is equal to either the simplex $\{y\}$ of $D(S)$ or the simplex $\{y, x\}$ of $D(S)$. This shows, in any case, that $\{y, w\}$ is a simplex of $D(S)$; that is to say, w is a vertex of $\mathrm{St}(y, D(S))$.

This proves that $\mathrm{St}_0(x, D(S)) \subset \mathrm{St}_0(y, D(S))$. By interchanging the roles of X and Y, it follows that $\mathrm{St}_0(y, D(S)) \subset \mathrm{St}_0(x, D(S))$. Hence, $\mathrm{St}_0(x, D(S)) = \mathrm{St}_0(y, D(S))$. This completes the argument for Case (2c).

Since, as indicated above, the remaining cases follow by similar arguments, it follows that (2) implies (1).

$\square$

Proposition 6.15. *Suppose that S is not a sphere with four holes. Let $\{x, y\}$ be a pair of distinct vertices of $D(S)$. Let $\varphi \in \mathrm{Aut}(D(S))$. Then $\{x, y\}$ is a biperipheral edge if and only if $\{\varphi(x), \varphi(y)\}$ is a biperipheral edge.*

Proof. Suppose, on the one hand, that $\{x, y\}$ is a biperipheral edge of $D(S)$. (Note that since $\{x, y\}$ is a biperipheral edge of $D(S)$, S has at least two boundary components.)

We may assume that x and y are represented by disjoint domains X and Y on S satisfying case (2a) of Proposition 6.14. Let A be the unique codomain of $X \cup Y$ which joins X to Y. Then $X \cup A \cup Y$ is a biperipheral pair of pants P on S which is isotopic on S to X. Since S is not a sphere with four holes, the unique codomain Q of P on S is nonelementary. Since P represents the vertex x of $D(S)$, it follows that $\mathrm{Lk}(x, D(S))$ has infinitely many vertices.

By Proposition 6.14, $\mathrm{St}(x, D(S)) = \mathrm{St}(y, D(S))$. Since $\varphi : D(S) \to D(S)$ is an automorphism of $D(S)$, it follows that $\{\varphi(x), \varphi(y)\}$ is an edge of $D(S)$, $\mathrm{Lk}(\varphi(x), D(S))$ has infinitely many vertices, and $\mathrm{St}(\varphi(x), D(S)) = \mathrm{St}(\varphi(y), D(S))$.

Since $\varphi(x) \neq \varphi(y)$ and $\mathrm{St}(\varphi(x), D(S)) = \mathrm{St}(\varphi(y), D(S))$, it follows from Proposition 6.14 that x and y are represented by disjoint domains X' and Y' satisfying one of the seven cases of Proposition 6.14.

Suppose that $\{\varphi(x), \varphi(y)\}$ is not a biperipheral edge of $D(S)$. Then X' and Y' satisfy one of cases (2c), (2d), (2e), (2f), or (2g) of Proposition 6.14. Note that in any case, since X' represents the vertex $\varphi(x)$ of $D(S)$, it follows that $\mathrm{Lk}(\varphi(x), D(S))$ has at most four vertices, which is a contradiction. (In fact, $\mathrm{Lk}(\varphi(x), D(S))$ has at most three vertices.) Hence, $\{\varphi(x), \varphi(y)\}$ is a biperipheral edge of $D(S)$.

Suppose, on the other hand, that $\{\varphi(x), \varphi(y)\}$ is a biperipheral edge of $D(S)$. Then, since $\varphi^{-1} : D(S) \to D(S)$ is an automorphism of $D(S)$, it follows from the above argument, that $\{x, y\}$ is a biperipheral edge of $D(S)$.

This completes the proof.

□

Remark 6.16. If S is a sphere with at most three holes, then $D(S)$, $D^2(S)$, and $C(S)$ are empty. Hence, Proposition 7.3 is vacuously true.

Proposition 6.17. *Suppose that S is a sphere with four holes. Let x and y be vertices of $D(S)$ and $\mathrm{Ann}(x)$ and $\mathrm{Ann}(y)$ be their annular links in $D(S)$. Then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ if and only if one of the following holds:*

(1) $x = y$

(2) *x and y are annular vertices of $D(S)$*

(3) *x and y are represented by the two pairs of pants of some pants decomposition of S.*

Proof. Note that there are two types of vertices of $D(S)$; those which are represented by a biperipheral pair of pants on S, and those which are represented by a regular neighborhood of a biperipheral curve on S. Let $x \in D_0(S)$ and X be a domain on S representing x. If X is a biperipheral pair of pants on S, then $\mathrm{Ann}(x)$ is the vertex of $D(S)$ represented by a regular neighborhood of the unique essential boundary component of X. If X is an annulus on S, then $\mathrm{Ann}(x) = \emptyset$. This implies that $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ in each of the three cases: (1), (2), and (3).

Conversely, suppose that $\mathrm{Ann}(x) = \mathrm{Ann}(y)$. Let X and Y be domains on S representing x and y. We may assume that X is a biperipheral pair of pants on S. It follows that $\mathrm{Ann}(y) = \mathrm{Ann}(x) = \{w\}$, where w is represented by a regular neighborhood of the unique essential boundary component ∂ of X. Since $\mathrm{Ann}(y)$ is nonempty, it follows that Y is also a biperipheral pair of pants on S. Since $\mathrm{Ann}(y) = \{w\}$, it follows that the unique essential boundary component ϵ of Y is isotopic to ∂. Hence, we may assume that X and Y are joined on S by an annulus A on S with boundary components ∂ and ϵ. SInce X and Y are biperipheral pairs of pants on S, it follows that $S = X \cup A \cup Y$. Hence, X and Y are pairs of pants of a pants decomposition of S. □

Remark 6.18. If S is a closed torus, then $C(S)$ is a countably infinite set of vertices. Each vertex of $D(S)$ is an annular vertex of $D(S)$ and, hence, $D^2(S)$ is equal to the image of $C(S)$ in $D(S)$ under the natural inclusion $\eta : C(S) \hookrightarrow D(S)$, and $D(S) = D^2(S)$. Moreover, each vertex of $D(S)$ has empty annular link. Hence, Proposition 7.3 is false.

Proposition 6.19. *Suppose that S is a torus with one hole. Let x and y be vertices of $D(S)$ and $\mathrm{Ann}(x)$ and $\mathrm{Ann}(y)$ be their annular links in $D(S)$. Then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ if and only if one of the following holds:*

(1) $x = y$

(2) *x and y are annular vertices of $D(S)$*

Proof. The proof follows along the same lines as the proof of Proposition 6.17. In this case the two types of vertices of $D(S)$ are those which are represented by monoperipheral pairs of pants on S and those which are represented by annuli on S. Each monoperipheral pair of pants on S has two essential boundary components joined by an annulus on S. The corresponding vertex of $D(S)$ has the vertex of $D(S)$ represented by this annulus as its annular link. The annular vertices of $D(S)$ have empty annular links. □

Proposition 6.20. *Suppose that S is a torus with two holes. Let x and y be vertices of $D(S)$ and $\mathrm{Ann}(x)$ and $\mathrm{Ann}(y)$ be their annular links in $D(S)$. Then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ if and only if one of the following holds:*

(1) $x = y$

(2) *x and y are represented by the two pairs of pants of an embedded pants decomposition of S.*

Proof. Again, the proof follows along the same lines as the proof of Proposition 6.17. In this case, there are four types of vertices of $D(S)$: those which are represented by annuli on S; those which are represented by monoperipheral pairs of pants on S; those which are represented by biperipheral pairs of pants on S; and those which are represented by tori with one hole on S.

Suppose that X is an annulus. Then $\mathrm{Ann}(x)$ is an infinite discrete set of vertices, corresponding to the complex of curves of a four holed sphere if X is a regular neighborhood of a nonseparating curve on S and to the complex of curves of a one-holed torus if X is a regular neighborhood of an essential separating curve on S.

Suppose that X is a monoperipheral pair of pants on S. Then X has a unique pair of essential boundary component α and β on S and a unique codomain Y on S. Moreover, Y is a monoperipheral pair of pants on S. Let u and v be the annular vertices of $D(S)$ corresponding to regular neighborhoods of α and β on S. Then $\mathrm{Ann}(x)$ and $\mathrm{Ann}(y)$ are both equal to the edge $\{u, v\}$ of $D(S)$.

Suppose that X is a biperipheral pair of pants on S. Then X has a unique essential boundary component ∂ and a unique codomain Y on S. Moreover Y is a one-holed torus on S. Let w be the annular vertex of $D(S)$ corresponding to a regular neighborhood of ∂ on S. Then $\mathrm{Ann}(x)$ is the join of w to the infinitely many annular vertices of $D(S)$ corresponding to $C(Y)$. In particular, $\mathrm{Ann}(x)$ is an infinite connected subcomplex of $D(S)$.

Suppose that X is a torus with one hole. Then X has a unique essential boundary component ∂ and a unique codomain Y on S. Moreover, Y is a biperipheral pair of pants on S. Let y be the nonannular vertex of $D(S)$ corresponding to Y and w be the annular vertex of $D(S)$ corresponding to a regular neighborhood W of ∂ on S. Then $\mathrm{Ann}(x) = \{w\}$ and $\mathrm{Lk}(x) = \{y, w\}$.

The result now follows by using the above descriptions of the annular links of the four types of vertices of $D(S)$ to compare $\mathrm{Ann}(x)$ with $\mathrm{Ann}(y)$.

□

Proposition 6.21. *Suppose that S is a closed surface of genus two. Let x and y be vertices of $D(S)$ and $\mathrm{Ann}(x)$ and $\mathrm{Ann}(y)$ be their annular links in $D(S)$. Then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ if and only if one of the following holds:*

(1) $x = y$

(2) *x and y are represented by the two pairs of pants of some embedded pants decomposition of S.*

Proof. Again, the proof follows along the same lines as the proof of Proposition 6.17. ☐

7 Automorphisms of the truncated complex of domains

7.1 Distinguishing vertices of $D(S)$ via their annular links

Definition 7.1. Let x be a vertex of $D(S)$. The *annular link of x in $D(S)$* is the subcomplex $\mathrm{Ann}(x)$ of $D(S)$ consisting of those simplices of $\mathrm{Lk}(x, D(S))$ all of whose vertices are annular.

Proposition 7.2. *Suppose that S is neither a sphere with four holes nor a torus with at most one hole. Let x and y be vertices of $D(S)$. Suppose that x is annular. Then the following are equivalent:*

(1) $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$

(2) *$x = y$ or there exist disjoint domains X and Y on S representing x and y such that X is an annulus on S, Y is a biperipheral pair of pants on S, and $X \cup Y$ has exactly two codomains, exactly one of which is an annulus joining X to Y.*

Proof. We begin by proving that (1) implies (2). Suppose that $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$.

Since x is annular, x is represented by a regular neighborhood X of an essential curve α on X.

Since S has an essential curve α, S is not a sphere with at most three holes. Since S is also not a closed torus, there exists a pants decomposition $\mathcal{C}$ of S containing α. Let R be a regular neighborhood of the support $|\mathcal{C}|$ of $\mathcal{C}$ on S and $\mathcal{P}$ be the collection of codomains of R on S. We may assume that X is the unique component of R which contains the element α of $\mathcal{C}$.

Note that each element of $\mathcal{P}$ is a pair of pants on S.

Let β be an element of $\mathcal{C}$ which is not equal to α. Then a regular neighborhood Z of β on S represents a vertex z of $\mathrm{Ann}(x)$ and, hence, of $\mathrm{Ann}(y)$. It follows that y is represented by a domain Y on S which is disjoint from and not isotopic to each of the components of a regular neighborhood W of the union of all the elements of $\mathcal{C}$ which are not equal to α. Since Y is connected, Y is contained in a codomain of W on S.

Note that the unique codomain V of W on S which contains X is equal to the union of X with those elements of $\mathcal{P}$ which share at least one common essential boundary component with X. If there is exactly one such element of $\mathcal{P}$, then W is a torus with one hole. Otherwise, there are exactly two such elements of $\mathcal{P}$ and W is a sphere with four holes.

Every other codomain U of W on S is a pair of pants on S all of whose essential boundary components are isotopic to elements of $\mathcal{C}$ which are not equal to α.

Suppose that Y is contained in one of these other codomains U of W on S. Since U is a pair of pants and Y is a domain on S contained in U, Y is isotopic on S to a domain Y_1 on S such that Y_1 is equal to U or Y_1 is a regular neighborhood of an essential boundary component of U on S. Note, in any case, that an essential boundary component δ of U is contained in Y_1.

By assumption, δ is isotopic to an element β of $\mathcal{C}$ which is not equal to α. Since α and β are distinct elements of the pants decomposition $\mathcal{C}$, α and β are disjoint nonisotopic essential curves on S. It follows from Proposition 2.3 that there exists a curve γ on S such that $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) \neq 0$. Since δ is isotopic to β, $i(\gamma, \delta) = i(\gamma, \beta)$. Hence, $i(\gamma, \delta) \neq 0$.

It follows that a regular neighborhood Z of γ on S represents a vertex z of $\mathrm{Ann}(x)$ and, hence, of $\mathrm{Ann}(y)$. Since Y_1 represents the vertex y of $D(S)$ and Z represents the vertex z of $D(S)$, it follows that Z is isotopic on S to a domain Z_1 on S which is disjoint from Y_1. Thus, γ is isotopic on S to a curve γ_1 on S which is disjoint from δ. Thus, $i(\gamma, \delta) = i(\gamma_1, \delta) = 0$, which is a contradiction.

Hence, Y is not contained in one of these other codomains U of W on S. It follows that Y is not isotopic on S to a domain which is contained in one of the other codomains U of W on S.

It follows that Y is contained in the unique codomain V of W on S which contains X.

Since S is not a sphere with four holes nor a torus with one hole, V has an essential boundary component δ on S. Note that δ is isotopic on S to an element β of $\mathcal{C}$ which is not equal to α.

As before, it follows from Proposition 2.3 that there exists a curve γ on S such that $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) \neq 0$. Since $i(\gamma, \alpha) = 0$, we may assume that γ is disjoint from α. Since δ is isotopic to β, $i(\gamma, \delta) = i(\gamma, \beta)$. Hence, $i(\gamma, \delta) \neq 0$.

It follows that a regular neighborhood Z of γ on S represents a vertex z of $\mathrm{Ann}(x)$ and, hence, of $\mathrm{Ann}(y)$. Hence, Y is isotopic on X to a domain Y_1 on X which is disjoint from Z and, hence, from γ. Note that X and Y_1 are both domains on S which are contained in V and are disjoint from γ.

We may assume that the number of points of intersection of γ with each essential boundary component ϵ of V is equal $i(\gamma, \epsilon)$. Since $i(\gamma, \delta) \neq 0$, it follows that $\gamma \cap V$ is a nonempty disjoint union of properly embedded essential arcs on V.

Suppose, on the one hand, that V is a torus with one hole. Then, since X and Y_1 are both domains on S contained in V and disjoint from a properly embedded essential arc on V, it follows that X and Y_1 are isotopic annuli on S and, hence, $x = y$.

Suppose, on the other hand, that V is a sphere with four holes. Then, since X and Y_1 are both domains on S contained in V and disjoint from a properly embedded essential arc on V, it follows that Y_1 is isotopic to a domain Y_2 on S which is contained in one of the two elements P of $\mathcal{P}$ which share an essential boundary component with the annulus X.

Suppose that Y_2 is isotopic to a regular neighborhood of an essential boundary component σ of P. Note that σ is isotopic to an element β of $\mathcal{P}$. Since Y_2 is not isotopic to a regular neighborhood of any element of $\mathcal{C}$ which is not equal to α, it follows that β is equal to α. This implies that X is isotopic to Y_2 on S and, hence, $x = y$.

Hence, we may assume that Y_2 is not isotopic to a regular neighborhood of any essential boundary component of P. Since Y_2 is a domain on S contained in the pair of pants P, it follows that Y_2 is isotopic to P on S.

Suppose that there exists an essential boundary component τ of P such that τ is not isotopic to α on S. Then τ is isotopic to an element β of $\mathcal{C}$ which is not equal to α.

As before, it follows from Proposition 2.3 that there exists a curve γ on S such that $i(\gamma, \alpha) = 0$ and $i(\gamma, \beta) \neq 0$.

It follows that a regular neighborhood Z of γ on S represents a vertex z of $\mathrm{Ann}(x)$ and, hence, of $\mathrm{Ann}(y)$.

Since P represents y, it follows that Z is isotopic to a domain on S which is disjoint from P. Hence, γ is isotopic to a curve γ_1 on S which is disjoint from τ.

This implies that $i(\gamma, \beta) = i(\gamma_1, \tau) = 0$, which is a contradiction.

Hence, each essential boundary component of P is isotopic to α on S.

It follows that P is either a monoperiperipheral pair of pants sharing both of its essential boundary components with X or a biperipheral pair of pants sharing its unique essential boundary component with X. In the former case, it follows that S is a torus with one hole, which is a contradiction. Hence, the latter case holds.

Since Y_2 is a nonannular domain on S contained in the biperipheral pair of pants P on S, it follows that Y_2 is a biperipheral pair of pants on S whose unique codomain on P is an annulus on S joining X to Y_2.

This completes the proof that (1) implies (2). It remains to prove that (2) implies (1).

If $x = y$, then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and, hence, $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$.

Suppose that x and y are represented by disjoint domains X and Y on S such that X is an annulus on S, Y is a biperipheral pair of pants on S, and $X \cup Y$ has exactly two codomains, exactly one of which is an annulus joining X to Y.

Let P be the unique codomain of X on S such that Y is contained in P. Note that P is a biperipheral pair of pants on S.

Suppose that z is an element of $\mathrm{Ann}(x)$. Then z is represented by an annulus Z on S which is disjoint from and not isotopic to X.

Since Z is connected and disjoint from X, Z is contained in a codomain Q of X on S.

Suppose that Q is equal to P. Then since Z is an annular domain on S and P is a biperipheral pair of pants on S, it follows that Z is isotopic to a regular neighborhood of the unique essential boundary component of P on S. Since every essential boundary component of a codomain of X on S is an essential boundary component of the annulus X, it follows that Z is isotopic to X on S, which is a contradiction. Hence, Q is not equal to P.

Since any two distinct codomains of X on S are disjoint, it follows that Z is disjoint from P and, hence, from Y. Note that the annulus Z is not isotopic on S to the pair of pants Y. Hence, the vertex z of $D(S)$ represented by the annulus Z is an element of $\mathrm{Ann}(y)$.

This proves that (2) implies (1), completing the proof.

$\square$

Proposition 7.3. *Suppose that S is neither a sphere with four holes nor a torus with at most one hole. Let x and y be vertices of $D(S)$. Suppose that x is annular. Then* $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ *if and only if $x = y$.*

Proof. It suffices to prove that $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ implies $x = y$.

To this end, suppose that $\mathrm{Ann}(x) = \mathrm{Ann}(y)$. Then x is annular and $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$. It follows from Proposition 7.2 that either (i) $x = y$ or (ii) x and y are represented by an annulus X on S and a biperipheral pair of pants Y on S such that $X \cup Y$ has exactly two codomains, exactly one of which is an annulus joining X to Y.

Suppose that (ii) holds. Then x is a vertex of $\mathrm{Ann}(y)$; that is to say, since $\mathrm{Ann}(x) = \mathrm{Ann}(y)$, x is a vertex of $\mathrm{Ann}(x)$. Since $\mathrm{Ann}(x)$ is a subcomplex of $\mathrm{Lk}(x, D(S))$, it follows that x is a vertex of $\mathrm{Lk}(x, D(S))$, which is a contradiction. Hence, (ii) does not hold.

It follows that (i) holds; that is to say, it follows that $x = y$, completing the proof.

$\square$

Proposition 7.4. *Let x and y be vertices of $D(S)$. Suppose that $\{x, y\}$ is a simplex of $D(S)$. Then* $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$ *if and only if either $x = y$ or x and y are represented by disjoint domains X and Y on S such that Y is a pair of pants with each of its essential boundary components on S joined to X by annuli.*

Proof. Suppose, on the one hand, that $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$.

We may assume that x is not equal to y. Then, since $\{x, y\}$ is a simplex of $D(S)$, $\{x, y\}$ is an edge of $D(S)$. It follows that x and y are represented by disjoint domains X and Y on S which are not isotopic to one another on S.

Suppose that Y is a nonelementary domain on S. It follows from Proposition 2.18 that there exist curves α and β on S such that $i(\alpha, \beta) \neq 0$ and α and β are contained in the interior of Y.

Let U and V be regular neighborhoods of α and β in the interior of Y. Suppose that V is isotopic to X on S. Then β is isotopic on S to a curve β_1 which is contained in the interior of X and, hence, is disjoint from Y. Since β is isotopic to β_1 on S, $i(\alpha, \beta) = i(\alpha, \beta_1)$. Since α is contained in Y and β_1 is disjoint from Y, it follows that α and β_1 are disjoint and, hence, $i(\alpha, \beta_1) = 0$. We conclude that $i(\alpha, \beta) = 0$ which is a contradiction.

ence, V is not isotopic to X on S. Since V is contained in Y and X and Y are disjoint, X and V are disjoint domains on S. It follows that V represents an annular vertex v of $\mathrm{Lk}(x, D(S))$. This implies that v is a vertex of $\mathrm{Ann}(x)$ and, hence, of $\mathrm{Ann}(y)$.

Since V represents v and Y represents y, it follows that V is isotopic on S to a domain on S which is disjoint from Y. Since β is contained in V, it follows that β is isotopic on S to a curve β_2 which is disjoint from Y. Again, this implies that $i(\alpha, \beta) = i(\alpha, \beta_2) = 0$, which is a contradiction.

It follows that Y is an elementary domain on S.

Suppose that Y is an annulus. Since X and Y are disjoint nonisotopic domains on S, it follows that the annular vertex y of $D(S)$ represented by Y is a vertex of $\mathrm{Ann}(x)$ and, hence of $\mathrm{Ann}(y)$. Since $\mathrm{Ann}(y)$ is a subcomplex of $\mathrm{Lk}(y, D(S))$, it follows that y is a vertex of $\mathrm{Lk}(y, D(S))$, which is a contradiction.

Hence, Y is not an annulus. Since Y is an elementary domain on S, it follows that Y is a pair of pants.

Let β be an essential boundary component of Y on S. Suppose that β is not isotopic to any essential boundary component of X on S. Then, by Proposition 2.3, there exists an essential curve γ on S such that $i(\gamma, \alpha) = 0$ for every essential boundary component α of X on S and $i(\gamma, \beta) \neq 0$.

Since Y is disjoint from X, it follows that a regular neighborhood W of γ on S represents a vertex w of $\mathrm{Ann}(x)$ and, hence, of $\mathrm{Ann}(y)$. It follows that γ is isotopic on S to a curve γ_1 on S which is disjoint from Y. It follows that $i(\gamma, \beta) = i(\gamma_1, \beta) = 0$, which is a contradiction.

Hence, the essential boundary component β of Y on S is isotopic on S to some essential boundary component α of X on S. Since X and Y are disjoint, it follows that there is an annulus A on S whose boundary components are α and β.

Since Y is a pair of pants, it follows that $A \cap Y = \beta$. Moreover, either $A \cap X = \alpha$ or $X \subset A$. In the former case, A is an annulus joining β to X.

Suppose $X \subset A$. Then X is an annulus contained in A. It follows that $A = X \cup B$, where B is an annulus joining β to X.

In any case, β is joined to X by an annulus.

This proves the "only if" direction. It remains to prove the "if" direction.

If $x = y$, then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$ and, hence, $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$.

Suppose that x and y are represented by disjoint domains X and Y on S such that Y is a pair of pants with each of its essential boundary components on S joined to X by annuli.

Since Y is disjoint from X on S and each of the essential boundary components of Y on S is joined to X by an annulus, it follows that Y is isotopic on S to the unique codomain Y_1 of X on S which contains Y.

Let z be an element of $\mathrm{Ann}(x)$. Then z is represented by an annulus Z on S which is disjoint from X and, hence, is contained in a codomain W of X on S.

Suppose that W is not equal to Y_1. Then Z is disjoint from Y_1 and, hence, from Y. Since Y is a pair of pants, it follows that the annulus Z is not isotopic to Y on S. Hence, the annular vertex z of $D(S)$ represented by Z is a vertex of $\mathrm{Ann}(y)$.

Suppose that W is equal to Y_1.

Since Z is an annular domain on S contained in the pair of pants Y_1 on S, it follows that Z is isotopic on S to an annulus Z_1 on S which is disjoint from Y_1 and, hence, from Y. Note that the annulus Z_1 is not isotopic to the pair of pants Y on S. Hence, the vertex z of $D(S)$ represented by Z_1 is a vertex of $\mathrm{Ann}(y)$.

In any case, z is an element of $\mathrm{Ann}(y)$.

Again, we conclude that $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$.

In any case, $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$.

This proves the "if" direction, completing the proof.

$\square$

Proposition 7.5. *Suppose that S is neither a sphere with four holes nor a torus with at most one hole. Let x and y be vertices of $D(S)$. Suppose that $\{x, y\}$ is not a simplex of $D(S)$. Then* $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$ *if and only if x and y are represented by domains X and Y on S such that Y is a domain on X.*

Proof. Since $\{x, y\}$ is not a simplex of $D(S)$, $x \neq y$.

Suppose that $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$.

Suppose that x is an annular vertex of $D(S)$. Since S is neither a sphere with four holes nor a torus with at most one hole and $x \neq y$, it follows from Proposition 7.2 that x and y are represented by disjoint domains X and Y on S. Hence, $\{x, y\}$ is a simplex of $D(S)$, which is a contradiction. Therefore, x is not an annular vertex of $D(S)$.

It follows that a regular neighborhood Z of any essential boundary component α of X represents a vertex z of $\mathrm{Ann}(x)$ and, hence, of $\mathrm{Ann}(y)$. It follows that y is represented by a domain Y on S which is not isotopic to a regular neighborhood of any essential boundary component of X on S and is disjoint from a regular neighborhood of the union of the essential boundary components of X on S.

Since Y is disjoint from a regular neighborhood of the union of the essential boundary components of X on S, either Y is disjoint from X or Y is contained in X. Since $\{x, y\}$ is not a simplex of $D(S)$, Y is not disjoint from X. Hence, Y is contained in X.

Since Y is a domain on S contained in the domain X on S, it follows from Proposition 2.13, that Y is isotopic on S to either X, or a domain on X, or a regular neighborhood of an essential boundary component of X.

Since $x \neq y$, Y is not isotopic on S to X. Since Y is not isotopic on S to a regular neighborhood of any essential boundary component of X on S, we conclude that Y is isotopic to a domain Y_1 on X.

Hence, x and y are represented by domains X and Y_1 on S such that Y_1 is a domain on X.

This proves the "only if" direction. It remains to prove the "if" direction.

Suppose that x and y are represented by domains X and Y on S such that Y is a domain on X.

Let z be a vertex of $\mathrm{Ann}(x)$. Then z is represented by an annulus Z on S which is disjoint from X. Since Y is contained in X, it follows that Z is disjoint from Y.

Suppose that Y is isotopic to Z on S. Then Y is an annulus on S. Hence, Y is a regular neighborhood of an essential curve α on S. Since Y is a domain on X, α is an essential curve on X.

It follows from Proposition 2.10, that there exists an essential curve β on X such that the geometric intersection number of α and β on S is not equal to zero.

Since Z is an annular domain on S, Z is a regular neighborhood of an essential curve γ on S. Since β is contained in X and Z is disjoint from X, it follows that β is disjoint from γ and, hence, $i(\gamma, \beta) = 0$. Since $i(\alpha, \beta) \neq 0$, it follows that α is not isotopic to γ on S. This implies that Y is not isotopic to Z. Since Z is an annulus disjoint from Y and not isotopic to Y on S, it follows that the vertex z of $D(S)$ represented by Z is a vertex of $\mathrm{Ann}(y)$.

This proves the "if" direction, completing the proof.

□

Proposition 7.6. *Suppose that S is neither a sphere with four holes nor a torus with at most one hole. Let x and y be vertices of $D(S)$. Then the following are equivalent:*

(1) $\mathrm{Ann}(x) = \mathrm{Ann}(y)$

(2) *$x = y$ or there exist disjoint domains X and Y on S, representing x and y, which belong to one of the following cases:*

((a) *S is a torus with two holes, X and Y are monoperipheral pairs of pants on S, and $X \cup Y$ has exactly two codomains, both of which are annuli joining X to Y.*

((b) *S is a closed surface of genus two, X and Y are pairs of pants on S, and $X \cup Y$ has exactly three codomains, all of which are annuli joining X to Y.*

Proof. We begin by proving the "only if" direction. Suppose that $\mathrm{Ann}(x) = \mathrm{Ann}(y)$.

If x is annular, then, since $\mathrm{Ann}(x) = \mathrm{Ann}(y)$, it follows from Proposition 7.3 that $x = y$. Likewise, if y is annular, then, since $\mathrm{Ann}(y) = \mathrm{Ann}(x)$, it follows from Proposition 7.3 that $y = x$. Hence, if either x or y is annular, then $x = y$.

Thus, we may assume that neither x nor y is annular.

We may assume that $x \neq y$.

Suppose that $\{x, y\}$ is not a simplex. Then, since $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$, it follows from Proposition 7.5, that x and y are represented by domains X and Y on S such that Y is a domain on X. Likewise, since $\mathrm{Ann}(y) \subset \mathrm{Ann}(x)$, it follows from Proposition 7.5, that y and x are represented by domains Y_1 and X_1 on S such that Y_1 is a domain on X_1. Thus X is isotopic to a domain on Y and Y is isotopic to a domain on X, which contradicts Proposition 2.11.

Hence, $\{x, y\}$ is a simplex of $D(S)$. Since $\mathrm{Ann}(x) \subset \mathrm{Ann}(y)$ and $x \neq y$, it follows from Proposition 7.4 that x and y are represented by disjoint domains X and Y on S such that Y is a pair of pants with each of its essential boundary components on S joined to X by annuli. This implies that the number of essential boundary components of Y on S is less than or equal to the number of essential boundary components of X on S.

Likewise, since $\mathrm{Ann}(y) \subset \mathrm{Ann}(x)$ and $y \neq x$, it follows from Proposition 7.4 that y and x are represented by disjoint domains Y_1 and X_1 on S such that X_1 is a pair of pants with each of its essential boundary components on S joined to Y_1 by annuli. Again, this implies that the number of essential boundary components of X_1 on S is less than or equal to the number of essential boundary components of Y_1 on S.

Since X and X_1 both represent x, X is isotopic to the pair of pants X_1 on S. This implies that X is a pair of pants on S with the same number of essential boundary components on S as X_1. Likwise, Y_1 is a pair of pants on S with the same number of essential boundary components on S as Y. Since the number of essential boundary components of X_1 on S is less than or equal to the number of essential boundary components of Y_1 on S, it follows that the number of essential boundary components of X on S is less than or equal to the number of essential boundary components of Y on S. Since the number of essential boundary components of Y on S is less than or equal to the number of essential boundary components of X on S, we conclude that X is a pair of pants on S with the same number of essential boundary components on S as the pair of pants Y on S.

Thus, X and Y are disjoint pairs of pants on S with the same number n of essential boundary components on S and the n essential boundary components of X on S are joined by disjoint annuli to the n essential boundary components of Y on S.

Since X is a pair of pants domain on S, $1 \leq n \leq 3$. If $n = 1$, then S is a sphere with four holes, which is a contradiction. Hence, $2 \leq n \leq 3$. If $n = 2$, then X and Y satisfy case (2a). If $n = 3$, then X and Y satisfy case (2b).

This completes the proof of the "only if" direction. It remains to prove the "if" direction.

If $x = y$, then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$.

Suppose that X and Y are as in case (2a). Note that the two codomains of $X \cup Y$ on S are annuli which are disjoint from and not isotopic on S to the pairs of pants X and Y on S. Hence, they represent vertices of $\mathrm{Ann}(x)$ and $\mathrm{Ann}(y)$.

Suppose that z is a vertex of $\mathrm{Ann}(x)$. Then z is represented by an annulus on S which is contained in the unique codomain Y_1 of X on S. Note that Y_1 is a pair of pants on S which is isotopic to Y on S. It follows that Z is isotopic to a regular neighorhood of an essential boundary component α of Y_1 on S. Since Y_1 is a codomain of X on S, α is an essential boundary component of X on S. It follows that Z is isotopic to one of the two codomains of $X \cup Y$ on S. This proves that $\mathrm{Ann}(x)$ is the edge of $D(S)$ whose vertices are represented by the two codomains of $X \cup Y$ on S. Likewise, $\mathrm{Ann}(y)$ is equal to this edge and, hence, $\mathrm{Ann}(x) = \mathrm{Ann}(y)$.

Similarly, if X and Y are as in case (2b), then $\mathrm{Ann}(x) = \mathrm{Ann}(y)$.

In any case, $\mathrm{Ann}(x) = \mathrm{Ann}(y)$.

This proves the "if" direction, completing the proof.

□

7.2 Automorphisms of $D^2(S)$ are geometric

In this section, we prove that if S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two, then each automorphism of $D^2(S)$ is induced by a self-homeomorphism of S which is uniquely defined up to isotopy on S. This will imply that, under the same hypothesis on S, $Aut(D^2(S)) \simeq \Gamma^*(S)$ and, if $b \leq 1$, $Aut(D(S)) \simeq \Gamma^*(S)$.

Lemma 7.7. *Suppose that S is not a torus with one hole. Let $i : C(S) \to D^2(S)$ be the natural inclusion corresponding to forming regular neighborhoods of essential curves on S. Let $\varphi : D^2(S) \to D^2(S)$ be an automorphism of*

$D^2(S)$. Then there exists an automorphism $\tau : C(S) \to C(S)$ such that $\varphi \circ i = i \circ \tau$.

Proof. Let a be a vertex of $C(S)$, $x = i(a)$, and $u = \varphi(x)$. Note that x is an annular vertex of $D^2(S)$. Since $\varphi \in \mathrm{Aut}(D^2(S))$, it follows by Corollary 6.11, that u is an annular vertex of $D^2(S)$. Hence, there exists a vertex b of $C(S)$ such that $i(b) = u$. Since $i : C(S) \to D(S)$ is injective, such a vertex is unique. It follows that the correspondence $a \mapsto b$ yields a well-defined function $\tau : C_0(S) \to C_0(S)$ such that $\varphi(i(a)) = i(\tau(a))$ for every vertex a of $C(S)$. Since curves on S are disjoint if and only if they have disjoint regular neighborhoods, it follows that $\tau : C_0(S) \to C_0(S)$ extends to a simplicial map $\tau : C(S) \to C(S)$. Since $\varphi(i(a)) = i(\tau(a))$ for every vertex a of $C(S)$, it follows that $\varphi \circ i = i \circ \tau : C(S) \to D^2(S)$. This shows that there exists a simplicial map $\tau : C(S) \to C(S)$ such that $\varphi \circ i = i \circ \tau : C(S) \to D^2(S)$. Likewise, there exists a simplicial map $\sigma : C(S) \to C(S)$ such that $\varphi^{-1} \circ i = i \circ \sigma : C(S) \to D^2(S)$. Since i is injective, it follows that σ is an inverse for τ. Hence, $\tau : C(S) \to C(S)$ is an automorphism of $C(S)$. ☐

Theorem 7.8. *Suppose that S is not a sphere with four holes, a torus with at most two holes, or a closed surface of genus two. Then the natural homomorphism $\rho : \Gamma^*(S) \to \mathrm{Aut}(D^2(S))$ corresponding to the action of $\Gamma^*(S)$ on $D^2(S)$ is an isomorphism.*

Proof. We begin by showing that ρ is surjective. To this end, we let $\varphi \in \mathrm{Aut}(D^2(S))$ and show that there exists a homeomorphism $H : S \to S$ such that $\varphi = H_* : D^2(S) \to D^2(S)$.

To simplify the exposition, we identify $C(S)$, via $i : C(S) \to D(S)$, with its image in $D^2(S)$ under $i : C(S) \to D(S)$. Since S is not a torus with one hole, using this identification, we may restate Lemma 7.7 as saying that φ restricts to an element τ of $Aut(C(S))$.

Since S is neither a sphere with at most four holes nor a torus with at most two holes, it follows from Theorem 1 of Ivanov [25] and Theorem 1 of Korkmaz [35] (see Luo [38] for a different proof) that there exists a homeomorphism $H : S \to S$ such that $\tau = H_* : C(S) \to C(S)$. Let $\psi = H_*^{-1} \circ \varphi : D^2(S) \to D^2(S)$. Note that ψ fixes every vertex of $C(S)$.

We shall now show that ψ is equal to the identity map of $D^2(S)$; that is to say, we shall show that $\varphi = H_* : D^2(S) \to D^2(S)$.

Let $v \in D^2(S)$. Since ψ is an automorphism of $D^2(S)$ preserving $C(S)$, $\psi(\mathrm{Ann}(v)) = \mathrm{Ann}(\psi(v))$. On the other hand, since $\mathrm{Ann}(v)$ is a subcomplex of $C(S)$ and ψ fixes each vertex of $C(S)$, $\psi(\mathrm{Ann}(v)) = \mathrm{Ann}(v)$. Hence, $\mathrm{Ann}(\psi(v)) = \mathrm{Ann}(v)$. Since S is not a sphere with four holes, a torus with at most two holes, or a closed surface of genus two, it follows from Proposition 7.6 that $\psi(v) = v$. This proves that $\varphi = H_* : D^2(S) \to D^2(S)$ and, hence, the natural homomorphism $\rho : \Gamma^*(S) \to D^2(S)$ is surjective.

It remains to show that $\rho : \Gamma^*(S) \to D^2(S)$ is injective. To this end, suppose that h is an element of the kernel of ρ. Let $H : S \to S$ be a homeomorphism representing h. Since $h \in ker(\rho)$, H induces the trivial automorphism of $D^2(S)$.

Let α be an essential curve on S and X be a regular neighborhood of α on S. It follows that $[X] = H_*[X] = [H(X)]$ and, hence, $H(X)$ is isotopic to X on S. This implies that $H(\alpha)$ is isotopic to α on S. Thus $H : S \to S$ preserves the isotopy class of every essential curve on S. In other words, h is in the kernel of the action of $\Gamma^*(S)$ on $D^2(S)$.

Since S is not a sphere with at most three holes, it follows from Proposition 2.6 that $H : S \to S$ is orientation-preserving. This implies that h is in the kernel of the action of $\Gamma(S)$ on $D^2(S)$. Since S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two, it follows from [31], Lemma 5.1 and Theorem 5.3, that h is equal to the identity element of $\Gamma^*(S)$.

This proves that $\rho : \Gamma^*(S) \to D^2(S)$ is injective, completing the proof. □

8 Automorphisms of the complex of domains

8.1 Exchange automorphisms of $D(S)$

Throughout the rest of this chapter, let $\mathcal{E}$ denote the set of biperipheral edges of $D(S)$.

Proposition 8.1. *Suppose that S is not a sphere with four holes. Then there exists a monomorphism $\Phi : \mathcal{B}(\mathcal{E}) \to \mathrm{Aut}(D(S))$ from the Boolean algebra $\mathcal{B}(\mathcal{E})$ of all subsets of $\mathcal{E}$ to Aut($D(S)$) such that for each collection $\mathcal{F}$ of biperipheral edges of $D(S)$, $\Phi(\mathcal{F}) = \varphi_{\mathcal{F}}$ exchanges the two vertices of each biperipheral edge in $\mathcal{F}$ and fixes every vertex of $D(S)$ which is not a vertex of some biperipheral edge in $\mathcal{F}$.*

Proof. It follows from Propositions 6.14 and 3.28 that $\mathcal{E}$ is a collection of exchangeable edges of $D(S)$. Since S is not a sphere with four holes, no two distinct edges in $\mathcal{E}$ have a common vertex. Hence, the result follows from Proposition 3.31. □

Following the language of Definition 3.32, we call the image of the Boolean algebra $\mathcal{B}(\mathcal{E})$ under the monomorphism Φ of Proposition 8.1 the *Boolean subgroup $B_{\mathcal{E}}$ of $D(S)$*. In particular, the Boolean subgroup $B_{\mathcal{E}}$ is naturally isomorphic to the Boolean algebra $\mathcal{B}(\mathcal{E})$.

Proposition 8.2. *Let $\varphi \in \mathrm{Aut}(D(S))$, $\mathcal{F} \subset \mathcal{E}$ and $\mathcal{G} = \varphi(\mathcal{F})$. Then $\mathcal{G} \subset \mathcal{E}$ and $\varphi \circ \Phi_{\mathcal{F}} \circ \varphi^{-1} = \Phi_{\mathcal{G}}$.*

Proof. This is an immediate consequence of Propositions 6.15 and 3.33. □

Proposition 8.3. *$B_{\mathcal{E}}$ is a normal subgroup of $Aut(D(S))$.*

Proof. This is an immediate consequence of Proposition 8.2. □

Proposition 8.4. *The monomorphism $\Phi : \mathcal{B}(\mathcal{E}) \to \mathrm{Aut}(D(S))$ is natural with respect to the action of the extended mapping class group $\Gamma^*(S)$ on $D(S)$. More precisely, if $h \in \Gamma^*(S)$ and $\mathcal{F} \subset \mathcal{E}$, then $\Phi(h_*(\mathcal{F})) = h_* \circ \Phi(\mathcal{F}) \circ h_*^{-1}$.*

Proof. This is an immediate consequence of Propositions 8.2 and 8.3. □

Proposition 8.5. *There is a natural monomorphism:*

$$\rho : \mathcal{B}(\mathcal{E}) \rtimes \Gamma^*(S) \longrightarrow \mathrm{Aut}(D(S))$$

corresponding to the action of $\Gamma^*(S)$ on $D(S)$ and the induced action on the set $\mathcal{E}$ of biperipheral edges of $D(S)$.

Proof. Since, by Proposition 8.4, the monomorphism $\Phi : \mathcal{B}(\mathcal{E}) \to B_{\mathcal{E}}$ is natural, there exists a natural homomorphism $\rho : \mathcal{B}(\mathcal{E}) \rtimes \Gamma^*(S) \longrightarrow \mathrm{Aut}(D(S))$. Since a pair of pants is not homeomorphic to an annulus, a geometric automorphism of $D(S)$ cannot exchange the vertices of any biperipheral edge of $D(S)$. It follows that the image of the extended mapping class group $\Gamma^*(S)$ in $Aut(D(S))$ under the natural homomorphism $\rho : \Gamma^*(S) \to \mathrm{Aut}(D(S))$ corresponding to the action of $\Gamma^*(S)$ on $D(S)$ has trivial intersection with the Boolean subgroup $B_{\mathcal{E}}$ of $Aut(D(S))$. Since the natural homomorphism $\Phi : \mathcal{B}(\mathcal{E}) \to B_{\mathcal{E}}$ is injective, it remains only to show that $\rho : \Gamma^*(S) \to \mathrm{Aut}(D(S))$ is injective. To this end, suppose that $h \in \Gamma^*(S)$ is in the kernel of ρ. Since $D^2(S)$ is a subcomplex of $D(S)$, it follows that h induces the trivial automorphism of $D^2(S)$. Since S is not a sphere with four holes, a torus with at most two holes, or a closed surface of genus two, it follows from Theorem 7.8 that h is equal to the identity element of $\Gamma^*(S)$. This proves that $\rho : \Gamma^*(S) \to \mathrm{Aut}(D(S))$ is injective, completing the proof. □

8.2 Automorphisms of $D(S)$

Throughout this section, let $\mathcal{E}$ denote the set of biperipheral edges of $D(S)$.

Proposition 8.6. *Suppose that S is not a sphere with four holes. Let $\pi : D(S) \to D^2(S)$ be the natural projection from $D(S)$ to $D^2(S)$ sending each* *vertex of $D(S)$ corresponding to a biperipheral pair of pants on S to the annular vertex of $D(S)$ corresponding to its unique essential boundary component on S. If $\varphi \in \mathrm{Aut}(D(S))$, then there exists a unique simplicial automorphism $\varphi_* : D^2(S) \to D^2(S)$ such that $\varphi_* \circ \pi = \pi \circ \varphi : D(S) \to D^2(S)$.*

Proof. Let $i : D^2(S) \to D^2(S)$ denote the inclusion map of the subcomplex $D^2(S)$ of $D(S)$ into $D(S)$ and $\varphi_* = \pi \circ \varphi \circ i : D^2(S) \to D^2(S)$. Note that $\varphi_* : D^2(S) \to D^2(S)$ is a simplicial map from $D^2(S)$ to $D^2(S)$.

We shall prove that $\varphi_* \circ \pi = \pi \circ \varphi : D(S) \to D^2(S)$. To this end, let x be a vertex of $D(S)$.

Suppose, on the one hand, that $x \in D^2(S)$. Then, by the definition of $\pi : D(S) \to D^2(S)$, $\pi(x) = x$ and, hence, $(\varphi_* \circ \pi)(x) = \varphi_*(\pi(x)) = \varphi_*(x) = (\pi \circ \varphi \circ i)(x) = \pi(\varphi(i(x))) = \pi(\varphi(x)) = (\pi \circ \varphi)(x)$.

Suppose, on the other hand, that x is not in $D^2(S)$. Since $x \in D(S)$ and x is not in $D^2(S)$, x is represented by a biperipheral pair of pants X on S. Let Y be a regular neighborhood of the unique essential boundary component of X on S and y be the vertex of $D(S)$ represented by Y. Then $\{x, y\}$ is a biperipheral edge of $D(S)$. It follows from the definition of $\pi : D(S) \to D(S)$, that $\pi(x) = y = \pi(y)$. Moreover, it follows from Proposition 6.15 that $\{\varphi(x), \varphi(y)\}$ is a biperipheral edge of $D(S)$. Hence, either $\varphi(x)$ is represented by a biperipheral pair of pants on S or $\varphi(y)$ is represented by a biperipheral pair of pants on S. In the former case,it follows from the definition of $\pi : D(S) \to D^2(S)$, that $\pi(\varphi(x)) = \varphi(y) = \pi(\varphi(y))$. In the latter case,it follows from the definition of $\pi : D(S) \to D^2(S)$, that $\pi(\varphi(x)) = \varphi(x) = \pi(\varphi(y))$. Hence, in any case, $\pi(\varphi(x)) = \pi(\varphi(y))$. It follows that $(\varphi_* \circ \pi)(x) = \varphi_*(\pi(x)) = \pi(\varphi(i(\pi(x))) = \pi(\varphi(\pi(x)) = \pi(\varphi(y)) = \pi(\varphi(x)) = (\pi \circ \varphi)(x)$.

This shows, in any case, that $(\varphi_* \circ \pi)(x) = (\pi \circ \varphi)(x)$ and, hence, $\varphi_* \circ \pi = \pi \circ \varphi : D(S) \to D^2(S)$.

Suppose that $\beta : D^2(S) \to D^2(S)$ is a simplicial map such that $\beta \circ \pi = \pi \circ \varphi : D(S) \to D^2(S)$. Then $\beta \circ \pi = \varphi_* \circ \pi : D(S) \to D^2(S)$. Since $\pi : D(S) \to D^2(S)$ is surjective, it follows that $\beta = \varphi_* : D(S) \to D^2(S)$. This proves that there exists a unique simplicial map $\varphi_* : D^2(S) \to D^2(S)$ such that $\varphi_* \circ \pi = \pi \circ \varphi : D(S) \to D^2(S)$.

It remains only to prove that $\varphi_* : D^2(S) \to D^2(S)$ is a simplicial automorphism of $D^2(S)$. To this end, consider the simplicial automorphism $\psi = \varphi^{-1} : D(S) \to D(S)$. of $D(S)$. Repeating the above argument, we conclude that there exists a unique simplicial map $\psi_* : D^2(S) \to D^2(S)$ such that $\psi_* \circ \pi = \pi \circ \psi : D(S) \to D^2(S)$.

It follows that $(\varphi_* \circ \psi_*) \circ \pi = \varphi_* \circ (\psi_* \circ \pi) = \varphi_* \circ (\pi \circ \psi) = (\varphi_* \circ \pi) \circ \psi = (\pi \circ \varphi) \circ \psi = \pi \circ (\varphi \circ \psi) = \pi : D(S) \to D^2(S)$. Since $\pi : D(S) \to D^2(S)$ is surjective, it follows that $\varphi_* \circ \psi_* : D^2(S) \to D^2(S)$ is the identity map of $D^2(S)$. Likewise, we conclude that $\psi_* \circ \varphi_* : D^2(S) \to D^2(S)$ is the identity map of $D^2(S)$. Hence, $\varphi_* : D^2(S) \to D^2(S)$ and $\psi_* : D^2(S) \to D^2(S)$ are inverse simplicial maps. This shows that $\varphi_* : D^2(S) \to D^2(S)$ is a simplicial automorphism of $D^2(S)$, completing the proof.

□

Proposition 8.7. *Suppose that S is not a sphere with four holes. Let $\pi : D(S) \to D^2(S)$ be the natural projection from $D(S)$ to $D^2(S)$ sending each vertex of $D(S)$ corresponding to a biperipheral pair of pants on S to the annular vertex of $D(S)$ corresponding to its unique essential boundary component on S. Then there exists a unique homomorphism $\rho : \mathrm{Aut}(D(S)) \to \mathrm{Aut}(D^2(S))$ such that for each automorphism $\varphi \in \mathrm{Aut}(D(S))$, $\rho(\varphi)$ is the unique simplicial automorphism $\varphi_* : D^2(S) \to D^2(S)$ such that $\varphi_* \circ \pi = \pi \circ \varphi : D(S) \to D^2(S)$. Moreover, there exists a natural exact sequence:*

$$1 \longrightarrow B_{\mathcal{E}} \longrightarrow \mathrm{Aut}(D(S)) \longrightarrow \mathrm{Aut}(D^2(S)).$$

Proof. By Proposition 8.6, there is a map $\rho : \mathrm{Aut}(D(S)) \to \mathrm{Aut}(D^2(S))$ such that for each automorphism $\varphi \in \mathrm{Aut}(D(S))$, $\rho(\varphi)$ is the unique simplicial automorphism $\varphi_* : D^2(S) \to D^2(S)$ such that $\varphi_* \circ \pi = \pi \circ \varphi : D(S) \to D^2(S)$. Suppose that $\varphi : D(S) \to D(S)$ and $\psi : D(S) \to D(S)$ are elements of $Aut(D(S))$. Since $\varphi_* : D^2(S) \to D^2(S)$ and $\psi_* : D^2(S) \to D^2(S)$ are automorphisms of $D^2(S)$, $(\varphi_* \circ \psi_*) \circ \pi = \varphi_* \circ (\psi_* \circ \pi) = \varphi_* \circ (\pi \circ \psi) = (\varphi_* \circ \pi) \circ \psi = (\pi \circ \varphi) \circ \psi = \pi \circ (\varphi \circ \psi)$. It follows from the uniqueness clause of Proposition 8.6 that $(varphi \circ \psi)_* = \varphi_* \circ \psi_* : D^2(S) \to D^2(S)$ and, hence, $\rho : \mathrm{Aut}(D(S)) \to \mathrm{Aut}(D^2(S))$ is a homomorphism. This proves the existence and uniqueness of such a homomorphism $\rho : \mathrm{Aut}(D(S)) \to \mathrm{Aut}(D^2(S))$.

Since $B_{\mathcal{E}}$ is by definition a subgroup of $Aut(D(S))$, the natural homomorphism $B_{\mathcal{E}} \to \mathrm{Aut}(D(S))$ is injective.

Suppose that $\varphi \in B_{\mathcal{E}}$. By the definition of $B_{\mathcal{E}}$, there exists a unique subset $\mathcal{F}$ of the collection $\mathcal{E}$ such that φ exchanges the two vertices of each pair of distinct vertices of $D(S)$ in the collection $\mathcal{F}$ and fixes every vertex of $D(S)$ which is not one of the two vertices of some pair of distinct vertices of $D(S)$ in the collection $\mathcal{F}$.

Suppose that z is a vertex of $D^2(S)$. Since $\pi : D(S) \to D^2(S)$ is a surjective simplicial map, there exists a vertex x of $D(S)$ such that $\pi(x) = z$.

Suppose, on the one hand, that x is one of the two vertices of some distinct pair of vertices $\{x, y\}$ of $D(S)$ in the collection $\mathcal{F}$. Since $\{x, y\}$ is in $\mathcal{F}$, φ interchanges x and y and, hence, $\varphi(x) = y$. Since $\mathcal{F}$ is a subset of $\mathcal{E}$, it follows from the definition of $\pi : D(S) \to D^2(S)$ that $\pi(y) = \pi(x)$. Hence, $\varphi_*(z) = \varphi_*(\pi(x)) = \pi(\varphi(x)) = \pi(y) = \pi(x) = z$.

Suppose, on the other hand, that x is not one of the two vertices of any distinct pair of vertices of $D(S)$ in the collection $\mathcal{F}$. Then, $\varphi(x) = x$. Hence, $\varphi_*(z) = \varphi_*(\pi(x)) = \pi(\varphi(x)) = \pi(x) = z$.

In any case, it follows that the simplicial automorphism $\varphi_* : D^2(S) \to D^2(S)$ of $D^2(S)$ fixes every vertex of $D^2(S)$ and is, hence, the identity map of $D^2(S)$. This proves that the image of the natural homomorphism $B_{\mathcal{E}} \to \mathrm{Aut}(D(S))$ is in the kernel of $\rho : \mathrm{Aut}(D(S)) \to \mathrm{Aut}(D^2(S))$.

Conversely, suppose that $\varphi : D(S) \to D(S)$ is in the kernel of $\rho : \mathrm{Aut}(D(S)) \to \mathrm{Aut}(D^2(S))$. Then $\varphi_* : D^2(S) \to D^2(S)$ is the identity map of $D^2(S)$. Let x be a vertex of $D(S)$, $y = \varphi(x)$, and $z = \pi(x)$. Since z is a vertex of $D^2(S)$, it follows that $z = \varphi_*(z) = \varphi_*(\pi(x)) = \pi(\varphi(x)) = \pi(y)$.

Hence, x and y are in the same fiber of $\pi : D(S) \to D^2(S)$. It follows from the definition of $\pi : D(S) \to D^2(S)$ that either $y = x$ or $\{x, y\}$ is a pair of distinct vertices of $D(S)$ in the collection $\mathcal{E}$.

Suppose that y is not equal to x. Then $\{x, y\}$ is a pair of distinct vertices of $D(S)$ in the collection $\mathcal{E}$. Let $w = \varphi(y)$. Repeating the previous argument, with (y, w, z) rather than (x, y, z). we conclude that either $w = y$ or $\{y, w\}$ is a pair of distinct vertices of $D(S)$ in the collection $\mathcal{E}$. Suppose that $w = y$. Then $\varphi(y) = w = y = \varphi(x)$. Since $\varphi : D(S) \to D(S)$ is a simplicial automorphism and, hence, injective, it follows that $y = x$, which is a contradiction. It follows that w is not equal to y and, hence, $\{y, w\}$ is a pair of distinct vertices of $D(S)$ in the collection $\mathcal{E}$. Since $\{x, y\}$ and $\{y, w\}$ are both pairs of distinct vertices of $D(S)$ in the collection $\mathcal{E}$ having at least one common vertex y and no two distinct pairs of vertices in the collection $\mathcal{E}$ have a common vertex, it follows that $\{x, y\} = \{y, w\}$. Since w is not equal to y, it follows that $w = x$; that is to say, $\varphi(y) = x$.

This proves that for each vertex x of $D(S)$, either $\varphi(x) = x$ or x is one of the two vertices of a pair $\{x, y\}$ of distinct vertices of $D(S)$ in $\mathcal{E}$ and φ exchanges x and y.

Let $\mathcal{F}$ be the subset of $\mathcal{E}$ consisting of all pairs of distinct vertices of $D(S)$ in $\mathcal{E}$ which are exchanged by φ. It follows that $\varphi : D(S) \to D(S)$ is equal to the generalized exchange $\varphi_{\mathcal{F}} : D(S) \to D(S)$ of $D(S)$ associated to $\mathcal{F}$. By the definition of $B_{\mathcal{E}}$, φ is an element of $B_{\mathcal{E}}$. Hence, the kernel of $\rho : \mathrm{Aut}(D(S)) \to \mathrm{Aut}(D^2(S))$ is in the image of the natural homomorphism $B_{\mathcal{E}} \to \mathrm{Aut}(D(S))$. This proves that the image of the natural homomorphism $B_{\mathcal{E}} \to \mathrm{Aut}(D(S))$ is equal to the kernel of $\rho : \mathrm{Aut}(D(S)) \to \mathrm{Aut}(D^2(S))$.

□

Theorem 8.8. *Suppose that S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two. Every automorphism of $D(S)$ is a composition of an exchange automorphism of $D(S)$ with a geometric automorphism of $D(S)$.*

More precisely, let $\mathcal{E}$ be the collection of biperipheral edges of $D(S)$. Let $\varphi \in \mathrm{Aut}(D(S))$. Then there exists a unique subset $\mathcal{F}$ of $\mathcal{E}$ and a unique element h of $\Gamma^*(S)$ such that φ is equal to the composition $\varphi_{\mathcal{F}} \circ h_*$ of the *exchange automorphism* $\varphi_{\mathcal{F}}$ *of* $D(S)$ *corresponding to* $\mathcal{F}$ *and the geometric automorphism* h_* *of* $D(S)$ *induced by* h.

Proof. Let $\varphi \in \mathrm{Aut}(D(S))$. We begin by proving the existence of such a factorization of φ. Since S is not a sphere with four holes, it follows from Proposition 8.6 that there exists a unique simplicial automorphism $\psi : D^2(S) \to D^2(S)$ such that $\psi \circ \pi = \pi \circ \varphi : D(S) \to D^2(S)$.

Since S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two, it follows from Theorem 7.8 that there exists a homeomorphism $H : S \to S$ such that $\psi([X]) = [H(X)]$ for every domain X on S which is not a biperipheral pair of pants.

Let $G = H^{-1} : S \to S$ and $G_* : D(S) \to D(S)$ be the geometric automorphism of $D(S)$ defined by the rule $G_*([X]) = [G(X)]$ for every domain X on S.

Note that $G_* \circ \varphi : D(S) \to D(S)$ is an automorphism of $D(S)$. We shall now show that $G_* \circ \varphi$ is an exchange automorphism.

Since S is not a sphere with four holes and $\varphi \in \mathrm{Aut}(D(S))$, it follows from Proposition 6.15 that $\varphi(\mathcal{E}) = \mathcal{E})$.

Suppose that X is a domain on S which is not a biperipheral pair of pants or a regular neighborhood of a biperipheral curve. Note that $[X]$ is not a vertex of an edge in $\mathcal{E}$. Since φ is an automorphism of $D(S)$ and $\varphi(\mathcal{E}) = \mathcal{E}$, it follows that $\varphi([X])$ is not a vertex of an edge in $\mathcal{E}$. Hence, $\varphi([X]) = [Y]$ where Y is a domain on S which is not a biperipheral pair of pants or a regular neighborhood of a biperipheral curve.

By the definition of the natural projection $\pi : D(S) \to D^2(S)$, $\pi([X]) = [X]$ and $\pi([Y]) = [Y]$. Hence, $\pi(\varphi([X]) = \varphi([X])$.

Hence, $\varphi([X]) = \pi(\varphi([X])) = \psi(\pi([X]) = \psi([X]) = [H(X)]$.

It follows that $(\varphi \circ G_*)([X]) = \varphi[G(X)] = [H(G(X))] = [X]$.

This shows that $\varphi \circ G_*$ fixes every vertex of $D(S)$ which is not a vertex of an edge in $\mathcal{E}$.

By a similar argument, it follows that if X is a domain on S which is a biperipheral pair of pants and Y is a regular neighborhood of the corresponding biperipheral curve, then either (i) $(\varphi \circ G_*)([X]) = [X]$ and $(\varphi \circ G_*)([Y]) = [Y]$ or (ii) $(\varphi \circ G_*)([X]) = [Y]$ and $(\varphi \circ G_*)([Y]) = [X]$.

Let $i = \varphi \circ G_*$. It follows that i is an exchange automorphism of $D(S)$.

Since $i = \varphi \circ G_*$, we conclude that $\varphi = \varphi_{\mathcal{F}} \circ h_*$ where $\mathcal{F}$ is the subcollection of $\mathcal{E}$ consisting of all biperipheral edges of $D(S)$ whose vertices are exchanged by i, and h is the isotopy class of $H : S \to S$.

This proves the existence of such a factorization of φ.

Suppose that $\Phi_{\mathcal{F}} \circ h_* = \Phi_{\mathcal{P}} \circ q_*$. Then $\Phi_{\mathcal{P} \triangle \mathcal{F}} = (g \cdot h^{-1})_*$.

Since an automorphism of $D(S)$ which is induced by a homeomorphism of S cannot exchange an annular vertex with a nonannular vertex of $D(S)$, it follows that $\mathcal{P} \triangle \mathcal{F} = \emptyset$. In other words, $\mathcal{F} = \mathcal{P}$.

Since $\Phi_{\mathcal{P}\triangle\mathcal{F}} = (g \cdot h^{-1})_*$ and $\mathcal{F} = \mathcal{P}$, it follows that $(g \cdot h^{-1})_*$ is the trivial automorphism $id : D(S) \to D(S)$ of $D(S)$. In other words, $g \cdot h^{-1}$ acts trivially on $D(S)$.

Since $D^2(S)$ is a subcomplex of $D(S)$, it follows that $g \cdot h^{-1}$ acts trivially on $D^2(S)$. Since S is not a sphere with four holes, a torus with at most two holes, or a closed surface of genus two, it follows from Theorem 7.8 that $g \cdot h^{-1}$ is equal to the identity element of $\Gamma^*(S)$. In other words, g is equal to h.

This proves the uniqueness of such a factorization of φ, completing the proof.

□

We can summarize the preceding results as follows.

Theorem 8.9. *Suppose that S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two. Then we have a natural commutative diagram of exact sequences:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{B}(\mathcal{E}) & \longrightarrow & \mathcal{B}(\mathcal{E}) \rtimes \Gamma^*(S) & \longrightarrow & \Gamma^*(S) & \longrightarrow & 1 \\
 & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\
1 & \longrightarrow & B_{\mathcal{E}} & \longrightarrow & \mathrm{Aut}(D(S)) & \longrightarrow & \mathrm{Aut}(D^2(S)) & \longrightarrow & 1
\end{array}$$

Proof. The exactness of the first row of the above diagram follows immediately from the definition of a semi-direct product.

The commutativity of the left-hand square follows from Propositions 8.1 and 8.5.

The commutativity of the right-hand square follows from Propositions 8.5 and 8.6.

The isomorphism $\mathcal{B}(\mathcal{E}) \xrightarrow{\simeq} B_{\mathcal{E}}$ follows from Proposition 8.1 and the definition of the Boolean subgroup $B_{\mathcal{E}}$ of $Aut(D(S))$.

Since S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two, it follows from Theorem 7.8 that the natural homomorphism $\Gamma^*(S) \xrightarrow{\simeq} \mathrm{Aut}(D^2(S))$ is an isomorphism.

Since the natural homomorphisms $\mathcal{B}(\mathcal{E}) \rtimes \Gamma^*(S) \longrightarrow \Gamma^*(S)$ and $\Gamma^*(S) \to \mathrm{Aut}(D^2(S))$ are both surjective, it follows from the commutativity of the right-hand square that the natural homomorphisms $Aut(D(S)) \to \mathrm{Aut}(D^2(S))$ is also surjective. Hence, since S is not a sphere with four holes, the exactness of the second row of the above diagram follows from Proposition 8.7.

This shows that the diagram is a commutative diagram of exact sequences. Since the vertical arrows on the left and right are both isomorphisms, it follows from standard results that the natural monomorphism $\mathcal{B}(\mathcal{E}) \rtimes \Gamma^*(S) \to \mathrm{Aut}(D(S))$ of Proposition 8.5 is an isomorphism, completing the proof.

□

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