THE VISUAL SPHERE OF TEICHMÜLLER SPACE AND A THEOREM OF MASUR-WOLF

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Abstract. In [M-W], Masur and Wolf proved that the Teichmüller space of genus $g > 1$ surfaces with the Teichmüller metric is not a Gromov hyperbolic space. In this paper, we provide an alternative proof based upon a study of the visual sphere of Teichmüller space.

1. Introduction

As observed in [M-W], the Teichmüller space of surfaces of genus $g > 1$ with the Teichmüller metric shares many properties with spaces of negative curvature. In his study of the geometry of Teichmüller space [Kr], Kravetz claimed that Teichmüller space was negatively curved in the sense of Busemann [Bu]. It was not until about ten years later, that Linch [L] discovered a mistake in Kravetz's arguments. This left open the question of whether or not Teichmüller space was negatively curved in the sense of Busemann. This question was resolved in the negative by Masur in [Ma].

A metric space X is negatively curved, in the sense of Busemann, if the distance between the endpoints of two geodesic segments from a point in X is at least twice the distance between the midpoints of these two segments. An immediate consequence of this definition is that distinct geodesic rays from a point in a Busemann negatively curved metric space must diverge. Masur proved that Teichmüller space is not negatively curved, in the sense of Busemann, by constructing distinct geodesic rays from a point in Teichmüller space which remain a bounded distance away from each other.

In [G], Gromov introduced a notion of negative curvature for metric spaces which, while less restrictive than that of Busemann, implies many of the properties which Teichmüller space shares with spaces of Riemannian negative sectional curvature. This raised the question of whether Teichmüller space was negatively curved in the sense of Gromov, (i.e. Gromov hyperbolic). According to one of the definitions of Gromov hyperbolicity, an affirmative answer to this question would rule out so-called "fat" geodesic triangles in Teichmüller space. In [M-W], Masur and Wolf resolved the Gromov hyperbolicity question in the negative by constructing such "fat" geodesic triangles.

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As observed in [M-W], the existence of distinct nondivergent rays from a point in Teichmüller space does not preclude Teichmüller space from being Gromov hyperbolic. Apparently for this reason, rather than taking Masur's construction of such rays as the starting point for their proof, Masur and Wolf found their motivation from another source. They, observed that the isometry group of the Teichmüller metric is the mapping class group [R], which is not a Gromov hyperbolic group, since it contains a free abelian group of rank 2. This fact, like Masur's result on the existence of distinct nondivergent rays from a point, is insufficient to imply that Teichmüller space is not Gromov hyperbolic. Nevertheless, it served as motivation for Masur and Wolf's construction of "fat" geodesic triangles.

In this paper, we provide an alternative proof of the result of Masur and Wolf. Our proof, unlike that of Masur and Wolf, builds upon Masur's construction of nondivergent rays from a point in Teichmüller space. On the other hand, unlike the proof of Masur and Wolf, our proof depends upon one of the deeper consequences of Gromov hyperbolicity. Namely, in order for Teichmüller space to be Gromov hyperbolic, the visual sphere of Teichmüller space would have to be Hausdorff. We show that, on the contrary, the visual sphere of Teichmüller space is not Hausdorff. The proof of this fact relies heavily upon the specific nature of Masur's construction of nondivergent rays. In this way, we show that the result of Masur and Wolf that Teichmüller space is not negatively curved in the sense of Gromov is latent in Masur's original proof that Teichmüller space is not negatively curved in the sense of Busemann.

The outline of the paper is as follows. In section 2, we review the prerequisites for our proof. In section 3, we prove our main result that the visual sphere of Teichmüller space is not Hausdorff and conclude that Teichmüller space is not Gromov hyperbolic.

2. Preliminaries

2.1. **Teichmüller space.** Let M denote a closed, connected, orientable surface of genus $g \geq 2$. The Teichmüller space T_g of M is the space of equivalence classes of complex structures on M, where two complex structures S_1 and S_2 on M are equivalent if there is a conformal isomorphism $h : S_1 \to S_2$ which is isotopic to the identity map of the underlying topological surface M.

The Teichmüller distance $d([S_1],[S_2])$ between the equivalence classes $[S_1]$ and $[S_2]$ of two complex structures S_1 and S_2 on M is defined as $\frac{1}{2}\log \inf_h K(h)$, where the infimum is taken over all quasiconformal homeomorphisms $h : S_1 \to S_2$ which are isotopic to the identity map of M and $K(h)$ is the maximal dilitation of h.

As shown by Kravetz [Kr], (T_g, d) is a *straight G-space* in the sense of Busemann ([Bu],[A]). Hence, any two distinct points, x and y, in T_g are joined by a unique geodesic segment (i.e. an isometric image of a Euclidean interval), [x,y], and lie on a unique geodesic line (i.e. an isometric image of $\mathbb{R}$), $\gamma(x, y)$.

Now, fix a conformal structure S on M and let $QD(S)$ be the space of holomorphic quadratic differentials on S. The geodesic rays (i.e. isometric images of $[0, \infty)$) which emanate from the point $[S]$ in T_g are described in terms of $QD(S)$. If q is a holomorphic quadratic differential on S, p is a point on S and z is a local parameter on S defined on a neighborhood U of p, then q may be written in the form $\phi(z)dz^2$ for some holomorphic function ϕ on U. If $\phi(p) \neq 0$ and $z_0 = z(p)$, then on a sufficiently small neighborhood V of p contained in U, we may define a branch $\phi(z)^{1/2}$ of the square root of ϕ. The integral $w = \Phi(z) = \int_{z_0}^{z} \phi(z)^{1/2} dz$ is a conformal function of z and determines a local parameter for S on a sufficiently small neighborhood W of p in V. This parameter w is called a *natural rectangular parameter* for q at the regular point p. In terms of this parameter w, q may be written in the form dw^2. For each nonzero quadratic differential q on S, there is a one-parameter family $\{S_K\}$ of conformal structures on M and quadratic differentials $\{q_K\}$ on S_K obtained by replacing the natural parameters w for q on S by natural parameters w_K for q_K on S_K. The relationship between w_K and w is given by the rule:

$$Rew_K = K^{1/2} Rew \quad Imw_K = K^{-1/2} Imw.$$

The Teichmüller distance from $[S_K]$ to $[S]$ is equal to $\log(K)/2$. The map $t \mapsto [S_{e^{2t}}]$ is a Teichmüller geodesic ray emanating from $[S]$ and every geodesic ray emanating from $[S]$ is of this form. Two nonzero quadratic differentials on S determine the same Teichmüller geodesic ray in T_g emanating from $[S]$ if and only if they are positive multiples of one another.

It is well-known that (T_g, d) is homeomorphic to $\mathbb{R}^{6g-6}$ and closed balls in (T_g, d) are homeomorphic to closed balls in $\mathbb{R}^{6g-6}$. In fact, using the previous description of geodesic rays, a homeomorphism can be constructed from the open unit ball of $QD(S)$ onto T_g. Suppose q is a point in the open unit ball of $QD(S)$. Then $q = kq_1$ where $0 \leq k < 1$ and q_1 is a quadratic differential in the unit sphere of $QD(S)$. Map q to the point $[S_K]$ on the geodesic ray through $[S]$ in the direction of q_1 where $K = (1+k)/(1-k)$. By the work of Teichmüller, this map is a homeomorphism from the open unit ball of $QD(S)$ onto T_g. Since $QD(S)$ is a complex vector space of dimension $3g - 3$, this proves that T_g is homeomorphic to $\mathbb{R}^{6g-6}$. Note also that this homeomorphism maps the closed ball of radius k centered at the origin of $QD(S)$ onto the closed ball of radius $\log(K)/2$ centered at the point $[S]$ in (T_g, d). This proves that closed balls in (T_g, d) are homeomorphic to closed balls in $\mathbb{R}^{6g-6}$.

We shall be particularly interested in the Jenkins-Strebel differentials. These are the quadratic differentials all of whose noncritical horizontal trajectories are closed. Let θ be a Jenkins-Strebel differential and F be the horizontal foliation of θ. The complement in M of the critical trajectories of F consists of p disjoint open annuli $A_1, \ldots, A_p$, where $1 \leq p \leq 3g - 3$.

Let σ_i be a core curve of the annulus A_i. The core curves $\sigma_1, \dots, \sigma_p$ are distinct, nontrivial, pairwise nonisotopic circles on M. Each annulus A_i is foliated by closed leaves of F isotopic to σ_i. Let M_i be the modulus of the annulus A_i. The basic existence and uniqueness theorem of Jenkins-Strebel ([J], [S]) states that there exists a unique quadratic differential θ in $Q(S)$ with prescribed isotopy classes $\gamma_i = [\sigma_i]$ of core curves and moduli M_i of the corresponding annuli A_i. Note that two Jenkins-Strebel differentials on S determine the same Teichmüller geodesic in T_g emanating from $[S]$ if and only if the horizontal foliations of these Jenkins-Strebel differentials are projectively equivalent.

Following Masur [Ma], we define a *Strebel ray* in T_g emanating from $[S]$ to be a Teichmüller geodesic ray determined by a Jenkins-Strebel differential on S. Suppose that θ_1 and θ_2 are Jenkins-Strebel differentials corresponding to the same isotopy classes of core curves, but not necessarily the same moduli, of corresponding annuli. Then, following Masur, we say that the Strebel rays determined by θ_1 and θ_2 are *similar*. Masur proved that similar Strebel rays emanating from the same point in T_g are nondivergent.

Theorem (Masur [Ma]**) .** *Let r and s be similar Strebel rays in T_g emanating from a point x in T_g. There exists $N < \infty$ such that if y and z are any two points on r and s which are equidistant from x, then $d(y,z) \leq N$.*

Since $g \geq 2$, there exist distinct similar Strebel rays r and s in T_g emanating from the same point $x = [S]$ in T_g. We may construct all such pairs of rays as follows. Choose a collection of disjoint, nontrivial, pairwise nonisotopic circles $\sigma_1, \dots \sigma_p$ on M, where $2 \leq p \leq 3g-3$. Let $a = (a_1, a_2, \dots, a_p)$ and $b = (b_1, \dots, b_p)$ be p-tuples of positive real numbers a_i and b_i such that a and b lie on distinct rays emanating from the origin in $\mathbb{R}^p$. Let θ be the Jenkins-Strebel differential on S corresponding to the isotopy classes $\gamma_i = [\sigma_i]$ of core curves and moduli a_i of corresponding annuli. Likewise, let ψ be the Jenkins-Strebel differential on S corresponding to the isotopy classes $\gamma_i = [\sigma_i]$ of core curves and moduli b_i of corresponding annuli. Finally, let r and s be the Strebel rays determined by θ and ψ.

Combining the observation of the previous paragraph with his theorem on nondivergence of similar Strebel rays, Masur constructed distinct, nondivergent Teichmüller geodesic rays emanating from the same point in T_g. Indeed, any pair of distinct similar Strebel rays emanating from the same point in T_g is such a pair of nondivergent rays. In this way, Masur proved that T_g is not negatively curved in the sense of Busemann [Ma]. The particular nature of Masur's construction of nondivergent rays will be crucial to our proof that T_g is not negatively curved in the sense of Gromov.

The modulus of a flat cylinder C of circumference l and height h is $Mod(C) = h/l$. Let S be a conformal structure on M. Every cylinder C embedded in M has a conformal structure induced from S. C is conformally equivalent to a unique flat cylinder up to change of scale. The modulus of C is the modulus of any such flat cylinder. Let γ be an isotopy class of nontrivial simple closed curves on M. The modulus $mod_S(\gamma)$ of γ is defined to be the supremum of the moduli of all cylinders embedded in M with core curve $\sigma \in \gamma$.

For each conformal metric ρ on S, let $\ell_\rho(\gamma)$ denote the infimum of the lengths, with respect to ρ, of simple closed curves $\sigma \in \gamma$. Let A_ρ denote the area, with respect to ρ, of M. The *extremal length* $ext_S(\gamma)$ of γ (with respect to the conformal structure S on M) is equal to $\sup_\rho (\ell_\rho(\gamma))^2/A_\rho$. The extremal length is related to the modulus by the equation $ext_S(\gamma) = 1/mod_S(\gamma)$.

According to Kerckhoff [K], the Teichmüller metric d may be expressed in terms of extremal length.

Theorem (Kerckhoff [K]**) .** *The Teichmüller distance between two points* $[S_1]$ *and* $[S_2]$ *in* T_g *is given by the rule:*

$$d([S_1],[S_2]) = \frac{1}{2}\log \sup_\gamma \frac{ext_{S_1}(\gamma)}{ext_{S_2}(\gamma)}$$

where the supremum ranges over all isotopy classes γ *of nontrivial simple closed curves on* M.

We recall that there is a unique hyperbolic conformal metric ρ on S. There exists a unique hyperbolic geodesic in the isotopy class γ. The hyperbolic length $\ell_\rho(\gamma)$ is the length of this hyperbolic geodesic. Maskit established the following comparisons between the hyperbolic length $\ell_\rho(\gamma)$ and the extremal length $ext_S(\gamma)$ [M].

Theorem (Maskit [M]**) .** *Let* γ *be an isotopy class of nontrivial simple closed curves on* M, S *be a conformal structure on* M *and* ρ *be the unique hyperbolic conformal metric on* S. *Let* ℓ *be the hyperbolic length* $\ell_\rho(\gamma)$ *and* m *be the extremal length* $ext_S(\gamma)$. *Then* $\ell \le m\pi$ *and* $m \le (1/2)\ell e^{\ell/2}$.

2.2. **Visual spheres and Gromov hyperbolicity.** Let X be a space equipped with a metric d. X is said to be *proper* if closed balls in X are compact. Since closed balls in (T_g, d) are homeomorphic to closed balls in $\mathbb{R}^{6g-6}$, (T_g, d) is proper. X is said to be *geodesic* if every pair of points $x, y \in X$ can be connected by a *geodesic segment* (i.e. an isometric embedding of an interval). By Kravetz' result that (T_g, d) is a straight G-space in the sense of Busemann discussed in (2.1), (T_g, d) is geodesic.

Let x be a point in X. A *geodesic ray* emanating from x is an isometric embedding $r : [0,\infty) \to X$ mapping 0 to x. If r_1 and r_2 are two geodesic rays in X emanating from x and the function $t \mapsto d(r_1(t), r_2(t))$ is bounded, then we say that r_1 and r_2 are *asymptotic* and write $r_1 \sim r_2$. In this way, we define an equivalence relation $\sim$ on the set R_x of geodesic rays in X emanating from x. Equip R_x with the topology of uniform convergence on compact sets. The *visual sphere* of X at x is the quotient space $\partial_{vis,x}X$ of R_x with respect to the equivalence relation $\sim$.

Gromov ([G], see also [C-D-P], [G-H]) introduced a notion of hyperbolicity for metric spaces which is now called Gromov hyperbolicity. Gromov hyperbolic metric spaces share many of the qualitative properties of hyperbolic space. We shall not need the precise definition of Gromov hyperbolicity. We shall, however, require the following result.

Theorem (Gromov [C-D-P]**) .** *Let X be a proper, geodesic, Gromov hyperbolic space and x be a point in X. Then the visual sphere $\partial_{vis,x}X$ of X at x is Hausdorff.*

Remark 2.3. In fact, the visual sphere of a proper, geodesic, Gromov hyperbolic space is metrizable. The visual sphere of such a space does not depend upon the base point x in X and is naturally isomorphic to the Gromov boundary ∂X of X [C-D-P]. Note that the visual sphere is defined for any metric space. The Gromov boundary, however, is only defined for a restricted class of metric spaces including Gromov hyperbolic spaces.

3. The visual sphere of Teichmüller space

In this section, we prove that the visual sphere of Teichmüller space is not Hausdorff and conclude that Teichmüller space is not Gromov hyperbolic.

Theorem 3.1. *Let S be a conformal structure on M representing a point x in T_g. Then the visual sphere $\partial_{vis,x}T_g$ of T_g at x, with respect to the Teichmüller metric d, is not Hausdorff.*

Proof. Let σ_0 and σ_1 be a pair of disjoint simultaneously nonseparating circles on M. For each real number t with $0 < t < 1$, let θ_t denote the unique Jenkins-Strebel differential on S with core curves σ_0 and σ_1 and moduli $M_0 = 1 - t$ and $M_1 = t$. Let θ_0 denote the unique Jenkins-Strebel differential on S with core curve σ_0 and modulus $M_0 = 1$. Let θ_1 denote the unique Jenkins-Strebel differential on S with core curve σ_1 and modulus $M_1 = 1$. Let r_t be the geodesic ray in T_g emanating from x corresponding to the nonzero quadratic differential θ_t. The family $\{r_t | 0 \leq t \leq 1\}$ is a continuous one-parameter family of geodesic rays in T_g emanating from x. Let $[r_t]$ denote the point in $\partial_{vis,x}T_g$ represented by r_t.

Note that r_t is similar to $r_{1/2}$ for all t such that $0 < t < 1$. By Masur's result on nondivergence of similar rays discussed in (2.1), it follows that r_t is asymptotic to $r_{1/2}$ for all t such that $0 < t < 1$. Let $x = [r_{1/2}]$. Then $x = [r_t]$ for all t such that $0 < t < 1$. By continuity of the quotient map from R_x to the visual sphere (recalling that the visual sphere is equipped with the quotient topology), and the convergence of the rays in R_x, $[r_0]$ and $[r_1]$ are contained in the closure of x in $\partial_{vis,x}T_g$.

We shall now show, using Maskit's comparison of extremal and hyperbolic lengths discussed in (2.1), that $[r_0]$ is not equal to $[r_1]$. Since σ_0 and σ_1 are simultaneously nonseparating circles on M, we may choose a nonseparating circle σ on M such that σ is disjoint from σ_1, transverse to σ_0, and meets σ_0 in exactly one point. Let γ_i denote the isotopy class of σ_i and γ denote the isotopy class of σ. Let $\{S_K^i\}$ denote the family of conformal structures on M determined by θ_i.

We recall Masur's description of the surfaces $\{S_K^i\}$ ([Ma]). The complement of the critical points of θ_i and the horizontal leaves of θ_i joining critical points of θ_i is a single annulus R^i foliated by closed horizontal leaves of θ_i homotopic to σ_i. We may assume that σ_i is the central curve of R^i. The surface S_K^i is formed from S by "fattening" R^i, by cutting M along σ_i and inserting a standard annulus of appropriate modulus. As K tends to infinity, the modulus of the inserted annulus tends to infinity. Hence, the modulus of γ_i on S_K^i tends to infinity. In other words, $ext_{S_K^i}(\gamma_i)$ tends to zero.

In particular, $ext_{S_K^0}(\gamma_0)$ tends to zero as K tends to infinity. Let ρ_K^0 denote the unique hyperbolic conformal metric on S_K^0. By Maskit's comparison theorem discussed in (2.1), $\ell_{\rho_K^0}(\gamma_0)$ tends to zero as K tends to infinity. Since σ_0 meets σ transversely and in a single point, the unique hyperbolic geodesics for the hyperbolic metric ρ_K^0 in the isotopy classes of σ_0 and σ also meet transversely and in a single point. $\ell_{\rho_K^0}(\gamma_0)$ and $\ell_{\rho_K^0}(\gamma)$ are the respective lengths of these hyperbolic geodesics. Hence, by Lemma 1 of Chapter 11, Section 3.3 of [A], $\ell_{\rho_K^0}(\gamma)$ tends to infinity as K tends to infinity. Again, by Maskit's comparison theorem, $ext_{S_K^0}(\gamma)$ tends to infinity as K tends to infinity.

On the other hand, note that σ is disjoint from σ_1. Let R be any annulus on S disjoint from σ_1 with core curve isotopic to σ. By the description of S_k^1 in terms of fattening R^1 along σ_1, the annulus R embeds conformally in S_k^1. Hence, the modulus of γ on S_k^1 is bounded below by the constant $C = mod_S(R)$. In other words, the extremal length of γ on S_k^1 is bounded above by the constant $1/C$.

We have shown that $ext_{S_K^0}(\gamma)$ tends to infinity and $ext_{S_K^1}(\gamma)$ remains bounded above as K tends to infinity. Hence, $ext_{S_K^0}(\gamma)/ext_{S_K^1}(\gamma)$ tends to infinity as K tends to infinity. By Kerckhoff's description of the Teichmüller metric in terms of extremal length discussed in (2.1), $d(S_K^0, S_K^1)$ tends to infinity as K tends to infinity. We conclude that r_0 is not asymptotic to r_1. In other words, $[r_0] \neq [r_1]$. Hence, we have a pair of distinct points $[r_0]$ and $[r_1]$ in the closure of a single point $[r_{1/2}]$ in the visual sphere $\partial_{vis,x}T_g$ of T_g at x. It follows that the visual sphere $\partial_{vis,x}T_g$ of T_g at x is not Hausdorff. $\square$

We are now ready to deduce the result of Masur and Wolf.

Corollary 3.2. *(Masur-Wolf* [M-W]*) Teichmüller space with the Teichmüller metric is not Gromov hyperbolic.*

Proof. Suppose that (T_g, d) is Gromov hyperbolic. Closed balls in (T_g, d) are compact and (T_g, d) is geodesic. By Gromov's theorem on the visual sphere of a proper, geodesic, Gromov hyperbolic space discussed in (2.2), it follows that the visual sphere of Teichmüller space is Hausdorff. This contradicts Theorem 3.1. Hence, (T_g, d) is not Gromov hyperbolic. $\square$

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