# Group Theory I
# Lecture Notes for MTH 912
# F10

Ulrich Meierfrankenfeld

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# Contents

# Chapter 1

# Group Actions

## 1.1 Groups acting on sets

**Definition 1.1.1.** *An action of a groups $G$ on set $\Omega$ is a function*

$$\begin{array}{rrcl} \bullet: & \Omega \times G & \to & \Omega \\ & (\omega, g) & \mapsto & \omega^g \end{array}$$

*such that*

*(i)* $\omega^1 = \omega$ *for all* $\omega \in \Omega$.

*(ii)* $(\omega^x)^y = \omega^{xy}$ *for all* $\omega \in \Omega$, $x, y \in G$.

*A $G$-set is a set $\Omega$ together with an action of $G$ on $\Omega$.*

**Example 1.1.2.** *Let $G$ be a group, $H$ a subgroup of $G$ and $\Omega$ a set. Each of the following are actions.*

*(a)*

$$\begin{array}{rrcl} \text{RM:} & G \times G & \to & G \\ & (a, b) & \mapsto & ab \end{array} \quad (\text{ action by right multiplication})$$

*(b)*

$$\begin{array}{rrcl} \text{LM:} & G \times G & \to & G \\ & (a, b)) & \mapsto & b^{-1}a \end{array} \quad (\text{ action by left multiplication})$$

*(c)*

$$\begin{array}{rrcl} \text{Conj:} & G \times G & \to & G \\ & (a, b) & \mapsto & b^{-1}ab \end{array} \quad (\text{ action by left conjugation})$$

*(d)*

$$\begin{array}{rrcl} \text{RM:} & G/H \times G & \to & G \\ & (Ha, b) & \mapsto & Hab \end{array} \quad \text{( action by right multiplication)}$$

*(e)*

$$\begin{array}{rrcl} \text{Nat:} & \Omega \times \mathrm{Sym}(\Omega) & \to & \Omega \\ & (\omega, \pi) & \mapsto & \omega\pi \end{array} \quad \text{( natural action )}$$

**Definition 1.1.3.** *Let $\bullet$ be an action of the group $G$ on the set $\Omega$. Let $g \in G$, $\omega \in \Omega$, $H \subseteq G$ and $\Delta \subseteq H$.*

*(a)* $\Delta^g := \{\omega^g \mid \omega \in \Delta\}$.

*(b)* $\omega^H = \{\omega^h \mid h \in H\}$.

*(c)* $C_H^\bullet(\Delta) := \{h \in H \mid \omega^h = \omega \textit{for all} \omega \in \Delta\}$.

*(d)* $C_H^\bullet(w) = \{h \in H \mid \omega^h = \omega$. *(So* $C_H^\bullet(\omega) = C_H(^\bullet(\{\omega\}$. *We will often write* $H_\omega^\bullet$ *for* $C_G^\bullet(\omega)$.

*(e)* $N_H^\bullet(\Delta) := \{h \in H \mid \Delta^h = \Delta\}$.

*(f)* $\Delta$ *is called $H$-invariant if* $\Delta = \Delta^h$ *for all* $h \in H$.

*(g)* $H$ *is called transitive on* $\Omega$ *if for all* $a, b \in \Omega$ *there exists* $h \in H$ *with* $a^h = b$.

Note that map $(\Delta, g) \mapsto \Delta^g$ defines an action for $G$ on the set of subsets of $\Omega$. If $H \leq G$ and $\Delta$ is an $H$-invariant subgroup of $G$, then $H$ acts in $\Delta$ via $(\omega, h) \to \omega^h$ for all $\omega \in \Delta$, $h \in H$. If there is no doubt about the action $\bullet$, we will often omit the superscript $\bullet$ in $C_H^\bullet(\omega)$, $N_H^\bullet(\Delta)$ and so on.

**Lemma 1.1.4.** *Let $\bullet$ be an action of the group $G$ on the set $\Omega$, $g \in G$, $H \subseteq G$ and $\Delta \subseteq \Omega$.*

*(a) If* $H \leq G$, *then* $C_H(\Delta) \leq N_H(\Delta) \leq H$.

*(b)* $C_H(\Delta)^g = C_{H^g}(\Delta^g)$.

*(c) Define* $\phi_g : \Omega \to \Omega, \omega \to \omega^g$ *and* $G^{\bullet\Omega} = \{\phi_g \mid g \in G\}$. *Then the map* $\phi : G \to \mathrm{Sym}(\Omega), g \to \phi_g$ *is a well-defined homomorphism of groups with* $\ker\phi = C_G^\bullet(\Omega)$ *and* $\mathrm{Im}\,\phi = G^{\bullet\Omega}$. *In particular,* $C_G^\bullet(\Omega)$ *is a normal subgroup of* $G$, $G^{\bullet\Omega}$ *is a subgroup of* $\mathrm{Sym}(\Omega)$ *and* $G/C_G^\bullet(\Omega) \cong G^{\bullet\Omega}$.

*Proof.* Readily verified. As an example we prove (b). Let $h \in H$ and $\omega \in \Delta$. Then

$$\begin{array}{cc} & h \in C_H(\omega) \\ \iff & \omega^h = \omega \\ \iff & \omega^{hg} = \omega^g \\ \iff & \omega^{gg^{-1}hg} = \omega^g \\ \iff & (\omega^g)^{h^g} = \omega^g \\ \iff & h^g \in C_{H^g}(\omega^g) \end{array}$$

Hence $C_H(\omega)^g = C_{H^g}(\omega^g)$. Intersecting over all $\omega \in \Delta$ gives (b). □

**Definition 1.1.5.** *Let $G$ be a group and let $\Omega_1$ and $\Omega_2$ be $G$-sets. Let $\alpha : \Omega_1 \to \Omega_2$ be a function.*

*(a) $\alpha$ is called $G$-equivariant if*

$$\omega^\gamma \alpha = \omega \alpha^g$$

*for all $\omega \in \Omega$ and $g \in G$.*

*(b) $\alpha$ is called a $G$-isomorphism if $\alpha$ is $G$-equivariant bijection.*

*(c) $\Omega_1$ and $\Omega_2$ are called $G$-isomorphic if there exists a $G$-isomorphism from $\Omega_1$ to $\Omega_2$.*

**Definition 1.1.6.** *Let $G$ be a group acting on a set $\Omega$.*

*(a) We say that $G$ acts transitively on $\Omega$ if for all $a, b \in \Omega$ there exists $g \in G$ with $a^g = b$.*

*(b) We say that $G$ acts regularly on $\Omega$ if for all $a, b \in G$ there exists exactly one $g \in G$ with $a^g = b$.*

*(c) We say that $G$ acts semi-regularly on $\Omega$ if for all $a, b \in G$ there exists at most one $g \in G$ with $a^g = b$.*

**Lemma 1.1.7.** *Suppose that the group $G$ acts transitively on the set $\Omega$ and let $\omega \in \Omega$. View $G/G_\omega$ as a $G$-set via right multiplication. Then the map*

$$\begin{array}{rccc} \phi : & G/G_\omega & \to & \Omega \\ & G_\omega g & \mapsto & \omega^g \end{array}$$

*is a well defined $G$-isomorphism. In particular, $|\Omega| = |G/G_\omega|$.*

*Proof.* Let $g, h \in G$. Then the following statements are equivalent:

$$\begin{aligned} \omega^g &= \omega^h \\ \omega^{gh^{-1}} &= \omega \\ gh^{-1} &\in G_\omega \\ G_\omega g &= G_\omega h \end{aligned}$$

The forward direction shows that $\phi$ is 1-1 and the backward direction shows that $\phi$ is well-defined. Since $G$ is transitive on $\Omega$, $\phi$ is onto. Note that

$$((G_\omega g)h)\phi = (G_\omega(gh)\phi = \omega^{gh} = (\omega^g)^h = ((G_\omega g)\phi)^g$$

and so $\phi$ is $G$-equivariant. □

**Proposition 1.1.8** (Frattini Argument)**.** *Let $G$ be a group acting on a set $\Omega$, $H$ a subgroup of $G$ and $\omega \in \Omega$.*

*(a) If $H$ acts transitively on $\Omega$, then $G = G_\omega H$.*

*(b) If $H$ acts semi-regularly on $\Omega$, then $G_\omega \cap H = 1$.*

*(c) If $H$ acts regularly on $\Omega$, then $H$ is a complement to $G_\omega$ in $G$, that is $G = G_\omega H$ and $G_\omega \cap H = 1$.*

*Proof.* (a) Let $g \in G$. Since $H$ acts transitively on $\Omega$, $\omega^g = \omega^h$ for some $h \in G$. Hence $gh^{-1} \in G_\omega$ and $g = (gh^{-1})h \in G_\omega H$.

(b) Let $h \in G_\omega \cap H$. Then $\omega^h = \omega = \omega^1$ and so $h = 1$ by definition of semi-regular.

(c) Since $H$ acts transitively and semi-regularly on $\Omega$, this follows from (a) and (b). □

**Definition 1.1.9.** *Let $G$ be group acting on a set $\Omega$*

*(a) $\omega \in \Omega$ and $H \subseteq G$. Then $\omega^H := \{\omega^h \mid h \in H\}$. $\omega^G$ is called an orbit for $G$ in $\Omega$. $\Omega/G$ denotes the set of orbits of $G$ on $\Omega$.*

*(b) Let $a, b \in G$. We say that $a$ is $G$-equivalent to $b$ if $b = a^g$ for some $g \in G$, that is if $b \in a^G$.*

**Lemma 1.1.10.** *Let $G$ be a group acting on set $\Omega$.*

*(a) $G-$ equivalence is an equivalence relation. The equivalence classes are the orbits of $G$ on $\Omega$.*

*(b) Let $\Delta$ be a set of representatives for the orbits of $G$ on $\Omega$, (that is $|\Delta \cap O| = 1$ for each orbit $O$ of $G$ on $\Omega$. Then $\Omega$ and $\uplus_{\omega \in \Delta} G/G_\omega$ are isomorphic $G$-sets. In particular,*

$$|\Omega| = \sum_{\omega \in \Delta} |G/G_\omega| \qquad \text{(Orbit equation)}$$

*Proof.* (a) Since $\omega = \omega^1$, the relation is reflexive. If $a, b \in \Omega$ with $a^g = b$, then $b^{g^{-1}} = a$ and the relation is symmetric. If $a^g = b$ and $b^h = c$, then $a^{gh} = c$ and the relation is transitive. $\omega^G$ is the set of elements if $G$ in relation with $\omega$ and so is an equivalence class.

(b) Since each elements of $\omega$ lies in exactly on orbits of $G$ on $\Omega$ and since each orbit contains exactly one element of $\Delta$ we have

$$\Omega = \uplus_{\omega \in \Delta} \omega^G$$

Observe that $G$ acts transitively on $\omega^G$ and so by 1.1.7 $\omega^G \cong G/G_\omega$ as a $G$-set. So (b) holds. □

**Lemma 1.1.11.** *Let $G$ be a group acting on a set non-empty set $\Omega$.*

*(a) $G$ acts regularly on $\Omega$ if and only if it acts transitively and semi-regularly.*

*(b) $G$ acts semi-regularly on $\Omega$ if and only if $G_a = 1$ for all $a \in \Omega$.*

*(c) $G$ acts transitively on $\Omega$ if and only if there exists $a \in \Omega$ such that for all $b \in \Omega$ there exists $g \in G$ with $a^g = b$.*

*(d) $G$ acts transitively on $G$ if and only if there exists $a \in \Omega$ such that for all $b \in \Omega$ there exists a unique $g \in G$ with $a^g = b$.*

*Proof.* (a): Follows immediately from the definitions.

(b) Note that $G_a = 1$ if and only if and only if the exists a unique element $g \in G$ with $a^g = 1$ (namely $g = 1$). So if $G$ acts semiregularly on $\Omega$, then $G_a = 1$ for all $a \in \Omega$.

Suppose now that $G_a = 1$ for all $a \in \Omega$. Let $a, b \in G$ and $g, h \in G$ with $a^g = b = a^h$. Then $a^{gh^{-1}} = a$, $gh^{-1} \in G_a$ and $g = h$. So $G$ acts regularly on $\Omega$.

(c) The forward direction is obvious. If $a^G = \Omega$, then $\Omega$ is an orbit for $G$ on $\Omega$. Thus any two elements of $\Omega$ are $\sim$-equivalent and so $G$ act transitively on $\Omega$. (d) The forward direction is obvious. Suppose now $a \in \Omega$ and for each $b$ there exists a unique $g \in G$ with $a^g = b$. For $b = a$ we see that $G_a = 1$. Thus $G_b = G_{a^g} = G_a^g = 1$. (c) and (b) now show that $G$ acts transitively and semiregularly on $\Omega$. So by (a), $G$ acts regularly on $\Omega$. □

**Lemma 1.1.12.** *Let $G$ be a group acting on a set $\Omega$. Then the map*

$$\Omega \times G/C_G(\Omega) \to \Omega, (\omega, C_G(\Omega)g) \to \omega^g$$

*is a well-defined, faithful action of $G/C_G(\Omega)$ on $\Omega$.*

*Proof.* Readily verified. □

**Lemma 1.1.13.** *Let $G$ be a group. The the following are equivalent:*

*(a) All subgroups of $G$ are normal in $G$.*

*(b) Whenever $G$ acts transitively on a set $\Omega$,then $G/C_G(\Omega)$ acts regularly on $\Omega$.*

*Proof.* Let $\Omega$ be a set on which $G$ acts transitively and put $\overline{G} = G/C_G(\Omega)$. Note that $\overline{G}_\omega = \overline{G_\omega}$ for all $\omega \in \Omega$. Then $\overline{G}$ acts regularly on $\Omega$ iff $\overline{G_\omega} = 1_{\overline{G}}$ and so iff $G_\omega = C_G(\Omega)$ for all $\omega$ in $\Omega$.

(a) $\Longrightarrow$ (b): Suppose all subgroups of $G$ are normal in $G$. Then $G_\omega = G_\omega^g = G_{\omega^g}$. Since $G$ acts transitively on $G$ this gives $G_\omega = G_\mu$ for all $\mu \in \Omega$ and so $G_\omega = C_G(\Omega)$. Thus $\overline{G}$ act regualrly on $G$.

(b) $\Longrightarrow$ (a): Suppose $G/C_G(\Omega)$ acts regularly on $\Omega$ for all transitive $G$-sets $\Omega$. Let $H \leq G$ and put $\Omega = G/H$. Let $\omega = H$ and note that $\omega \in \Omega$ and $G_\omega = H$. Thus $H = C_G(\Omega)$ and since $C_G(\Omega)$ is a normal subgroup of $G$, $H$ is normal in $G$. □

**Definition 1.1.14.** *Let $G$ and $H$ be groups. An actions of $G$ on $H$ is an action $\cdot$ of $G$ on the set $H$ such that*

$$(ab)^g = a^g b^g$$

*for all $a, b \in H$ and $g \in G$.*

If $G$ acts on the group $H$ and $g \in G$, then the map $\phi_g : H \to H, h \mapsto h^g$ is an automorphism of $H$. Hence we obtain an homomorphism $\phi : G \to \mathrm{Aut}(H), g \mapsto \phi_g$ and $G^{\cdot H}$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$. Conversely every homomorphism from $G$ to $Aut(G)$ gives rise to an action of $G$ on $H$.

**Example 1.1.15.** *Let $G$ be a group and $N$ a normal subgroup of $G$.*

*(a) $N \times G \to N, (n, g) \to g^{-1}ng$ is an action of $G$ on the group $N$.*

*(b) $G \times G \to G, (h, g) \to hg$ is not an action of $G$ on the group $G$ (unless $G = 1$).*

*(c) $G \times \mathrm{Aut}(G) \to G, (g, \alpha) \to g\alpha$ is an action of $Aut(G)$ on the group $G$.*

**Lemma 1.1.16** (Modular Law)**.** *Let $G$ be a group and $A$, $B$, and $U$ subsets of $G$. If $UB^{-1} \subseteq U$, then $U \cap AB = (A \cap U)B$.*

*Proof.* Let $u \in U \cap AB$. Then $u = ab$ for some $a \in A$ and $b \in B$. Since $UB^{-1} \subseteq U$, $a = ub^{-1} \in U$. Thus $a \in A \cap U$ and so $U \cap AB = (A \cap U)B$. □

**Definition 1.1.17.** *Let $K$ be a groups and $G$ and $H$ subgroups of $G$.*

*(a) $G$ is called a complement to $H$ in $K$ if $K = GH$ and $G \cap H = 1$.*

*(b) $K$ is called the internal semidirect product of $H$ by $G$ if $H$ is normal in $G$ and $G$ is a complement of $H$ in $G$.*

**Lemma 1.1.18.** *Let $G$ be a group, $H \leq G$ and $K_1$ and $K_2$ complements to $H$ in $G$.*

*(a) $G = HK_1$.*

*(b) If $K_1 \leq K_2$, then $K_1 = K_2$.*

*(c) $K_1$ and $K_2$ are conjugate under $G$ if and only if they are conjugate under $H$.*

*Proof.* (a) Let $g \in G$. Then $g^{-1} = kh$ for some $k \in K$ and $h \in H$. Thus $g = (g^{-1})^{-1} = (kh)^{-1} = h^{-1}k^{-1} \in HK_1$.

(b) Since $K_1 \leq K_2$ and $G = K_1H$, Dedekind 1.1.16 implies $K_2 = K_1(K_2 \cap H)$. As $K_2 \cap H = 1$ we infer that $K_1 = K_2$.

(c) Let $g \in G$ with $K_1^g = K_2$. Then $g = k_1h$ with $k_1 \in K_1$ and so $K_2 = K_1^g = K_1^{kh} = K_1^h$. $\square$

Suppose that $K$ is the internal direct product of $H$ be $G$. Then $G$ acts on $H$ by conjugation and every element in $K$ can be uniquely written as $gh$ with $g \in G$ and $h$ in $H$. Moreover

$$(g_1h_1)(g_2h_2) = g_1g_2g_2^{-1}h_1g_2h_2 = (g_1g_2)(h_1^{g_2})h_2$$

This leads to the following definition.

**Definition 1.1.19.** *Let $G$ be a group acting on the group $H$. Then $G \ltimes H$ is the set $G \times H$ together with the binary operation*

$$(G \times H) \times (G \times H) \to (G \times H)$$
$$(g_1, h_1), (g_2, h_2 \mapsto (g_1g_2, h_1^{g_2}h_2)$$

**Lemma 1.1.20.** *Let $G$ be a group acting on the group $H$ and put $G^* = \{(g, 1) \mid g \in G\}$ and $H^* = \{(1, h) \mid h \in H\}$.*

*(a) $G \ltimes H$ is a group.*

*(b) The map $G \to G \ltimes H, g \to (g, 1)$ is an injective homomorphism with image $G^*$.*

*(c) The map $H \to G \ltimes H, h \to (1, h)$ is an injective homomorphism with image $H^*$.*

*(d) $(1, h)^{(g,1)} = (1, h^g)$ for all $g \in G$, $h \in H$.*

*(e) $G \rtimes H$ is the internal semidirect product of $H^*$ by $G^*$.*

*Proof.* (a) Clearly $(1, 1)$ is an identity. We have

$$((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1g_2, h_1^{g_2}h_1)(g_3, h_3) = (g_1g_2g_3, (h_1^{g_2}h_2))^{g_3}h_3) =$$
$$(g_1g_2g_3, (h_1^{g_2})^{g_3}h_2^{g_3}h_3) = (g_1g_2g_3, h_1^{g_2g_3}h_2^{g_3}h_1) = (g_1, h_1)(g_2g_3, h_2^{g_3}h_3) = (h_1, g_1)((h_2, g_2)(h_3, g_3))$$

and so the multiplication is associative.

We have $(g, h)(x, y) = (1, 1)$ iff $gx = 1$ and $h^xy = 1$ iff $x = g^{-1}$ and $y = (h^x)^{-1} = h^{-1})^x$ and so iff $x = g^{-1}$ and $y = (h^{-1})g^{-1}$. Thus the inverse of $(g, h)$ is $(g^{-1}, h^{-1})^{g^{-1}}$.

(b) Since for any $g \in G$, the map $H \to H, h \to h^g$ is a homomorphism, we have $1^g = 1$. Thus $(g_1, 1)(g_2, 1) = (g_1g_2, 1^{g_2}1) = (g_1g_2, 1)$.

(c) $(1, h_1)(1, h_2) = (1, h_1^1 h_2) = (1, h_1h_2)$.

(d) $(1, h)^{(g,1)} = (g^{-1}, 1)(1, h)(g, 1) = (g^{-1}, h)(g, 1) = (g^{-1}g, h^g 1) = (1, h^g)$.

(e) Clearly $G^* \cap H^* = 1$. Since $(g, 1)(1, h) = (g, h)$, $G \ltimes H = G^*H^*$ and so $G^*$ is a complement to $H^*$ in $G \ltimes H$. By (d), $G^*$ normalizes $H^*$ and so $H^*$ is normal in $G^*H^* = G \ltimes H$. □

As an example consider $H = C_2 \times C_2$ and $G = \mathrm{Aut}(H)$. Note that $G \cong \mathrm{Sym}(3)$. We claim that $G \ltimes H$ is isomorphic to Sym(4). Sym(4) has $H^* := \{1, (12)(34), (13)(24), (14)(23)\}$ has a normal subgroup and $H^* \cong C_2 \times C_2$. Observe that $H^*$ acts regularly on $\Omega = \{1, 2, 3, 4\}$ and so by 1.1.8, $G^* := \mathrm{Sym}(4)_4$ is a complement to $H^*$ in Sym(4). Also $G^* \cong \mathrm{Sym}(3) \cong \mathrm{Aut}(H^*)$ and $C^*_G(H^*) = 1$. It is now not to difficult to verify that $G \rtimes H \cong G^* \rtimes H^* \cong \mathrm{Sym}(4)$.

**Lemma 1.1.21.** *Let $G$ be a group, and $K, H \leq G$ with $G = KH$. $G$. Let $U \leq G$ with $K \leq U$. Then*

*(a) $U = K(H \cap U)$. In particular, if $H$ is a complement to $K$ in $G$, then $(H \cap U)$ is a complement to $K$ in $U$.*

*(b) The map $\alpha : G/U \to H/H \cap U, X \to X \cap H$ is a well defined bijection with inverse $\beta : H/H \cap U \to G/U, Y \to KY$.*

*(c) Let $T \subseteq H$. Then $T$ is a transversal to $H \cap U$ in $H$ if and only if $T$ is a transversal to $U$ in $G$.*

*Proof.* (a) By 1.1.16 since $UK^{-1} = K$, $U = U \cap G = U \cap KH = K(U \cap H)$. Also if $K \cap H = 1$ we get $(U \cap H) \cap K = U \cap (H \cap K) = 1$.

(b) Let $X$ be a coset of $U$ in $G$. Then $K^{-1}X \subseteq UX = X$ and so by 1.1.16, $X = X \cap KH = K(X \cap H)$. In particular, there exists $h \in X \cap H$ and so $X \cap H = Uh \cap H = (U \cap H)h$. Hence $X \cap H$ is indeed a coset of $X \cap H$ and $\alpha$ is well defined. Moreover, $X = K(X \cap H)$ means that $\alpha \circ \beta$ is the identity on $G/U$.

Now let $Y = (U \cap H)y \in H/U \cap H$. Then $KY = K(U \cap H)u = Uy$ and so $KY$ is a coset of $U$ in $G$ and $\beta$ is well defined. Moreover, since $Hy^{-1} = H$, 1.1.16 gives $KY \cap H = Uy \cap H = (U \cap H)y = Y$ and so $\beta \circ \alpha$ is the identity on $H/H \cap U$.

(c) $T$ is a transversal to $U$ in $G$ if and only if $|T \cap X| = 1$ for all $X \in G/U$. Since $T \subseteq H$, this holds if and only if $T \cap (X \cap H) = 1$ for all $X \in G/U$. By (b), the latter is equivalent to $|T \cap Y| = 1$ for all $Y \in H/H \cap U$ and so equivalent to $T$ being a transversal to $H \cap U$ in $U$. □

## 1.2 Complements

**Lemma 1.2.1.** *Let $G$ be a group, $K \leq G$ and $U \trianglelefteq K$ such that $K/U$ is Abelian and $G/K$ is finite. Let $\mathcal{S}$ be the set of transversals to $K$ in $G$. (A transversal to $K$ in $G$ is a subset*

*$T$ of $G$ such $|T \cap C| = 1$ for all $C \in G/K$.). For $R, S \in \mathcal{S}$ define*

$$R \mid S := \prod_{\substack{(r,s)\in R\times S\\ Kr=Ks}} rs^{-1}U$$

*Then for all $R, S, T \in \mathcal{S}$,*

*(a) $R \mid S$ is an elements of $K/U$ and independent of the order of multiplication.*

*(b) $(R \mid S)^{-1} = S \mid R$.*

*(c) $R \mid S)(S \mid T) = R \mid T$.*

*(d) The relation $\sim$ on $\mathcal{S}$ defined by $R \sim S$ if $R \mid S = U = 1_{K/U}$ is an equivalence relation.*

*(e) $G$ acts on $\mathcal{S}$ be right multiplication.*

*(f) $N_G(K)^{\mathrm{op}}$ acts on $\mathcal{S}$ by left multiplication.*

*(g) Let $x \in N_G(K) \cap N_G(U)$. Then $xR \mid xS = x(R \mid S)x^{-1}$. In particular, $\sim$ is $N_G(K) \cap N_G(U)$-invariant with respect to the action by left multiplication.*

*(h) Put $m = G/K$. Then $kR \mid S = k^m(R \mid S)$.*

*Proof.* (a) If $Kr = Ks$, then $rs^{-1} \in K$ and so $R \mid S \in K/U$. Since $K/U$ is abelian, the product is independent of the order of multiplication. (b) Follows from $(rs^{-1})^{-1} = sr^{-1}$.

(c) Follows from $(rs^{-1})(st^{-1}) = rt^{-1}$.

(d) Since $rr^{-1} = 1$, $R \mid R = 1U = U$ and so $\sim$ is reflexive. If $R \mid S = 1_{K/U}$, then by (b), $S \mid R = (R \mid S)^{-1} = 1_{K/U}^{-1} = 1_{K/U}$. So $\sim$ is symmetric. If $R \sim S$ and $S \sim T$, then by (c),

$$R \mid T = (R \mid S)(S \mid T) = 1_{K/U} \cdot 1_{K/U} = 1_{K/U}$$

and so $\sim$ is transitive.

(e) Let $g \in G$ and $C \in G/K$. Then $|Sg \cap C| = |S \cap Cg^{-1}| = 1$ and so $Sg \in \mathcal{S}$.

(f) Let $g \in N_G(K)$ and $C = Kh \in G/K$. Then $|gS \cap Kh| = |S \cap g^{-1}Kh| = |S \cap Kg^{-1}h| = 1$ and so $gS \in \mathcal{S}$.

(g) Let $r \in R$, $s \in S$ and $x \in N_G(K) \cap N_G(U)$. Then $Kr = Ks$ iff $xKr = xKS$ and iff $K(xr) = K(xr)$. Thus

$$xR \mid xS = \prod_{\substack{(r,s)\in R\times S\\ Kxr=Kxs}} (xr)(xs)^{-1}U = x\left(\prod_{\substack{(r,s)\in R\times S\\ Kr=Ks}} rs^{-1}\right)x^{-1}U$$
$$= x\left(\prod_{\substack{(r,s)\in R\times S\\ Kr=Ks}} rs^{-1}\right)Ux^{-1} = x(R \mid S)x^{-1}$$

(h) Let $k \in K$. Then $Kkr = Kr$ and so $Kr = Ks$ if and only if $K(kr) = Ks$. Thus

$$kR \mid S = \prod_{\substack{(r,s)\in R\times S\\ Kkr=Ks}} (kr)s^{-1}U = \left( \prod_{\substack{(r,s)\in R\times S\\ Kr=Ks}} krs^{-1} \right) U$$

$$= k^m \left( \prod_{\substack{(r,s)\in R\times S\\ Kr=Ks}} rs^{-1} \right) U = k^m (R \mid S)$$

□

**Theorem 1.2.2** (Schur-Zassenhaus)**.** *Let $G$ be a group and $K$ an Abelian normal subgroup of $G$. Suppose that $m := |G/K|$ is finite and that one of the following holds:*

*2 $K$ is finite and* $\gcd |K||G/K| = 1$.

*2 The map $\alpha : K \to K, k \to k^m$ is a bijection.*

*Then there exists a complement to $K$ in $G$ and any two such complements are conjugate under $K$.*

*Proof.* Suppose that (?) holds. Observe that $\alpha$ is a homomorphism. Also if $k \in K$ with $k^m = 1$, then $|k|$ divides $|K|$ and $m$ and so $|k| = 1$ and $k = 1$. Thus $\alpha$ is injective. Since $K$ is finite we conclude that $\alpha$ is a bijection.

So (?) implies (?) and we may assume that (?) holds.

We know apply 1.2.1 with $U = 1$. Since $K \trianglelefteq G$, we have $G = N_G(K) = N_G(K) \cap N_G(U)$. Thus by 1.2.1(f),g, $G^{\mathrm{op}}$ acts on $\mathcal{S}/\sim$ via left multiplication. Let $R, S \in \mathcal{S}$ and $k \in K$. Then

$$[kR] = S \iff (kR) \mid S = 1 \iff k^m (R \mid S) = 1 \iff k^m = (R \mid S)^{-1}$$

Since $\alpha$ is a bijection, for any $R, S \in \mathcal{S}$ there exists a unique such $k \in K$. Thus $K$ acts regularly on $\mathcal{S}/\sim$ and so by the Frattini argument, $G_{[R]}$ is a complement to $K$.

Now let $H$ be any complements of $K$ in $G$. Then $H$ is a transversal in $K$ in $G$ and so $H \in \mathcal{S}$. Since $hH = H$ for all $h \in H$ we conclude that $H \leq G_{[H]}$. Now both $H$ and $G_{[H]}$ are complements to $K$ in $G$ and 1.15.7(b) implies that $H = G_{[H]}$. Thus the map $[S] \to G_{[S]}$ is a $G$-isomorphism from $\mathcal{S}/\sim$ to the set of complements to $K$ in $G$. Since $K$ acts transitively on $\mathcal{S}/\sim$ it also acts transitively on the set of complements to $K$ in $G$. □

**Theorem 1.2.3** (Gaschütz)**.** *Let $G$ be a group, $K$ an Abelian normal subgroup of $G$ and $K \leq U \leq G$. Suppose that $m := |G/U|$ is finite and that one of the following holds:*

*2 $K$ is finite and* $\gcd |K||G/U| = 1$.

*2 The map $\alpha : K \to K, k \to k^m$ is a bijection.*

*Then*

*(a) There exists a complement to $K$ in $G$ if and only if there exists a complement to $K$ in $G$.*

*(b) Let $H_1$ and $H_2$ be complements of $K$ in $G$. Then $H_1$ and $H_2$ are conjugate in $G$ of and only if $H_1 \cap U$ and $H_2 \cap U$ are conjugate in $U$.*

*Proof.* (a) '$\Longrightarrow$:' Let $H$ be a complement to $K$ in $G$. Then by 1.1.21, $H \cap U$ is a complement to $K$ in $U$

(b) '$\Longrightarrow$:' Suppose $H_1$ and $H_2$ are conjugate in $G$. Then by 1.1.18(c), $H_2 = H_1^k$ for some $k \in K$. Since $k \in U$ we get $(H_1 \cap U)^k = H_1^k \cap U^k = H_2 \cap U$ and so $H_1 \cap U$ and $H_2 \cap U$ are conjugate in $U$.

(a) '$\Longleftarrow$:' Let $\mathcal{B}$ be the set of left transversals to $U$ in $G$ and fix $S_0$ in $\mathcal{B}$

Suppose that there exists a complement $A$ to $K$ in $U$. Let $g \in G$. We claim that

**1°.** *There exists uniquely determined $s_g \in \mathcal{S}_0, k_h \in K$ and $a_g \in A$ with $g = s_g k_g a_g$.*

Indeed since $S_0$ is a left transversal to $U$ there exists uniquely determined $s_g \in S_0$ and $u_g \in U$ with $g = s_g u_g$. Since $A$ is a complement to $K$ in $U$, there exists uniquely determined $k_g \in K$ and $a_g \in A$ with $u_g = k_g a_g$.

**2°.** *Put $g_0 = s_g k_g$. Then $g_0$ is the unique element of $S_0 K$ with $gA = g_0 A$.*

If $g_1 \in S_0 K$ with $gA = g_1 A$, then $g_1 = s_1 k_1$ and $g = g_1 a_1$ for some $s_1 \in S_0, k_1 \in K$ and $a_1 \in A$. Then $g = g_1 k_1 a_1$ and so $g_1 = g_0$ by (1°).

Define $\mathcal{S} = \{T \in \mathcal{B} \mid T \subseteq S_0 K\}$. For $T \subseteq G$ put $T_0 := \{t_0 \mid t \in T\}$

**3°.** *If $L \in \mathcal{B}$, then $L_0$ is the unique element of $\mathcal{S}$ with $LA = L_0 A$.*

Since $l_0 A = lA$ we have $lU = lU$ and so $L_0$ is a left transversal to $U$. Moreover, $L_0 A = LA$ and so $L_0 \in \mathcal{S}$. Now suppose $L_1 \in \mathcal{S}$ with $LA = L_1 A$. Then for each $l \in L$ there exists $l_1 \in L_1$ with $lA = l_1 A$. Since $l_1 \in L_1 \subseteq S_0 K$ we conclude that $l_1 = l_0$. Thus $L_0 \subseteq L_1$ and since $L_0$ and $L_1$ are both transversals to $U$ we conclude that $L_0 = L_1$.

Let $x \in G$ and $T \subseteq G$ define $x * T := (xT)_0$.

**4°.** *Let $L \in \mathcal{B}$. Then $x * L \in \mathcal{S}$ and $x * L = x * L_0$.*

Since $xL$ is a left transversal to $U$ in $G$, (2°) implies that $x * L \in \mathcal{S}$. Now

$$(x * L)A = (xL)_0 A = xLA = x(LA) = x(L_0 A) = (xL_0)A = (xL_0)_0 A = (x * L_0)A$$

Since both $x*$ and $x * L_0$ are contained in $\mathcal{S}$, the uniqueness statement in (3°) shows that $x * L = x * L_0$.

**5°.** *$\mathcal{S} \times G \to \mathcal{S}, (L, x) \to x * L$ is an action of $G^{\text{op}}$ on $\mathcal{S}$.*

By (4°), $x * L \in \mathcal{S}$. Since $L = L_0$ for all $L \in \mathcal{S}$ we have $1 * L = (1L)_0 = L_0 = L$. Let $x, y \in G$. Then

$$x * (y * L) = x * (yL)_0 \overset{(4^\circ)}{=} x * yL = (x(yL))_0 = ((xy)L)_0 = xy * L$$

and so $*$ is indeed an action.

For $R, S \in \mathcal{S}$ define

$$R \mid S := \prod_{\substack{(r,s) \in R \times S \\ rU = sU}} rs^{-1}$$

Since $RK = S_0K = SK$ for each $r \in R$ there exists $s \in S$ with $rK = sK$. Then also $rU = sU$. Since $K$ is normal in $H$, $Kr = Ks$ and $rs^{-1} \in K$. Hence $R \mid S \in K$ and since $K$ is Abelian, the definition of $R \mid S$ does not depend on the chosen order of multiplication. Define the relation $\sim$ in $\mathcal{S}$ by $R \sim S$ if $R \mid S = 1$. As in 1.2.1

**6°.** *$\sim$ is an equivalence relation on $\mathcal{S}$.*

Next we show

**7°.** *$k * S = kS$ for all $k \in K$, $S \in \mathcal{S}$.*

Indeed

$$kSK = kS_0K = kKS_0 = KS_0 = S_0K$$

and so $kS \in \mathcal{S}$. As $kSA = kSA$, the definition of $(kS)_0$ implies $(kS)_0 = kS$ and so (7°) holds.

**8°.** *$x * R \mid x * S = x(R \mid S)x^{-1}$ for all $x \in G$ and $R, S \in \mathcal{S}$.*

Let $r \in R$. Since $R, S \in \mathcal{S}$ we have $RK = S_0K = SK$ and so $r = sk$ for some $s \in S$ and $K$. Then $s$ is the unique element of $S$ with $rU = sU/$ Note that $xr = xsk$ and $xrU = xsU$. We have $xs = xsk = (s_{xs}k_{xs}a_{ks}k = s_{ks}(k_{xs}k^{a_{ks}^{-1}})a_{ks}$ and so $a_{rs} = a_{ks}$.

Also $(xr)_0 = s_{xr}k_{xr} = (xr)a_{xr}^{-1} = (xr)a_{xs}^{-1}$, $(xs)_0 = (xs)a_{xs}^{-1}$ and $(xr)_0U = xrU = xsU = (xs)_0U$. Thus

$$\begin{aligned}
x * R \mid x * S &= \prod_{\substack{(\tilde{r},\tilde{s}) \in ((xR)_0,(xS)_0) \\ \tilde{r}U = \tilde{s}U}} \tilde{r}\tilde{s}^{-1} &&= \prod_{\substack{(r,s) \in (R,S) \\ rU = sU}} (xr)_0(xs)_0^{-1} \\
&= \prod_{\substack{(r,s) \in (R,S) \\ rU = sU}} (xra_{xs}^{-1})(xsa_{xs}^{-1})^{-1} &&= \prod_{\substack{(r,s) \in (R,S) \\ rU = sU}} xra_{xs}^{-1}a_{xs}s^{-1}x^{-1} \\
&= x\left(\prod_{\substack{(r,s) \in (R,S) \\ rU = sU}} rs^{-1}\right)x^{-1} &&= x(R \mid S)x^{-1}
\end{aligned}$$

**9°.** *$\sim$ is $G$-invariant and $K$ acts regularly on $\mathcal{S}/modsim$.*

In view of (7°) and (8°) this follows as in the proof of 1.2.2

From (9°) and 1.1.8 we conclude that there exists a complement to $K$ in $G$.

(b) '$\Longleftarrow$': Let $H_0$ and $H_1$ be complements to $K$ in $G$ such that $H_0 \cap U$ is conjugate to $H_1 \cap U$ in $U$. Then $H_1 \cap U = (H_0 \cap U)^u = H_0^u \cap U$ for some $u \in U$. If $H_1$ is conjugate to $H_0^u$ in $G$, then $H_1$ is also conjugate to $H_0$ in $G$. So it suffices to show that $H_1$ and $H_0^u$ are conjugate in $G$ and replacing $H_0$ by $H_1$ we may assume that $H_1 \cap U = H_0 \cap U$. Put $A = H_0 \cap U$. Then $A$ is a complement to $K$ in $U$.

**10°.** *Let $T \subseteq H_i$. Then $T$ is a left transversal to $A$ in $H_i$ if and only if $T$ is a left transversal to $U$ in $G$.*

This follows from 1.1.21(c).

Let $S_0$ be a left transversal to $A$ in $H_0$. Then by (10°), $S_0$ is left transversal to $U$ in $G$ and we can use it to define the set $\mathcal{S}$. Let $s \in S_0$. Since $G = H_1K$, $s = hk$ for some $h \in H_1$ and $k \in K$. Then $sk^{-1} = h \in H_1$. Put $l_s = k^{-1}$ and note that $sk_s \in H_1$. Put $S_1 = \{sk_s \mid s \in S_0$. Since $k_s \in U$ and $S_0$ is a left transversal to $U$ in $G$, also $S_1$ is a left transversal to $U$ in $G$. So by (10°), $S_1$ is a left transversal to $A$ in $H_1$. Moreover, $S_1 \subseteq S_0K$ and so $S_1 \in \mathcal{S}$.

**11°.** *Let $L_i$ be a left transversal to $A$ in $H_i$. Then $(L_i)_0 = S_i$*

We have $L_iA = H_i = S_iA$ and so $S_i = (L_i)_0$ by (3°).

Let $x \in H_i$. Then $xS_i$ is a left transversal to $A$ in $H_i$ and so by (11°), $x{*}S_i = (xS_i)_0 = S_i$. Thus $H_i \leq G_{[S_i]}$ and hence by 1.1.18(b), $H_i = G_{[S_i]}$. Since $K$ acts transitively on $\mathcal{S}//\sim$ there exists $k \in K$ with $[k * S_0] = [S_1]$. Then $H_0^k = G_{[S_0]}^k = G_{[k*S_0]} = G_{[S_0]} = H_1$ and so $H_0$ and $H_1$ are conjugate in $G$. □

In the proof of the Gaschütz Theorem we used a complement $A$ to $K$ in $U$ to find a complement $H$ to $K$ in $G$. Then of course $H \cap U$ is a complement to $U$ in $K$. But these operation are not inverse to each other, that is $H \cap U$ can be different from $A$ and might not even be conjugate to $A$ in $U$. In fact, there are examples where there does not exist any complement $\tilde{H}$ to $K$ in $G$ with $A = U \cap \tilde{H}$.

Suppose now that we start with a complement $H$ to $K$ in $G$, then the proof of '(b) $\Longrightarrow$' shows that the complement to $U$ in $G$ constructed from the complement $H \cap U$ to $K$ in $U$, is conjugate to $H$, as long as one choose the left transversal $S_0$ to be contained in $H$.

## 1.3 Frobenius Groups

**Lemma 1.3.1.** *Let $G$ be a group acting on a set $\Omega$ and $N$ a normal subgroup of $G$ acting regularly on $\Omega$. Fix $a \in \Omega$ and for $b \in \Omega$, let $n_b$ be the unique element of $N$ with $a^{n_b} = b$. Then for all $b \in \Omega$ and $g \in G_a$*

$$(n_b)^g = n_{b^g}$$

*Thus the action of $G_a$ on $\Omega$ is isomorphic to the action of $G_a$ on $N$, and the action of $G_a$ on $\Omega \setminus \{a\}$ is isomorphic to the action of $G_a$ on $N^\sharp$.*

*Proof.* Since $g \in G_a$, also $g^{-1} \in G_a$ and so $a^{g^{-1}} = a$. Thus

$$a^{n_b^g} = a^{g^{-1} n_b g} = a^{n_b g} = b^g$$

and so $n_b^g = n_{b^g}$. □

**Definition 1.3.2.** *Let $G$ be a group acting on a set $\Omega$. We say that $G$ is a Frobenius group on $\Omega$ if*

*(a) $G$ acts faithfully and transitively on $\Omega$.*

*(b) $G$ does no act regularly on $\Omega$.*

*(c) For all $a \in \Omega$, $G_a$ acts semi-regularly on $\Omega \setminus \{a\}$.*

*$K\sharp_G(\Omega)$ consists of all the $g \in G$ such that $\langle g \rangle$ acts semi-regularly on $\Omega$.*

**Lemma 1.3.3.** *Let $G$ be a group, $H \leq G$ and put $\Omega = G/H$. Then the following are equivalent:*

*(a) $G$ is a Frobenius group on $\Omega$.*

*(b) $1 \neq H \neq G$ and $H \cap H^g = 1$ for all $g \in G \setminus H$.*

*In this case $K_G^\sharp(\Omega) = \{g \in G^\sharp \mid C_\Omega(g) = \emptyset\} = G \setminus \bigcup H^G =$*

*Proof.* (a) $\Longrightarrow$ (b): If $H = 1$, then $G$ acts regularly on $\Omega$, contrary to the definition of a Frobenius group on $\Omega$. So $H \neq 1$ and in particular, $G \neq 1$. If $G = H$, then $G$ acts trivially and so not faithfully on $\Omega$. Thus $H \neq G$. Let $g \in G \setminus H$. Then $H \neq Hg$. Since $H = G_H$ acts semiregularly on $\Omega \setminus \{H\}$ we conclude that $H \cap H^g = G_H \cap G_{Hg} = 1$.

(a) $\Longrightarrow$ (b): $H \neq 1$, $G$ does not act regularly on $\Omega$. Since $H \neq G$, there exists $g \in G \setminus H$. Hence $C_G(\Omega) \leq G_H \cap G_{Hg} = H \cap H^g = 1$ and $G$ acts faithfully on $\Omega$. Let $Ha, Hb \in \Omega$ with $Ha \neq Hb$. Then $ab^{-1} \notin H$ and so

$G_{Ha} \cap G_{Hb} = H^a \cap H^b = (H^{ab^{-1}} \cap H)^b = 1^b = 1$. Thus $G_{Ha}$ acts semiregulary on $\Omega \setminus Ha$ and so $G$ is a Frobenius group on $G$.

Suppose now that (a) and let $1 \neq g \in K_\Omega(G)$. Since $\langle g \rangle$ acts semiregulary on $\Omega$, $\langle g \rangle \cap G_\omega = 1$ for all $\omega \in \Omega$ and so $g \notin G_\omega$ and $\omega \notin C_\Omega(g)$. Thus $K_G^\sharp(\Omega) subseteq \{g \in G^\sharp \mid C_\Omega(g) = \emptyset\}$.

Let $g \in G$ with $C_\Omega(g) = \emptyset$ and let $l \in G$. Then $Hl \notin C_\Omega(g)$ and so $g \notin G_{Hl} = H^l$. Thus $\{g \in G^\sharp \mid C_\Omega(g) = \emptyset\} = G \setminus \bigcup H^G$.

Let $g \in G \setminus \bigcup H^G$. Since $1 \in H$, $g \neq 1$. Let $\omega \in \Omega$. Since $G_\omega \in H^G$ we have $g \notin G_\omega$. Hence $\omega \neq \omega^g$ and so $G_\omega \cap G_\omega^g = 1$. Since $g$ normalizes every subgroup of $\langle g \rangle$.

$$\langle g \rangle \cap G_\omega = (\langle g \rangle \cap G_\omega)^g \leq G_{\omega^g}$$

and so $\langle g \rangle \cap G_\omega = 1$. Thus $\langle g \rangle$ acts semiregularly on $\Omega$ and $g \in K_G^\sharp(\Omega)$. □

**Definition 1.3.4.** *Let $G$ be group and $H$ a subgroup of $G$. Put $K_G(H) = G \setminus \bigcup H^{\sharp G}$. We say that $G$ is a Frobenius group with Frobenius complement $H$ and Frobenius kernel $K_G(H)$ if*

*(a)* $1 \neq H \neq G$.

*(b)* $H \cap H^g = 1$ *for all* $g \in G$.

**Theorem 1.3.5** (Frobenius)**.** *Let $G$ be a finite Frobenius group with complement $H$ and kernel $K$. Then $G$ is the internal semidirect product of $K$ by $H$.*

We will proof this theorem only in the case that $|H|$ has even order. Currently all the proves available for Frobenius Theorem use character theory.

**Lemma 1.3.6.** *Let $G$ be a finite Frobenius group with complement $H$ and kernel $K$. Put $\Omega = |G/H|$.*

*(a) Let $g \in G \setminus K$. Then $g$ has a unique fixed point on $\Omega$*

*(b)* $|\Omega = |G/H| = |K|$ *and* $|K| \equiv 1 \mod |H|$.

*(c)* $G|/ \setminus K| \geq \frac{|G|}{2} > \frac{|G^\sharp|}{2}$.

*Proof.* (a) Since $g \notin K$, $g \neq 1$ and $g \in H^l$ for some $l \in G$. Thus $g$ fixes $Hl$. Since $G_{Hl}$ acts semiregulary on $\Omega \setminus \langle Hl \rangle$, $Hl$ is the only fix-point of $g$ on $\Omega$.

(b) Clearly $\Omega = |G/H|$. Since $H$ acts semiregularly on $\Omega \setminus \{H\}$, all orbits of $H$ on $\Omega \setminus \{H\}$ are regular and so have length $|H|$. Thus $|\Omega| \equiv 1 \mod |H|$.

From $H \cap H^g = 1$ for all $g \in G \setminus H$ we conclude that $N_G(H) = H$ and so $|H^G| = |G/N_G(H)| = |G/H|$. Moreover,$H^g \cap H^l = 1$ for all $g, l \in G$ with $H^g \neq H^l$ and so $\bigcup H^{\sharp G}$ is the disjoint union of $H^{\sharp g}, g \in G$. There are $|G/H|$ such conjugates and each conjugate as $|H| - 1$ elements.

Thus $|\bigcup H^{\sharp G}| = |G/H|)|H| - 1 = |G| - |G/H|$. Since $K = G \setminus \bigcup H^{\sharp G}$ this gives $|K| = |G/H$.

(c) Since $H \neq 1$, $|H| \geq 2$ and so $|K| = |G/H| \leq \frac{|G|}{2}$. This implies (c). □

**Lemma 1.3.7.** *Let $G$ be a finite Frobenius group with complement $H$ and kernel $K$.*

*(a) Let $U \leq G$. Then one of the following holds:*

1. $H \cap U = 1$.
2. $U \leq H$.
3. *$U$ is Frobenius groups with complement $U \cap H$ and kernel $U \cap K$.*

*(b) Let $H_0$ be any Frobenius complement of $G$. Then there exists $g \in G$ with $H \leq H_0^g$ or $H_0 \leq H^g$.*

*Proof.* (a) We may assume that $H\cap U\neq 1$ and $H\not\leq U$. Then $H\cap U\neq U$. If $g\in U\setminus(U\cap H)$, then $g\notin H$ and so $(U\cap H)\cap(U\cap H)^g\leq H\cap H^g=1$. Thus $U$ is a Frobenius group with complement $U\cap H$ and kernel (say) $\tilde{K}$. Note that $(U\cap K)\cap(U\cap H)^g\leq K\cap H^g=1$ for all $g\in U$ and so $U\cap K\subseteq\tilde{K}$. Let $u\in U\setminus(U\cap K)$. Then $u\in H^g$ for some $g\in U$. Since $u\neq 1$ we have $U\cap H^g\neq 1$. Suppose that $U\leq H^g$, then $1\neq U\cap H\leq H^g$ and so $H=H^g$ and $U\leq H$, a contradiction. Hence $U\neq H^g$ and as seen above $U\cap H^g$ is a Frobenius complement of $U$. From 1.3.6(c) applied to the Frobenius complements $U\cap H$ and $U\cap H^g$ of $U$,

$$|\bigcap(U^\sharp\cap H)^U|+|\bigcup(U^\#\cap H^g)^U|>\frac{|U^\sharp}{2}+\frac{|U^\sharp|}{2}=|U^\sharp|$$

Thus the two subsets $\bigcap(U^\sharp\cap H)^U$ and $\bigcap(U^\sharp\cap H^g)^U$ of $U$ cannot be disjoint and there exists $u_1,u_2\in U$ with $(U\cap H)^{u_1}\cap(U\cap H^g)^{u_2}\neq\emptyset$. Thus $H^{u_1}\cap H^{gu_2}\neq 1$. It follows that $H^{u_1}=H^{gu_2}$ and so $H^g=H^{u_1u_2^{-1}}\in H^U$. Thus $u\in\bigcup(U^\sharp U)^U$ and $u\notin\tilde{K}$. We proved that $U\cap K\setminus\tilde{K}$ and $(U\setminus K)\subseteq U\setminus\tilde{K}$. Thus $U\cap K=\tilde{K}$ and (a) holds.

(b) We may assume that $H_0\not\leq H^g$ for all $g\in G$. If $H_0\cap H^g\neq 1$ for some $g\in G$ then (a) shows that $H_0\cap K$ is a Frobenius kernel for $H_0$ and so $H_0\cap K\neq 1$. If $H_0\cap H^g=1$ for all $g\in G$, then $H_0\subseteq K$. So in any case $H_0\cap K\neq 1$. Put $m=|H_0^\sharp\cap K|$. Then $m$ is a positive integer.

Let $g\in G$. If $g\in H_0$, then $(H_0\cap K)^g=H_0\cap K$ and if $g\notin H_0$, then $(H_)\cap K)\cap(H_0\cap K)^g\leq H_0\cap H_0^g=1$. Thus $\bigcap(H_0^\sharp\cap K)^G$ is the disjoint union of $|G/H_0|$ sets, each of size $|H_0^\sharp\cap K|=m$. Thus

$$|\bigcap(H_0^\sharp\cap K)^G|=m|G/H_0|$$

and

$$|G/H|=|K|\geq|\bigcup(H_0\cap K)^G|\geq m|G/H_0|+1\geq|G/H_0|$$

Hence $|H|<|H_0|$. So if $H_0\not\leq H^g$ for all $g\in G$, then $|H|<|H_0|$. By symmetry, if $H\not\leq H_0^g$ for all $g\in G$, then $|H_0|<|H|$. This proves (b). □

**Theorem 1.3.8.** *Let $G$ be a finite Frobenius group with complement $H$ and kernel $K$. If $|H|$ is even, $G$ is the internal semidirect product of $K$ by $H$.*

*Proof.* Since $H$ has even order there exists $t\in H$ with $|t|=2$. We will first show that

**1°.** *$tt^g\in K^\sharp$ for all $g\in G$.*

Since $g\notin H$. $H\cap H^g=1$ and so $t^g\neq t^{-1}$. Thus $a:=tt^g\neq 1$. Observe that both $t$ and $t^g$ invert $a$. Suppose for a contradiction that $a\in H^x$ for some $x\in G$. Then $a=(a^t)^{-1}\in H^{xt}$ and similarly $a\in H^{xt^g}$. Thus $a\in H^x\cap H^{xt}\cap H^{xt^g}$ and so $H^x=H^{xt^g}$ and both $t$ and $t^g$ are contained in $H^x$. So $t\in H\cap H^x$ and $t^g\in H^g\cap H^x$. It follows that $H=H^x=H^g$, a contradiction to $g\notin H$. This completes the proof of (1°).

Let $S$ be a transversal to $H$ in $G$ with $1 \in S$. Let $r, s \in S$ with $tt^r = tt^s$, then $t^s = t^r \in H^r \cap H^s$. Thus $H^r = H^s$, $H^{rs^{-1}} = H$, $rs^{-1} \in H$ and $Hr = Hs$. Since $S$ is a tranversal, $r = s$. It follows that $|tt^S| = |S| = |G/H| = |K|$. Since $tt^1 = 1 \in K$ and $s \notin H$ for all $s \in S \setminus 1$, (1°) shows that $tt \subseteq K$. Hence $tt^S = K$. Since $K = K^{-1}$ we get $K = (tt^S)^{-1} = t^S t$. Let $k_1, k_2 \in K$. Then $k_1 = t^{s_1} t$ and $k_2 = tt^{s_2}$ for some $s_1, s_2 \in S$. Thus

$$k_1 k_2 = (t^{s_1} t t t^{s_2} = t^{s_1} t^{s_2} = (tt^{s_2 s_1^{-1}})^{s_1}$$

Note that either $s_2 = s_1$ or $s_2 s_1^{-1} \notin H$. Thus by (1°), $tt^{s_2 s_1^{-1}} \in K$. Since $K$ is invariant under conjugation, we conclude that $k_1 k_2 \in K$. Thus $K$ is closed under multiplication. Clearly $K$ is closed under inverses and under conjugation by elements of $G$. Hence $K$ is a normal subgroup of $G$. Since $K \cap H = 1$ and $|K||H| = |G/H||H| = |G|$ we get $G = HK$ and so $G$ is the internal semidirect product of $K$ by $H$. □

**Lemma 1.3.9.** *Let $G$ be the internal semidirect products of $K$ by $H$. Suppose that $K \neq 1$ and $H \neq 1$. Put $\Omega = G/H$. Then*

*(a) $K$ acts regularly in $\Omega$.*

*(b) $K_G^\sharp(H) = \{hk \mid h \in H, k \in K \setminus \{[h,l] \mid l \in K\}$. In particular, $K \subseteq K_G(H)$ and $K = K_G(H)$ if and only if for all $h \in H^\sharp$, $K = \{[h,l] \mid l \in K\}$.*

*(c) $h \in H$ acts fixed-point freely on $\Omega \setminus \{H\}$ if and only if $C_K(h) = 1$. In particular, $H$ is a Frobenius complement in $G$ if and only if $C_K(h) = 1$ for all $h \in H^\sharp$.*

*(d) If $G$ is finite, $H$ is a Frobenius complement if and only if $K = K_G(H)$.*

*(e) $K_G(H)$ is a subgroup of $G$ if and only if $K = K_G(H)$.*

*Proof.* (a) Since $K$ is normal in $G$ and $K \cap H = 1$ we have $K \cap H^g = 1$ for all $g \in G$ and so $K$ acts regularly on $\Omega$. Since $G = HK$, $K$ acts transitively on $\Omega$ and so (a) holds. (b) Let $g \in G^\sharp$. Then $g \in K_G(H)$ if and only if $g \neq a^r$ for all $a \in H^\sharp$ and $r \in G$. Since $G = HK$ and $H \cap K = 1$ there exists uniquely determined $k, l \in K$ and $h, b \in H$ with $g = hk$ and $r = bl$. Then

$$a^r = a^{bl} = (a^b)^l = a^b (a^b)^{-1} l = a^b [a^b, l]$$

Thus $g = a^r$ if and only if $h = a^b$ and $k = [a^b, l] = [h, l]$. So $hk \notin K_G(H)$ if and only if $k = [h, l]$ for some $l \in K$. Hence (b) holds.

(c) By (a) $K$ acts regularly on $\Omega$. So by 1.3.1, the action of $H$ on $\Omega \setminus \{H\}$ is isomorphic to the action of $H$ on $K^\sharp$. This implies (c).

(d) Let $h \in H$. Then the map $K/C_K(h) \to \{[h, l \mid l \in L\}$, $C_K(h)l \to [k, l]$ is a well-defined bijection. Hence $C_K(h) = 1$ if and only if $|K/C_K(h)| = |K|$ and if and only if $\{[h, l] \mid l \in L\} = K$. So (d) holds.

(e) Suppose that $K_G(H)$ is a subgroup of $G$. Since $K \leq K_G(H)$ and $G = HK$ we have $K_G(H) = (K_G(H) \cap H)K = 1K = K$. □

**Example 1.3.10.** *(a) Let $K$ be a non-trivial Abelian group with no elements of order 2. Let $t \in \mathrm{Aut}(A)$ defined by $a^t = a^{-1}$ for all $a \in A$. Let $H = \langle t \rangle$ and let $G$ be the semidirect product of $K$ by $H$. Then $G$ is a Frobenius group with complement $H$. $K_G(H)$ is a subgroup of $G$ if and only if $K = \{k^2 \mid k \in K\}$. In particular, the infinte dihedral group is an example of a Frobenius group, where the kernel is not a subgroup.*

*(b) Let $\mathbb{K}$ be a field, $K = (\mathbb{K}, +)$ and $H = (\mathbb{K}^\sharp, \cdot)$. Then $H$ acts on $K$ via right multiplication and $H \ltimes K$ is a Frobenius group with complement $H$.*

*Proof.* (a) We have $a^t = a$ iff $a^{-1} = a$ iff $a^2 = 1$ iff $a = 1$. Thus $C_K(t) = 1$ and so $H$ is a Frobenius complement in $G$. Since $[t, l] = (l^{-1})^{t^{-1}} l = ll = l^2$, $K = K_G(H)$ if and only if $K = \{l^2 \mid l \in L$.

If $K \cong (\mathbb{Z}, +)$, then $G$ is the infinite dihedral group. Since $2\mathbb{Z} \neq \mathbb{Z}$ we conclude that $K_G(H)$ is not a subgroup of $G$.

(b) If $h \in H^\sharp$ and $k \in K\sharp$, then $0 \neq k$ and $0 \neq h \neq 1$. Thus $hk \neq h$ and so $C_K(h) = 0$. Moreover $[h, k] = (-k)(h^{-1}) + k = k(1 - h^{-1})$. Since $1 - h^{-1} \neq 0$, we conclude that every element of $K$ is of the form $[h, k]$ with $k \in K$ and so $K = K_G(H)$. □

## 1.4 Imprimitive action

**Definition 1.4.1.** *Let $G$ be a group acting on set $\Omega$.*

*(a) A system of imprimitivity for $G$ on $\Omega$ is a $G$-invariant set $\mathcal{B}$ of non-empty subsets of $\Omega$ with $A = \uplus \mathcal{B}$.*

*(b) A set of imprimitivity for $G$ on $\Omega$ is a subset $B$ of $\Omega$ such that $B = B^g$ or $B \cap B^g = \emptyset$ for all $g \in G$.*

*(c) . A system of imprimitivity $\mathcal{B}$ for $G$ on $\Omega$ is called proper if $\mathcal{B} \neq \{\Omega\}$ and $\mathcal{B} \neq \{\{\omega\} \mid \omega \in \Omega\}$.*

*(d) A set of imprimitivity $B$ is called proper if $|B| \geq 2$ and $B \neq \Omega$.*

*(e) $G$ acts primitively on $G$ if there does not exit a proper set of imprimitivity for $G$ on $\Omega$*

**Lemma 1.4.2.** *Let $G$ be a group acting on a set $\Omega$.*

*(a) Let $\mathcal{B}$ be a set of imprimitivity for $G$ on $\Omega$ and $B \in \mathcal{B}$. Then $B$ is a set of imprimitivity. $\mathcal{B}$ is proper if and only of $\mathcal{B}$ contains a proper set of imprimitivity.*

*(b) Let $B$ be set of imprimitivity for $G$ on $\Omega$. Define $\mathcal{B} = B^G$ if $\bigcup B^G = \Omega$ and $\mathcal{B} = B^G \cap \{\Omega \setminus \bigcup B^G$ otherwise. Then $\mathcal{B}$ is system of imprimitivity for $G$ on $\Omega$. $\mathcal{B}$ is proper if and only if either $B$ is proper or $|\Omega \setminus \bigcup B^G| \geq 2$..*

*(c) $G$ acts primitively on $\Omega$ if and only if there does not exit a proper set of imprimitivity for $G$ on $\Omega$,*

*Proof.* (a) Let $g \in G$. Then $B, B^g$ both are contained in $\mathcal{B}$ and so either $B = B^g$ or $B \cap B^g = \emptyset$. Thus $B$ is a set of imprimitivity. Observe $\mathcal{B} \neq \Omega$ iff an only if $B \neq \Omega$ for all $B \in \mathcal{B}$ and iff $B \neq \Omega$ for some $B \in \mathcal{B}$. Also $\mathcal{B} \neq \{\{\omega\} \mid \omega \in \Omega$ if and only if $\mathcal{B}$ contains an element $\mathcal{B}$ with $|B| \geq 1$. Thus $\mathcal{B}$ is prober if and only if it contains an element $B$ with $|B| \geq 2$ and $B \neq \Omega$.

(b) Clearly $\mathcal{B}$ is $G$ invariant. Since $B = B^g$ or $B \cap B^g = \emptyset$ for all $g \in G$ we have $\Omega = \cup\mathcal{B}$. $\mathcal{B}$ is proper iff if contains a proper set of imprimitivity, iff $B$ is proper or $\Omega \setminus \bigcup B^G$ is proper and iff $B$ is proper or $|\Omega \setminus \bigcup B^G| \geq 2$.

(c) Suppose $G$ is not imprimitive on $\Omega$ iff there exists a proper system of imprimitivity for $G$ on $\Omega$. By (a) and (b) this is the case if and only if the exists a proper set of imprimitivity for $G$ on $\Omega$. □

**Lemma 1.4.3.** *Suppose $G$ acts primitively on a set $\Omega$. Then either $G$ acts transitively on $\Omega$ or $|\Omega| = 2$ and $G$ acts trivially on $\Omega$.*

*Proof.* Suppose that $G$ does not act transitively on $\Omega$. Let $O$ be an orbit for $G$ on $\Omega$. Then $\{O, \Omega \setminus O\}$ is system of imprimitivity for $G$ on $\Omega$. Since $G$ acts primitively, $|O| = 1 = |\Omega \setminus O$. Thus $|\Omega| = 2$ and $G$ acts trivially on $\Omega$. □

**Lemma 1.4.4.** *Let $G$ be a group acting transitively on $\Omega$ and let $\omega \in \Omega$. Then the map*

$$H \to \omega^H$$

*is a bijection between the subgroups of $G$ containing $G_\omega$ and the sets of imprimitivity containing $\omega$. The inverse of this bijection is given by*

$$\Delta \to N_G(\Delta)$$

*Moreover, $\omega^H$ is a proper, if and only if $G_\omega \lneq H \lneq G$.*

*Proof.* Let $G_\omega \leq H \leq G$. We will first show that $\omega^H$ is indeed a set of imprimitivity. For this let $g \in G$ with $\omega^H \cap \omega^{Hg} \neq \emptyset$. Then $\omega^{h_1} = \omega^{h_2 g}$ for some $h_1, h_2 \in \Omega$. Hence $h_2 g h_1^{-1} \in G_\omega \leq H$ and so $\omega^{Hg} = \omega^H$.

Now let $\Delta$ be a set of imprimitivity with $\omega \in \Delta$. Since $\omega = \omega^g \leq \Delta \cap \Delta^g$ for all $g \in G_\omega$ we get $\Delta = \Delta^g$ and so $G_\omega \leq N_G(\Delta)$.

We showed that both of our maps are well-defined. Next we show that they are inverse to each other.

Clearly $H \leq N_G(\omega^H)$. Since $N_G(\omega^H)$ acts on $\omega^H$ and $H$ acts transitively on $\omega^H$, the Frattini argument gives $N_G(\omega^H) \leq G_\omega H = H$.

Clearly $\omega^{N_G(\Delta)} \subseteq \Delta$. Let $\mu \in \Delta$. Since $G$ acts transitively on $G$, $\mu = \omega^g$ for some $g \in G$. Then $\mu = \omega^{\in} \Delta \cap \Delta^g$ and so $\Delta = \Delta^g$ and $g \in N_G(\Delta)$ and $\mu \in \omega^{N_G(\Delta)}$. Hence $\omega^{N_G(\Delta)} = \Delta$.

We proved that the two maps are inverse to each other and so are bijection. The non-proper sets of imprimitivity containing $\omega$ are $\{\omega\}$ and $\Omega$. Since $N_G(\{\omega\}) = G_\omega$ and $N_G(\Omega) = G$ we conclude that $\omega^H$ is proper if and only if $H \neq G_\omega$ and $H \neq G$. □

**Corollary 1.4.5.** *Suppose $G$ acts transitively on $\Omega$ and let $\omega \in \Omega$. Then $G$ cats primitively on $\Omega$ iff $G_\omega$ is a maximal subgroup of $G$.*

*Proof.* Since $G$ acts transitively on $\Omega$ there exists a proper set of imprimitivity for $G$ on $\Omega$ iff ther exists a proper set of imprimitivity containing $\omega$. Thus $G$ acts primitively on $G$ if and only if $\{\omega\}$ and $\Omega$ are the only sets of imprimitivity containing $\omega$. By 1.4.4, this holds iff $G_\omega$ and $G$ are the only subgroups of $G$ containing $G_\omega$ and so iff $G_\omega$ is a maximal subgroup of $G$. □

**Lemma 1.4.6.** *Let $G$ be a group and $N$ a normal subgroup of $G$. The the orbits of $N$ on $G$ form a system of imprimitivity for $G$ on $\Omega$. This system is proper unless $N$ acts transitively on $\Omega$ or trivially on $\Omega$.*

*Proof.* Let $\omega^N$ be an orbit for $N$ on $G$ and $g \in G$. Then
$omega^{Ng} = \omega^{gN}$ is also an orbit for $N$ on $\Omega$. Thus the set of orbits of $N$ on $G$ is $G$-invariant. $\Omega$ is the disjoint union of these orbits and so the orbits indeed form a system of imprimitivity. The system is proper unless one of the orbits is equal to $\Omega$ or all orbits have size 1. So unless $N$ acts transitively on $\Omega$ or acts trivially on $\Omega$. □

**Corollary 1.4.7.** *Let $G$ be a group acting faithfully and primitively on a set $\Omega$. Then all non-trivial normal subgroups of $G$ act transitively on $\Omega$.*

*Proof.* Since $G$ acts faithfully on $G$, a non-trivial subgroup cannot act transitively on $\Omega$. Thus the Corollary follows from 1.4.6 □

## 1.5 Wreath products

**Lemma 1.5.1.** *Let $G$ be a group acting on set $\Omega$ and $H$ a group. Then $G$ acts on the group $H^\Omega$ via $f^g(\omega) = f(\omega^{g^{-1}})$ for all $f \in H^\Omega$, $\omega \in \Omega$.*

*Proof.* Readily verified. □

**Definition 1.5.2.** *Let $G$ be a group acting on set $\Omega$ and $H$ a group. The $G \wr_\Omega H$ denotes the semidirect product of $H^\Omega$ by $G$ with respect to the action defined in 1.5.1*

**Lemma 1.5.3.** *Let $G$ be a group and $H$ a subgroups of $G$. Let $S$ be a transversal to $H$ in $G$ and for $a \in G/H$ let $\tau(a)$ be the unique element of $a \cap S$. Put $\Omega = G/H$. Then the map*

$$\begin{array}{rccc} \rho_S : & G & \to & G \wr_\Omega H \\ & g & \to & (g, f_g) \end{array}$$

*where $f_g \in H^\Omega$ is defined by $f_g(a) = \tau(ag^{-1})g\tau(a)^{-1}$ is a well defined monomorphism. Moreover, if $T$ is a another transversal to $H$ in $G$, then there exists $b \in G \wr_\Omega H$ with $\rho(T)(g) = \rho_S(g)^b$ for all $g \in G$.*

*Proof.* Observe first that $G \wr_\Omega H$ is a subgroup of $G \wr_\Omega G$. For $g \in G$, define $c_g\omega : G \to G^\Omega, \omega \to g$. Then $c_a^b = c_a$ for all $a, b \in G$ and so the map $c : G \to G \wr_\Omega G, g \to (g, c_g)$ is a monomorphism. Note that the function $\tau : a \to \tau(g)$ is an element of $G^\Omega$ and so $(1,\tau)$ is an element of $G \wr_\Omega G$. Let $g \in G$. Then

$$(1,\tau)(g,c_g)(1,\tau)^{-1} = (g, \tau^g c_g)(1,\tau^{-1}) = (g, \tau^g c_g \tau^{-1})$$

and $\tau^g c_g \tau^{-1})(a) = \tau(ag^{-1} g\tau(g)^1 = f_g(a)$. Thus $\rho_S$ is the composition of the monomorphism $c$ and the inner automorphism of $G \wr_\Omega G$ induced by $\tau^{-1}$. So $\rho_S$ is a monomorphism. Note that $H\tau(ag^{-1})g = H(ag^{-1})g = Ha = H\tau(a)$. Hence $f_g(a) \in H$ and so $\rho_S(G) \leq G \wr_\Omega H$.

Now let $T$ be another transversal. Write $\tau_S$ and $\tau_T$ for the function from $\Omega$ to $G$ corresponding to $S$ and $T$, respectively. Since $H\tau_S(a) = Ha = H\tau_T(a)$, $\tau_S\tau_T^{-1}$ is an element of $H^\Omega$. Choosing $b = (1, \tau_S\tau_T^{-1}) \in G \wr_\Omega H$ we see that the lemma holds. □

**Lemma 1.5.4.** *Let $G$ be a group acting on a group $H$ and let $\Omega$ be a set such that $G$ and $H$ act on $\Omega$. Suppose that for all $g \in G$, $h \in H$ and $\omega$ in $\Omega$,*

$$((\omega^{g^{-1}})^h)^g = \omega^{h^g}$$

*Then $G \ltimes H$ acts on $\Omega$ via $\omega^{(g,h)} = (\omega^g)^h$.*

*Proof.* Let $\omega \in \Omega$, $g, \tilde{g} \in G$ and $h, \tilde{h} \in H$. We will write $\omega^{abc\cdots}$ for $(((\omega^a)^b)^c)^{\cdots}$.

We have

$$\omega^{(1,1)} = (\omega^1)^1 = \omega$$

and

$$\omega^{(g,h)(\tilde{g},\tilde{h})} = \omega^{gh\tilde{g}\tilde{h}} = \omega^{g\tilde{g}(\tilde{g}^{-1}h\tilde{g})\tilde{h}} = \omega^{g\tilde{g}h^{\tilde{g}}h} = \omega^{(g\tilde{g},h^{\tilde{g}}h)} = \omega^{((g,h)(\tilde{g},\tilde{h}))}$$

and so the lemma is proved. □

**Lemma 1.5.5.** *Let $G$ be a group acting on set $A$ and $H$ a group acting on a set $B$. Then*

*(a) $G \wr_A H$ acts on $A \times B$ via $(a,b)^{(g,f)} = (a^g, b^{f(a^g)}$.*

*(b) $\{a \times B \mid a \in A\}$ is a system of imprimitivity for $G \wr +AH$ and for $G$ on $A \times B$.*

*Proof.* (a) Clearly $G$ acts on $A \times B$ via $(a,b)^g = (a^g, b)$ and $H^A$ acts on $\Omega$ via $(a,b)^f = (a, b^f(a))$. Also

$$(a,b)^{g^{-1}fg} = (a^{g^{-1}}, b)^{fg} = (a^{g^{-1}}, b^{f(a^{g^{-1}})})^g = (a, b^{f(a^{g^{-1}})}) = (a, b^{f^g(b)}) = (a,b)^{f^g}$$

Hence by 1.5.4 $G \wr_A H = G \ltimes H^\Omega$ acts on $A \times B$ via

$$(a,b)^{(g,f)} = (a,b)^{gf} = (a^g,b)^f = (a^g, b^{f(a^g)})$$

Clearly $A \times B = \uplus_{a\in A} a \times B$ Moreover for $g \in G$, $(a\times B)^g = a^g \times B$ and for $f \in H^\Omega$, $(a,B)^f = a \times B$. So $\{a \times B \mid a \in A\}$ is $G \wr_A H$-invariant. □

**Lemma 1.5.6.** *Let $G$ be a group and $H$ a subgroup of $G$. Suppose $H$ acts on set $B$ and put $A = G/H$. Let $S$ be transversal to $H$ in $G$ and let $\rho_S$ and $f_g$ be as in 1.5.3.*

*(a) $G$ acts in $A \times B$ via $(a,b)^g = (a,b)^{\rho_S}(g) = (ag, b^{f_g(ag)})$.*

*(b) $\{a \times B \mid a \in A\}$ is a system of imprimitivity for $G$ on $A \times B$.*

*(c) $\tilde{B} := \{(H,b) \mid b \in B\}$ is an set of imprimitivity for $G$ on $A \times B$, $N_G(\tilde{B}) = H$ and $\tilde{B}$ is $H$-isomorphic to $B$.*

*Proof.* Since $\rho_S$ is a homomorphism, (a) and (b) follow from 1.5.5.

Note that $\tilde{B}^g = \tilde{B}$ iff $Hg = H$ and so iff $g \in H$. Thus (a) holds. □

**Lemma 1.5.7.** *Let $G$ be a group acting on a set $\Omega$, $B$ a set of imprimitivity for $G$ on $\Omega$, $H = N_G(B)$, $A = G/H$ and let $\mathcal{S}$ be a transversal to $H$ in $G$. Define $\tau$ and $\rho = \rho_S$ as in 1.5.3. Also define an action of $G \wr_A H$ on $A \times B$ as 1.5.5. Define*

$$\epsilon : A \times B \to \Omega, (a,b) \to b^{\tau(a)}$$

*Then $\epsilon$ is injective $G$-equivariant map with image $\bigcup B^G$ and $\epsilon(\{(H,b), b \in B\}) = B$.*

*Proof.* Suppose that $b^{\tau(a)} = \tilde{b}^{\tau\tilde{a}}$. Note that this element of $\Omega$ lies in $B^{\tau(a)}$ and $B^{\tau(\tilde{a})}$ and since $B$ is set of imprimitivity, $B^{\tau(a)} = B^{\tau(\tilde{b})}$. Thus $\tau(a)\tau(\tilde{a})^{-1} \in H$ and so $a = H\tau(a) = H\tau(\tilde{a}) = \tilde{a}$. Hence also $b = \tilde{b}$ and $\epsilon$ is injective. If $g \in G$, then $B^g = B^{Hg} = B^{\tau(Hg)}$ and so the image of $\epsilon$ is $\bigcup B^G$.

Let $f_g$ be as 1.5.3. Then$f_g(ag) = \tau(agg^{-1})g\tau(ag)^{-1} = \tau(a)g\tau(ag)^{-1}$. We compute

$$\epsilon((a,b)^g)) = \epsilon((a,b)^{\rho(g)}) = \epsilon((a,b)^{(g,f_g)}) = \epsilon((ag, b^{f_g(ag)}) = b^{\tau(a)g\tau(ag)^{-1}\tau(ag)}) = b^\tau(a)g = \epsilon(a,b)^g$$

□

## 1.6 Multi-transitive action

**Definition 1.6.1.** *(a) Let $\Omega$ be a set and $n \in \mathbb{N}$. Then $\Omega^n_{\neq} = \{(\omega_1, \omega_2, \ldots, \omega_n) \in \Omega^n \mid \omega_i \neq \omega_j$ for all $1 \leq i < j \leq n\}$.*

*(b) Let $G$ be a group acting on a set $\Omega$ and $n \in \mathbb{N}$ with $n \leq \Omega$. We say that $G$ acts $n$-transitive on $\Omega$ if $G$ acts transitively on $\Omega^n_{\neq}$. (Note here that $G$ acts on $\Omega^n$ via $(\omega_1, \omega_2, \ldots, \omega_n)^g = (\omega_1^g, \ldots, \omega_n^g)$ and $\Omega^n_{\neq}$ is an $G$-invariant subset of $\Omega_n$.*

(c) *Let* $\Omega$ *be a set,* $n \in \mathbb{N}$ *and* $\omega = (\omega_1, \ldots, \omega_n) \in \Omega^n$*. Then* $\underline{\omega} = \{\omega_1, \omega_2, \ldots, \omega_n\}$*.*

**Lemma 1.6.2.** *Let* $G$ *be a group acting on a finite set* $\Omega$*. Then* $G$ *acts* $|\Omega|$*-transitive on* $\Omega$ *if and only of* $G^{\cdots\Omega} = \mathrm{Sym}(\Omega)$*.*

*Proof.* Let $\omega, \mu \in \Omega^n_{\neq}$. Then there exists exactly one element $\pi \in \mathrm{Sym}(\Omega)$ with $\omega\pi = \mu$. So $\mathrm{Sym}(\Omega)$ acts $|\Omega|$-transitive. Conversely suppose $G$ acts $|\Omega|$-transitive on $\Omega$ and let $\pi \in \mathrm{Sym}(\Omega)$. Then there exists $g \in G$ with $\omega^g = \omega\pi$ and so the image of $g$ in $\mathrm{Sym}(\Omega)$ is $\pi$. Thus $G^{\cdots\Omega} = \mathrm{Sym}(\Omega)$. □

**Example 1.6.3.** *Let* $\mathbb{K}$ *be a field and* $V$ *a non-zero vector space over* $\mathbb{K}$*. Let* $\mathrm{GL}_{\mathbb{K}}(V)$ *be the group of* $\mathbb{K}$*-linear automorphism of* $V$*.*

(a) $\mathrm{GL}_{\mathbb{K}}(V)$ *acts transitively on* $V^\sharp$*.*

(b) *If* $|\mathbb{K}| = 2$ *and* $\dim_{\mathbb{K}} V \geq 2$*, then* $\mathrm{GL}_{\mathbb{K}}(V)$ *acts* 2*-transitive on* $V^\sharp$*.*

(c) *If* $|\mathbb{K}| = 2$ *and* $\dim_{\mathbb{K}} V = 2$ *then* $\mathrm{GL}_{\mathbb{K}}(V)$ *acts* 3*-transitive on* $V^\sharp$*.*

(d) *If* $\mathbb{K}| = 3$ *and* $\dim_{\mathbb{K}} V = 3$*, then* $\mathrm{GL}_{\mathbb{K}}(V)$ *acts* 3*-transitive on* $V^\sharp$*.*

(e) *If* $\dim_{\mathbb{K}}(V) \geq 2$*, then* $\mathrm{GL}_{\mathbb{K}}(V)$ *acts 2-transitive on the set of* 1*-dimensional subspace of* $V$*.*

**Lemma 1.6.4.** *Let* $G$ *group acting* $n$*-transitive on a set* $\Omega$*.*

(a) $G$ *acts* $m$*-transitively on* $\Omega$ *for all* $1 \leq m \leq n$*.*

(b) *Let* $m \in \mathbb{Z}^+$ *with* $n + m \leq |\Omega|$ *and* $\omega \in \Omega^n_{\neq}$*. Then* $G$ *acts* $n + m$*-transitive on* $\Omega$ *if and only if* $G_\omega$ *acts* $m$*-transitively on* $\Omega \setminus \underline{\omega}$

*Proof.* (a) is obvious.

(b) $\Longrightarrow$: Suppose that $G$ acts $n+m$-transitive on $\Omega$ and let $\alpha, \beta \in (\Omega \setminus \underline{\omega})^m_{\neq}$. The $(\omega, \alpha)$ and $(\omega, \beta)$ both are contained in $\Omega^{n+m}_{\neq}$. Thus there exists $g \in G$. With $\omega^g = \omega$ and $\alpha^g = \beta$. So $g \in G_\omega$ and $G_\omega$ acts $m$-transitive on $\Omega \setminus \underline{\omega}$.

$\Longleftarrow$:. Suppose that $G_\omega$ acts $m$-transitive on $\Omega \setminus \underline{\omega}$ and let $\alpha, \beta \in \Omega^{n+m}_{\neq}$. Let $\gamma \in \{\alpha, \beta\}$. Pick $\gamma_1 \in \Omega^n_{\neq}$ and $\gamma_2 \in (\Omega \setminus \underline{\gamma_1})^m_{\neq}$ with $\gamma = (\gamma_1, \gamma_2)$. Since $G$ acts $n$-transitive on $\Omega$ there exists $g_\gamma \in G$ with $\gamma_1^{g_\gamma} = \omega$ and so $\gamma_2^{g_\gamma} \in (\Omega \setminus \underline{\omega})^m_{\neq}$. Since $G_\omega$ acts $m$-transitive on $\Omega \setminus \underline{\omega}$ there exists $h \in G_\omega$ with $(\alpha_2^{g_\alpha})^h = \beta_2^{g_\beta}$. Put $g = g_\alpha h g_\beta^{-1}$. Then

$$\alpha_1^g = \alpha_1^{g_\alpha h g_\beta^{-1}} = \omega^{h g_\beta^{-1}} = \omega^{g_\beta^{-1}} = \beta_1$$

and

$$\alpha_2^g = \alpha_2^{g_\alpha h g_\beta^{-1}} = \beta_2$$

Thus $\alpha^g = \beta$ and $G$ acts $n+m$-transitive on $\Omega$. □

**Lemma 1.6.5.** *Let $G$ be the internal semidirect product of $K$ by $H$. Put $\Omega = G/H$.*

*(a) $K$ acts regularly on $\Omega$.*

*(b) Let $n \in \mathbb{Z}^+$ with $n < |\Omega|$. Then $G$ acts $n+1$ transitive on $\Omega$ if and only if $H$ acts $n$-transitive on $K^\sharp$.*

*Proof.* (a) Since $K \cap H^g = (K^{g^{-1}} \cap H)^g = 1$ for all $g \in G$, $K$ acts semiregularly on $\Omega$. Since $G = HK$, $K$ acts transitively on $\Omega$.

(b) By 1.6.4 $G$ acts $n+1$-transitive on $\Omega$ if and only if $H$ acts $n$-transitive on $\Omega \setminus \{H\}$. By 1.3.1, the action of $H$ on $\Omega \setminus \{H\}$ is isomorphic to the action of $H$ on $K^\sharp$. So (b) holds. □

**Example 1.6.6.** *Let $\mathbb{K}$ be a field and $V$ a non-zero vector space over $\mathbb{K}$. Let $G$ be the external semidirect product of $V$ by $\mathrm{GL}_\mathbb{K}(V)$ and let $\Omega = G/H \times \{1\}$. Then*

*(a) $G$ acts 2-transitive on $\Omega$.*

*(b) If $|\mathbb{K}| = 2$, then $G$ acts 3-transitive on $\Omega$.*

*(c) If $|\mathbb{K}| = 2$ and $\dim_\mathbb{K} V = 2$, then $G$ acts 4-transitive on $\Omega$.*

*(d) If $|\mathbb{K}| = 3$ and $\dim_\mathbb{K} V = 1$, then $G$ acts 3-transitive on $\Omega$.*

*Proof.* This follows from 1.6.3 and 1.6.5. □

Note that in 1.6.6(c), $|\Omega| = |V| = 4$. Since $G$ acts 4-transitive on $\Omega$ we conclude from 1.6.2 that $G \cong \mathrm{Sym}(4)$. So $\mathrm{Sym}(4)$ is isomorphic to the external semidirect product of $\mathbb{F}_2^2$ by $\mathrm{GL}_{\mathbb{F}_2}(\mathbb{F}_2^2)$.

**Lemma 1.6.7.** *Let $n \in \mathbb{Z}^+$ and $G$ a group acting $n$-transitive on a finite set $\Omega$. Suppose there exists a normal subgroup $N$ of $G$ acting regularly on $\Omega$. Then $n \leq 4$ and one of the following holds.*

*1. $n = 1$.*

*2. $n = 2$ and $N$ is an elementary abelian $p$-group for some prime $p$.*

*3. $n = 3$ and either $N$ is an elementary abelian 2-group or $|N| = 3 = |\Omega$ and $G^{\cdot\Omega} = \mathrm{Sym}(\Omega)$.*

*4. $n = 3$, $N$ is elementary abelian of order 4, $|\Omega| = 4$ and $G^{\cdot\Omega} = \mathrm{Sym}(\Omega)$.*

*Proof.* We may assume that $n \geq 2$. Let $\omega \in \Omega$. Then $G$ is the internal semidirect product of $N$ by $G_\omega$ and so by 1.6.5, $G_\omega$ acts $n-1$-transitive on $N^\sharp$. Let $p$ be a prime divisor of $|N|$. Since $G_\omega$ acts transitively on $N^\sharp$ and $N$ has an element of order $p$, all non-trivial elements of $N$ have order $p$. Thus $N$ is a $p$-group and so $Z(N) \neq 1$. Since $Z(N)^\sharp$ is invariant under $G_\omega$ we conclude that $N = Z(N)$ and so $N$ is an elementary abelian $p$-group.

If $n = 2$ we conclude that (2) holds and if $n = 3$ and $p = 2$, (3) holds. So it remains to consider the cases $n \geq 3$ and $p$ is odd and $n \geq 4$ and $p$ is even.

Suppose $n \geq 3$ and $p$ is odd. Then $C_{G_\omega}(x)$ acts transitively on $N^\sharp \setminus \{x\}$. Since $x^{-1} \in N^\sharp \setminus \{x\}$ and $C_{G_\omega}(x)$ fixes $x^{-1}$ we get $N^\sharp \setminus \{x\} = \{x^{-1}\}$. Thus $N = \{1, x, x^{-1}\}$, $|\Omega| = |N| = 3$ and $n = 3$. By 1.6.2, $G^{\cdot\Omega} = \mathrm{Sym}(\Omega)$ and (3) holds.

Suppose next that $n \geq 4$ and $p = 2$. Let $y \in N \setminus \langle x \rangle$. Then $xy \in N^\sharp \setminus \{x, y\}$ and $C_{G_\Omega}(\{x, y\})$ acts transitively on $N^\sharp \setminus \{x, y\}$. Since $C_{G_\Omega}(\{x, y\})$ fixes $xy$, this implies that $N^\sharp \setminus \{x, y\} = \{xy\}$. Thus $|\Omega| = |N| = 4$. So also $n = 4$ and by 1.6.2, $G^{\cdot\Omega} = \mathrm{Sym}(\Omega)$. Hence (4) holds in this case. □

## 1.7 Hypercentral Groups

**Lemma 1.7.1.** *Let $G$ be a group and $a, b, c$ in $G$. Then*

*(a)* $[a, b] = a^{-1}a^b$, $a^b = a[a, b]$ *and* $ab = ba[a, b]$

*(b)* $[a, bc] = [a, c][a, b]^c$.

*(c)* $[ab, c] = [a, c]^b[b, c]$

*(d)* $[b, a] = [a, b]^{-1} = [a^{-1}, b]^a = [a, b^{-1}]^b$.

*(e)* $[a, b^{-1}, c]^b[b, c^{-1}, a]^c[c, a^{-1}, b]^a = 1$

*Proof.* (a) follows immediately from $[a, b] = a^{-1}b^{-1}ab$ and $a^b = b^{-1}ab$.

(b) $[a, c][a, b]^c = (a^{-1}c^{-1}ac)c^{-1}(a^{-1}b^{-1}ab)c = a^{-1}c^{-1}b^{-1}abc = [a, bc]$

(c) $[a, c]^b[b, c] = b^{-1}(a^{-1}c^{-1}ac)b(b^{-1}c^{-1}bc) = b^{-1}a^{-1}c^{-1}abc = [ab, c]$.

(d)

$$a^{-1}, b]^a = a^{-1}(ab^{-1}a^{-1}b)a = b^{-1}a^{-1}ba = [b, a]$$

,

$$[b, a] = b^{-1}a^{-1}ba = (a^{-1}b^{-1}ab)^{-1} = [a, b]^{-1},$$

and

$$[a, b^{-1}]^b = (a^{-1}bab^{-1})^b = b^{-1}a^{-1}ba = [b, a]$$

(e)

$$[a, b^{-1}, c]^b = [a^{-1}bab^{-1}, c]^b = (ba^{-1}b^{-1}ac^{-1}a^{-1}bab^{-1}c)^b = \left(a^{-1}b^{-1}ac^{-1}a^{-1}\right)\left(bab^{-1}cb\right) = (aca^{-1}ba)^{-1}(bab^{-1}cb)$$

Put $x = aca^{-1}ba$, $y = bab^{-1}cb$ and $z = bab^{-1}cb$. Then

$$[a, b^{-1}, c]^b = x^{-1}y$$

Cyclicly permuting $a$, $b$ and $c$ gives

$$[b, c^{-1}, a]^c = y^{-1}z$$

and

$$[c, a^{-1}, b]^a = z^{-1}x$$

Since $(x^{-1}y)(y^{-1}z)(z^{-1}x) = 1$, (e) holds. □

**Lemma 1.7.2.** *Let $G$ be a group and $A$ and $B$ subsets of $G$.*

*(a)* $[A, B] = [B, A]$.

*(b) If* $1 \in B$, *then* $\langle A^B \rangle = \langle A, [A, B] \rangle$.

*(c) If $A$ is a subgroup of $G$ and* $1 \in B$, *then* $[A, B] \trianglelefteq \langle A^B \rangle$ *and* $\langle A^B \rangle = A[A, B]$

*(d) If $A$ and $B$ are subgroups of $G$, then $B$ normalizes $A$ if and only if* $[A, B] \leq A$.

*(e)* $[\langle A \rangle, B] = \langle [A, B]^{<A>} \rangle$.

*(f)* $[\langle A \rangle, \langle B \rangle] = \langle [A, B]^{\langle A \rangle \langle B \rangle}$.

*(g) If $A$ and $B$ are $A$-invariant, then* $[\langle A \rangle, B] = [A, B]$.

*(h) If* $a \in G$ *with* $B^a = B$, *then* $[a, B] = [\langle a \rangle, B]$.

*(i)* $[A, G] = [\langle A^G \rangle, G]$.

*Proof.* Let $a \in A$ and $b, c \in B$. (a) Follows from $[a, b]^{-1} = [b, a]$.

(b) Then $a^b = a[a, b] \in \langle \langle A, [A, B] \rangle$. So $\langle A^B \rangle \leq \langle A, [A, B] \rangle$. Since $1 \in B$ we have $a = a^1 \in \langle A^B \rangle$ and so also $[a, b] = a^{-1}a^b \in \langle A^B \rangle$. Thus $\langle A, [A, B] \leq \langle A, B \rangle$ and (a) holds.

(c) Let $d \in A$. By 1.7.1(d), $[ad, b] = [a, b]^d[d, b]$ and so $[a, b]^d = [ad, b][d, b]^{-1} \in [A, B]$. Thus $A$ normalizes $[A, B]$. Since also $[A, B]$ normalizes $[A, B]$, (b) implies that $[A, B] \trianglelefteq \langle A^B \rangle$. Hence $\langle A^B \rangle = \langle A, [A, B] \rangle = A[A, B]$.

(d) $B$ normalizes $A$ iff $A = \langle A^B \rangle$ iff $A = \langle A, [A, B] \rangle$ and iff $[A, B] \leq A$.

(e) Put $H = \langle [A, B]^{<A>} \rangle$. Since $[A, B] \leq [\langle A \rangle, B]$ and $\langle A \rangle$ normalizes $[\langle A \rangle, B]$ we conclude that $H \leq [\langle A \rangle, B]$. Define $D := \{d \in \langle A \rangle \mid [d, B] \leq H\}$. We will show that $D$ is a subgroup of $G$. Let $d, e \in D$ and $b$ in $D$. Observe that $H$ is an $\langle A \rangle$ invariant subgroup of $G$. Thus

$$[de, b] = [d, b]^e[e, b] \in H \text{ and } [d^{-1}, b] = ([d, b]^{-1})^{d^{-1}} \in H$$

Hence $de \in D$ and $d^{-1} \in D$. Thus $D$ is a subgroup of $G$. Since $A \subseteq D \leq \langle A \rangle$, this gives $D = \langle A \rangle$ and so $[\langle A \rangle, B] \leq H$.

(f) By (e)

$$[\langle A \rangle, \langle B \rangle] = \langle [\langle A \rangle, B]^{\langle B \rangle} \rangle = \langle \langle [A, B]^{\langle A \rangle} \rangle^{\langle B \rangle} = \langle [A, B]^{\langle A \rangle \langle B \rangle} \rangle$$

(g) If $A$ and $B$ are $A$-invariant, then $A$, $B$ and $[A, B]$ are $\langle A \rangle$-invariant. Thus (g) follows from (e).

(h) Follows from (g) applied to $A = \{a\}$.

(f) By (d), $G$ normalizes $[A, G]$. Thus $[A, G] = [A^G, G]$. Since $G$ and $A^G$ are $A$-invariant, (g) gives $[A^G, G] = [\langle A^G \rangle, G]$. □

**Definition 1.7.3.** *Let $G$ be a group and $A \leq G$. Then $Z(G, A) = \{g \in G \mid [G, g] \leq A$.*

Note that by 1.7.2(f), $Z(G, A)$ is a normal subgroup of $G$.

**Lemma 1.7.4.** *Let $G$ be a group and $A \leq G$. Put $B = \bigcup A^G$. Then*

$$Z(G, A) = \{g \in G \mid Agh = Ahg \text{ for all} h \in G\}$$

*and*

$$Z(G, A)/B = Z(G/B)$$

*Proof.* By 1.7.2(h), $[G, g] = [G, \langle g \rangle] = [G, g^{-1}]$. Hence $g \in Z(G, A)$ iff $[G, g] \leq A$, iff $[G, g^{-1}] \leq A$ iff $[h^{-1}, g^{-1}] \in A$ for all $h \in G$, iff $hgh^{-1}g^{-1} \in A$ for all $h \in G$, and iff $Ahg = Agh$ for all $h \in G$.

Since $[G, g]$ is a normal subgroup of $G$, $[G, g] \leq A$ iff $[G, g] \leq B$ and iff $Bg \leq Z(G/B)$. □

**Lemma 1.7.5.** *Let $G$ be a group and $A \leq B \leq G$.*

*(a) $Z(G, A) \leq Z(G, B)$.*

*(b) $Z(G, A) \leq N_G(A)$.*

*(c) If $A \trianglelefteq G$, then $Z(G, B)/A = Z(G/A, B/A)$*

*Proof.* (a): $[Z(G, A), G] \leq A \leq B$.

(b) $[Z(G, A), A] \leq [Z(G, A), G] \leq A$.

(c) $[g, h] \in A$ if and only if $[gN, hN]N \in A/N$. □

**Definition 1.7.6.** *Let $G$ be a group and $\alpha$ an ordinal. Define the groups $Z_\alpha(G)$ and $L^\alpha(G)$ inductively via*

$$Z_\alpha(G) = \begin{cases} 1 & \text{if } \alpha = 0 \\ Z(G, Z_\beta(G)) & \text{if } \alpha = \beta + 1 \text{ for some ordinal } \beta \\ \bigcup_{\beta < \alpha} Z_\beta(G) & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

*and*

$$L_\alpha(G) = \begin{cases} G & \text{if } \alpha = 0 \\ [L_\beta(G), G] & \text{if } \alpha = \beta + 1 \text{ for some ordinal } \beta \\ \bigcap_{\beta < \alpha} L_\beta(G) & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

*Let $z_G$ be the smallest ordinal $\alpha$ with $Z_\alpha(G) = Z_{\alpha+1}(G)$ and $l_G$ be smallest ordinal $\alpha$ with $L_\alpha(G) = L_{\alpha+1}(G)$. Put $Z_*(G) = Z_{z_G}(G)$ and $L_*(G) = L_{l_G}(G)$.*

*$G$ is called hypercentral of class $z_G$ if $G = Z_*(G)$ and $G$ is called hypocentral of class $l_G$ if $G = L_*(G)$. $Z_*(G)$ is called the hypercenter of $G$ and $L_*(G)$ the hypocenter of $G$.*

**Example 1.7.7.** *The hypercenter and hypocenter of $D_{2^\infty}$.*

Let $C_{p^k}$ be a cyclic group of order $p^k$ and view $C_{p^k}$ has a subgroup of $C_{p^{k+1}}$. Put $C_{p^\infty} = \bigcup_{k=0}^{\infty} C_{p^k}$. ($C_{p^\infty}$ is called the Prüfer group for the prime $p$. Let $t \in \mathrm{Aut}(C_{p^\infty})$ be defined by $x^t = x^{-1}$ for all $x \in C_{p^\infty}$. Let $D_{p^\infty} = \langle t \rangle \ltimes C_{p^\infty}$.

Observe that for $k \in \mathbb{N} \cup \{infty\}$, $C_{p^k}$ is a normal subgroup of $D_{p^\infty}$. If $k$ is finite, $D_{p^\infty}/C_{p^k} \cong D_{p^\infty}$. Also $D_{p^\infty}/C_{p^\infty} \cong C_2$.

We will now compute $Z_\alpha(D_{2^\infty})$. Since $C_{2^\infty}$ is Abelian and $t$ does not centralize $C_{2^\infty}$ we have $C_{D_{2^\infty}}(C_\infty) = C_{2^\infty}$. Thus $Z(D_{2^\infty}) \leq C_{2^\infty}$ and $Z(D_{2^\infty}) = C_{C_{2^\infty}}(t)$. $x \in C_{2^\infty}$ is centralized by $t$ if and only if $x^t = x^{-1} = x$ and so iff $x^2 = 1$. Thus $Z(C_{2^\infty}) = C_2$. Let $k \in \mathbb{N}$ and suppose inductively that $Z_k(C_{2^\infty}) = C_{2^k}$. Since $D_{2^\infty}/C_{2^k} \cong D_{2^\infty}$ we get

$$Z(D_{2^\infty}/C_{2^k}) = C_{2^{k+1}}/C_{2^k}$$

and so $Z_{k+1}(C_{2^\infty}) = C_{2^{k+1}}$. Let $\omega$ be the first infinite ordinal. Then

$$Z_\omega(D_{2^\infty}) = \bigcup_{k<\omega} Z_k(D_{2^\infty}) = \bigcup_{k<\omega} C_{2^k} = C_{2^\infty}$$

Since $D_{2^\infty}/C_{2^\infty}$ is isomorphic to $C_2$ and so is Abelian, we conclude that $Z_{\omega+1}(D_{2^\infty} = D_{2^\infty}$ and so $D_{2^\infty}$ is hypercentral of class $\omega + 1$.

Since $D_{2^\infty}/C_{2^\infty}$ is abelian, $L_1(D^{2^\infty}) \leq C_{2^\infty}$. Let $x \in C_{2^\infty}$. Then $[x, t] = x^{-1}x^t = x^{-2}$. Since each element if $C_{2^k}$ is the square of an element in $C_{2^{k+1}}$ we conclude that $C_{2^\infty} = [C_{2^\infty}, t] = [C_{2^\infty}, D_{2^\infty}] \leq L_1(D_{2^\infty})$. Thus

$L_1(D_{2^\infty}) = C_{2^\infty}$ and

$$L_2(D_{2^\infty}) = [C_{2^\infty}, D_{2^\infty}] = C_{2^\infty} = L_1(D_{2^\infty})$$

So $C_2\infty$ is the hypocenter of $D_{2^\infty}$ and $D_{2^\infty}$ is not hypocentral.

**Lemma 1.7.8.** *Let $G$ be a group and $H$ a subgroup of $G$ with $H \not\leq Z_*(G)$. Let $\alpha$ be the first ordinal with $Z_\alpha(G) \not\leq H$. Then $Z_\alpha(G) \leq Z(G, H)$. In particular, $Z(G, H) \not\leq H$ and $H \lneq N_G(H)$.*

*Proof.* Since $Z_0(G) = 1 \leq H$, $\alpha \neq 0$. If $\alpha$ is a limit ordinal, then $Z_\alpha(G) = \bigcup_{\beta<\alpha} Z_\beta(G) \leq H$, a contradiction. Thus $\alpha = \beta + 1$ for some ordinal $\beta$. Then $Z_\beta(G) \leq H$ by minimality of $\alpha$ and so

$$[Z_\alpha(G) = Z(G, Z_\beta(G)) \leq Z(G, H)$$

□

**Lemma 1.7.9.** *Let $G$ be a group and $N$ a non-trivial normal subgroup of $G$ with $N \leq Z_*(G)$. Then $N \cap Z(G) \neq 1$.*

*Proof.* Since $N \cap Z_*(G) = N \neq 1$ there exists an ordinal $\alpha$ minimal with $N \cap Z_\alpha(G) \neq 1$. Since $Z_0(G) \cap N = 1 \cap N = 1$, $\alpha \neq 0$. If $\alpha$ is a limit ordinal, then $Z_\alpha(G) \cap N = \bigcup_{\beta<\alpha} Z_\beta(G) \cap N = 1$, a contradiction. Thus $\alpha = \beta + 1$ for some ordinal $\beta$. Then

$$[Z_\alpha(G) \cap N, G] \leq Z_\beta \cap N = 1$$

and so $Z_\alpha(G) \cap N \leq Z(G)$. □

**Lemma 1.7.10.** *Let $G$ be a group. Then the following are equivalent:*

*(a) $G$ is hypercentral.*

*(b) $Z(G, H) \not\leq H$ for all $H \lneq G$.*

*(c) $Z(G/N) \neq 1$ for all $N \lhd G$.*

*Proof.* (a) $\Longrightarrow$ (b): Suppose $G$ is hypercental and $H \not\leq G$. Then $Z_*(G) = G \not\leq H$ and so by 1.7.8, $Z(G, H) \not\leq H$.

(b) $\Longrightarrow$ (c): Follows from $Z(G/N) = Z(G, N)/N$.

(c) $\Longrightarrow$ (a): Let $\alpha = z_G$. Then $Z(G/Z_\alpha(G)) = Z_{\alpha+1}(G)/Z_\alpha(G) = 1$ and so (b) implies $Z_\alpha(G) = G$. □

**Lemma 1.7.11.** *Let $G$ be a group and $\alpha$ an ordinal.*

*(a) Let $H \leq G$. Then $Z_\alpha(G) \cap H \leq Z_\alpha(H)$.*

*(b) Let $N \trianglelefteq G$. then $Z_\alpha(G)N/N \leq Z_\alpha(G/N)$.*

*(c) Let $\beta$ be an ordinal. Then $Z_\alpha(G/Z_\beta(G)) = Z_{\beta+\alpha}(G)/Z_\beta(G)$.*

*Proof.* In each case we assume that the statement holds for all ordinals less than $\alpha$.

(a) For $\alpha = 0$ the group on each side of the equation is trivial group. Suppose $\alpha = \beta + 1$ for some ordinal $\beta$. Then

$$[Z_\alpha(G) \cap H, H] \leq [Z_\alpha(G), G] \cap H \leq Z_\beta(G) \cap H \leq Z_\beta(H)$$

and so $Z_\alpha(G) \cap H \leq Z_\alpha(H)$.

Suppose $\alpha$ is limit ordinal.

$$Z_\alpha(G) \cap H = (\bigcup_{\beta<\alpha} Z_\beta) \cap H = \bigcup_{\beta<\alpha} Z_\beta(G) \cap H \leq \bigcup_{\beta<\alpha} Z_\beta(H) = Z_\alpha(H)$$

(b) For $\alpha = 0$ the group on each side of the equation is trivial group. Suppose $\alpha = \beta + 1$ for some ordinal $\beta$. Then

$$[Z_\alpha(G)N/N, G]] = [Z_\alpha, G]N/N \leq Z_\beta(G)N/N \leq Z_\beta(G/N)$$

and so $Z_\alpha(G)N/N \leq Z_\alpha(G/N)$.

Suppose $\alpha$ is limit ordinal. Then

$$Z_\alpha(G)N/N = (\bigcup_{\beta<\alpha} Z_\beta)N/N = \bigcup_{\beta<\alpha} Z_\beta(G)N/N \leq \bigcup_{\beta<\alpha} Z_\beta(G/N) = Z_\alpha(G/N)$$

(b) For $\alpha = 0$ the group on each side of the equation is trivial group. Suppose $\alpha = \gamma + 1$ for some ordinal $\gamma$. Then

$$\begin{array}{rclcl} Z_{\beta+\alpha}(G)/Z_\beta(G) & = & Z_{\beta+(\gamma+1)}/Z_\beta(G) & = & Z_{(\beta+\gamma)+1}(G)/Z_\beta(G) \\ & = & Z(G, Z_{\beta+\gamma}(G))/Z_\beta(G) & = & Z(G/Z_\beta(G), Z_{\beta+\gamma}(G)/Z_\beta(G) \\ & = & Z(G/Z_\beta(G), Z_\gamma(G/Z_\beta)(G) & = & Z_{\gamma+1}(G/Z_\beta(G)) \\ & = & = Z_\alpha(G/Z_\beta(G)) & & \end{array}$$

Suppose $\alpha$ is a limit ordinal. Then

$$\begin{array}{rclclcl} Z_\alpha(G/Z_\beta(G)) & = & \bigcup_{\gamma<\alpha} Z_\gamma(G/Z_\beta(G)) & = & \bigcup_{\gamma<\alpha} Z_{\beta+\gamma}(G)/Z_\beta(G) & = & \bigcup_{\beta\leq\rho<\beta+\alpha} Z_\rho(G)/Z_\beta(G) \\ & = & \bigcup_{\rho<\beta+\alpha} Z_\rho(G)/Z_\beta(G) & = & Z_{\beta+\alpha}(G)/Z_\beta(G) & & \end{array}$$

□

**Corollary 1.7.12.** *Let $G$ be a hypercentral group of class $\alpha$. The all subgroups and all quotients of $G$ are hypercentral of class at most $\alpha$.*

*Proof.* Let $H \leq G$. Then $H = H \cap G = H \cap Z_\alpha(G) \leq Z_\alpha(H)$ and so $H$ is hypercentral of class at most $\alpha$. Let $N \trianglelefteq G$. Then $G/N = Z_\alpha(G)/N = Z_\alpha(G/N)$ and $G/N$ is hypercentral of class at most $\alpha$. □

**Lemma 1.7.13.** *Let $G$ be a hypercentral group and $M$ a maximal subgroup of $G$. Then $M \trianglelefteq G$ and $G/M \cong C_p$ for some prime $p$.*

*Proof.* By 1.7.8 $M \lneq N_G(M)$. Since $M$ is a maximal subgroups, $N_G(M) = G$. So $M \trianglelefteq G$. Since $M$ is a maximal subgroups of $G$, $G/M$ has no proper subgroups. Thus $G/M \cong C_p$ for some prime $p$. □

**Lemma 1.7.14.** *Let $G$ be a hypercentral group and $A$ a maximal abelian normal subgroup of $G$. Then $C_G(A) = A$.*

*Proof.* Since $A$ is Abelian, $A \leq C_G(A)$. Suppose that $A < C_G(A)$. Then $C_G(A)/A$ is a non-trivial normal subgroup of the hypercentral group $G/A$. Thus by 1.7.9

$$Z(G/A) \cap C_G(A)/A \neq 1$$

Hence there exists $b \in C_G(A) \setminus A$ with $bA \in Z(G/Z)$. Then $[b, G] \leq A$ and it follows that $A\langle b\rangle$ is an abelian normal subgroup of $G$, a contradiction to maximality of $A$. □

**Definition 1.7.15.** *Let $A$ and $B$ be subsets of a group $G$ and $\alpha$ an ordinal. Define $[A, B; \alpha]$ inductively via*

$$[A, B; \alpha] = \begin{cases} \langle A^B \rangle & \text{if } \alpha = 0 \\ [[A, B; \beta], A] & \text{if } \alpha = \beta + 1 \text{ for some ordinal } \beta \\ \bigcap_{\beta < \alpha} [A, B; \beta] & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

Observe that $[G, G; \alpha] = L_\alpha(G)$. Morever, if $\alpha$ is finite $[A, B; \alpha + 1] = [[A, B], B, \alpha]$.

**Lemma 1.7.16.** *Let $n \in \mathbb{N}$, $G$ a group and $H \leq G$. Then $H \leq Z_n(G)$ if and only if $[H, G; n] = 1$.*

*Proof.* If $n = 0$ both statements say that $H = 1$. Suppose inductively that for all $A \leq G$, $A \leq Z_n(G)$ if and only if $[A, G; n] = 1$. Then
$H \leq Z_{n+1}(G)$ iff $[H, G] \leq Z_n(G)$ iff $[[H, G], G; n]] = 1$ and iff $[H, G; n + 1] = 1$. □

**Corollary 1.7.17.** *Let $G$ be a groups and $n \in \mathbb{N}$. Then $G = Z_n(G)$ if and only if $L_n(G) = 1$.*

*Proof.* We have $G = Z_n(G)$ iff $[G, G; n] = 1$ iff $L_n(G) = 1$. □

**Definition 1.7.18.** *Let $G$ be a group. Then $G$ is called nilpotent if $G = Z_n(G)$ for some $n \in \mathbb{N}$. The smallest such $n$ is called the nilpotency class of $G$.*

Let $n \in \mathbb{N}$. Observe that $G$ is nilpotent of class $n$ if and only if $G$ is hypercentral of class $n$ and if and only if $G$ is hypocentral of class $n$. In particular if $G$ is nilpotent of class $n$ the all subgroups and all quotient of $G$ are nilpotent of class at most $n$.

**Definition 1.7.19.** *Let $\pi$ be a set of prime and $G$ a group.*

*(a) $\pi(G)$ is the set or prime divisors of the elements of finite order in $G$.*

*(b) $G$ is called a $\pi$-group for all $g \in G$, $|g|$ is finite and $\pi(G) \subseteq \pi$.*

*(c) $O_\pi(G)$ is the largest normal $\pi$-subgroup of $G$.*

*(d) $\pi'$ is the set of all the primes not contained in $\pi$.*

**Lemma 1.7.20.** *Let $G$ be group and $N$ normal in $G$.*

*(a) Let $\alpha$ and $\beta$ be ordinals such that $N \leq Z_\alpha(G)$ and $G/N$ is hypercentral of class $\beta$. Then $G$ is hypercentral of class at most $\alpha + \beta$.*

*(b) $G$ is hypercentral if and only of $N \leq Z_*(G)$ and $G/N$ is hypercentral.*

*(c) $G$ is nilpotent if and only if $N \leq Z_n(G)$ for some $n \in \mathbb{N}$ and $G/N$ is nilpotent.*

*Proof.* (a) Note that $G/Z_\alpha(G)$ is isomorphic to a quotient of $G/N$ and so is hypercentral of class at most $\beta$. Thus $G/Z_\alpha(G) = Z_\beta(G/Z_\alpha(G)) = Z_{\alpha+\beta}(G)/Z_\beta(G)$. Hence $G = Z_{\alpha+\beta}(G)$ and (a) holds, (b) If $G$ is hypercentral, then $N \leq G = Z_*(G)$ and $G/N$ is hypercentral. Now suppose that $N \leq Z_*(G)$ and $G/N$ is hypercentral. Then by (a), $F$ is hypercentral. (c) If $G$ is nilpotent, then $G = Z_n(G)$ for some $n \in \mathbb{N}$. Thus $N \leq Z_n(G)$ and $G/N$ is nilpotent.

If $N \leq Z_n(G)$ and $G/N$ is nilpotent of class $m$, then by (a), $G$ is hypercentral of class at most $n+m$. Thus $G$ is nilpotent. □

**Lemma 1.7.21.** *Let $G_i, i \in I$ be a family of groups and put $G = \bigtimes_{i\in I} G_i$.*

*(a) Let $\alpha$ be an ordinal. Then $Z_\alpha(G) = \bigtimes_{i\in I} Z_\alpha(G_i)$.*

*(b) $G$ is hypercentral if and only if each $G_i$ is hypercentral.*

*(c) $G$ is nilpotent if and only if each $G_i$ is nilpotent and $\sup_{i\in I} z_{G_i}$ is finite.*

*Proof.* (a) Follows from $Z(G) = \bigtimes_{i\in I} Z(G_i)$ and induction on $\alpha$.

(b) and (c) follow from (a). □

**Lemma 1.7.22.** *Let $p$ be a prime and $G$ a finite $p$-group. Then $G$ is nilpotent.*

*Proof.* Let $N \trianglelefteq G$ with $N \neq G$. Then $G/N$ is a non-trivial $p$-group and so $Z(G/N) \neq 1$. Thus by 1.7.10, $G$ is hypercentral. Since $G$ is finite, $z_G$ is finite and so $G$ is nilpotent. □

**Lemma 1.7.23.** *Let $G$ be a finite group. Then the following are equivalent:*

*(a) $G$ is nilpotent.*

*(b) $H \lneq N_G(H)$ for all $H \lneq G$.*

*(c) $S \trianglelefteq G$ for all Sylow subgroups $S$ of $G$.*

*(d) $G = \bigtimes_{p\in\pi(G)} O_p(G)$, where $\pi(G)$ is the set of prime divisors of $G$.*

*(e) $G = \bigtimes_{i=1}^n G_i$, where $G_i$ is a $p_i$-subgroup for a prime $p_i$.*

*Proof.* (a) $\Longrightarrow$ (b): Since $G$ is nilpotent, then $G = Z_*(G)$ and so $H \lneq N_G(H)$ by 1.7.8

(b) $\Longrightarrow$ (c): Let $S$ be a Sylow $p$-subgroups of $G$ and put $H = N_G(S)$. Then $S$ is the only Sylow $p$-subgroups of $H$ and so $S$ is a characteristic subgroup of $G$. In particular, $S \trianglelefteq N_G(H)$ and so $N_G(H) = H$. Hence $H = G$ and $S \trianglelefteq G$.

(c) $\Longrightarrow$ (d): Let $p \in \pi(G)$ and $S_p \in \mathrm{Syl}_p(G)$. Since $S_p \trianglelefteq G$ we get $S_p \leq O_p(G)$ and so $O_p(G) = S_p$. Put $K_p = \langle O_r(G) \mid p \neq r \in \pi(G)\rangle$ and $K = \langle O_p(G) \mid p \in \pi(G)\rangle$. Note that $[O_p(G), O_{p'}(G)] \leq O_p(G) \cap O_{p'} = 1$ and since Observe that $K_p \leq O_{p'}(G)$ also $[O_p(G), K_p] \leq O_p(G) \cap K_p = 1$. Thus $K = \bigtimes_{p\in\pi(G)} O_p(G)$. Moreover, $|K| = \prod_{p\in\pi(G)} |O_p(G)| = \prod_{p\in\pi(G)} |S_p| = |G|$ and so $G = K$. So (d) holds.

(d) $\Longrightarrow$ (e): Obvious.

(e) $\Longrightarrow$ (a): By 1.7.22 each $G_i$ is nilpotent. So by 1.7.21, $G$ is nilpotent. □

**Lemma 1.7.24.** *Let $G$ be a group, $p$ a prime and $A$ and $B$ finite $p$-subgroups of $G$ with $|[A,B]| \leq p$. Then $|A/C_A(B)| = |B/C_B(A))$.*

*Proof.* Put $Z = [A,B]$ and let $|A/C_A(B)| = p^n$. Then there exist $a_i, 1 \leq i \leq n$ in $A$ with $A = \langle a_1, a_2 \ldots a_n \rangle C_A(B)$. Let $x \in a_i Z$ and $b \in B$. Then $x^b = x[x,b] \in xZ$. Thus $a_i Z$ is $B$ invariant. Since $|a_i Z| = |Z| \leq p$ and $B$ is a $p$-group, all orbits of $B$ on $a_i B$ have size 1 or $p$. Thus $|B/C_B(a_i)| \leq p$ and so $|B/\bigcap_{i=1}^n C_B(a_i)| \leq p^n$. Since $\bigcap_{i=1}^n C_B(a_i)$ centralizes $\langle a_1, \ldots a_n \rangle C_A(B) = A$ we get $|B/C_B(A)| \leq p^n = |A/C_A(B)$. By symmetry, $B/C_B(A)| \leq |A/C_B(A)$ and the lemma is proved. □

**Lemma 1.7.25.** *Let $p$ be a prime, $P$ a finite $p$-group with $|P'| = p$ and $A$ a maximal abelian subgroup of $P$. Then $A \trianglelefteq P$, $P' \leq Z(P) \leq A$, $|A/Z(P)| = |P/A|$ and $|P/Z(P)| = |A/Z(P)|^2$.*

*Proof.* Since $|P'| = p$ and $P' \trianglelefteq P$, $P' \leq Z(P)$. Since $A$ is a maximal abelian subgroup of $P$ and $A\langle x \rangle$ is abelian for all $x \in C_P(A)$, $C_P(A) = A$. In particular, $Z(P) \leq A$ and so $C_A(P) = Z(P)$. By 1.7.24 applied to $A$ and $B = P$, we have

$$|P/C_P(A)| = |A/C_A(P)|$$

and so

$$|P/A| = |A/Z(P)| \text{ and } |P/Z(P)| = |P/A||A/Z(P)| = |A/Z(P)|^2$$

□

## 1.8 The Frattini subgroup

**Definition 1.8.1.** *Let $G$ be a group, then $\Phi(G)$ is the intersection of the maximal subgroups of $G$, with $\Phi(G) = G$ if $G$ has no maximal subgroups. $\Phi(G)$ is called the Frattini subgroup of $G$.*

**Definition 1.8.2.** *Let $G$ be a group. Then a generating set for $G$ is a subset $H$ of $G$ with $G = \langle H \rangle$.*

**Lemma 1.8.3.** *Let $G$ be a finite group and $H \leq G$. If $G = H\Phi(G)$, then $H = G$.*

*Proof.* Otherwise $H \leq M$ for some maximal subgroup $M$ of $G$. But then also $\Phi(G) \leq M$ and $G = H\Phi(G) \leq M$, a contradiction. □

**Lemma 1.8.4.** *Let $G$ be a finite group and $H \subseteq G$. Then $H$ is a generating set for $G$ if and only if $\{\Phi(G)h \mid h \in H\}$ is a generating set for $G/\Phi(G)$.*

*Proof.* By 1.8.3 we have $G = \langle H \rangle$ iff $G = \langle H \rangle \Phi(G)$ and so iff $G = \langle \Phi(G)h \mid h \in H \rangle$ □

**Lemma 1.8.5.** *Let $G$ be a group and $N \trianglelefteq G$. Then $\Phi(G)N/N \leq \Phi(G/N)$.*

*Proof.* Let $\mathcal{A}$ be the set of maximal subgroups of $G$ and $\mathcal{B}$ the set of maximal subgroups of $G$ containing $N$. Then $\{M/N \mid M \in \mathcal{B}\}$ is the set of maximal subgroups of $G/N$. Since $\mathcal{B} \subseteq \mathcal{A}$, $\bigcap \mathcal{A} \subseteq \bigcup \mathcal{B}$. Thus

$$\Phi(G)N/N = (\bigcup \mathcal{A})N/N \leq \bigcap \mathcal{B}/N = \bigcap_{M\in\mathcal{B}} B/N = \Phi(G/N)$$

□

**Lemma 1.8.6.** *Let $G$ be a finite group.*

*(a) Let $H \trianglelefteq G$. Then $H$ is nilpotent if and only if $H\Phi(G)/\Phi(G)$ is nilpotent.*

*(b) $\Phi(G)$ is nilpotent.*

*(c) $G$ is nilpotent if and only if $G/\Phi(G)$ is nilpotent.*

*Proof.* (a) If $H$ is nilpotent then also $H/H \cap \Phi(G) \cong H\Phi(G)/\Phi(G)$ is nilpotent. Put $\overline{G} = G/\Phi(G)$ and suppose that $\overline{H}$ is nilpotent. Let $p$ be a prime and $S$ be a Sylow $p$-subgroup of $H\Phi(G)$. The $\overline{S}$ is a Sylow $p$-subgroup of $G$. Since $\overline{H}$ is nilpotent, $\overline{S}$ is the only Sylow $p$-subgroup of $oH$ and so is a characteristic subgroup of $\overline{H}$. Since $H \trianglelefteq G$, $\overline{H} \trianglelefteq oG$ and so also $\overline{S} \trianglelefteq G$ and $S\Phi(G) \trianglelefteq G$. The Frattini argument shows that

$$G = N_G(S)S\Phi(G) = N_G(S)\Phi(G)$$

Hence by 1.8.3, $G = N_G(S)$ and $S \trianglelefteq G$. So by 1.7.23, $H\Phi(G)$ is nilpotent. Thus also $H$ is nilpotent.

(b) Since $\Phi(G)/\Phi(G) = 1$ is nilpotent, this follows from (a) applied to $H = \Phi(G)$.

(c) This is the special case $H = G$ of (a). □

**Lemma 1.8.7.** *Let $A$ be an elementary Abelian $p$-group for some prime $p$. Then $\Phi(A) = 1$.*

*Proof.* Let $1 \neq b \in A$ and put $B = \langle b \rangle$. By Zorn's Lemma there exists a subgroup $D$ of $A$ maximal with $b \notin D$. Then $B \cap D \lneq B$ and since $|B| = |b| = p$, $B \cap D = 1$. Let $a \in A \setminus D$ and Put $E = \langle a > B$. Note that $|E/B| = p$. By maximality of $D$, $b \in E$ and so $E = \langle b > D = BD$. Thus $a \in DB$ and so $A = BD$. It follows that $|A/D| = p$ and so $D$ is a maximal subgroup of $A$. Since $b \notin D$ this gives $a \notin \Phi(A)$. This hold for any $1 \neq b \in A$ and so $\Phi(A) = 1$. □

**Lemma 1.8.8.** *Let $p$ be a prime and $P$ a $p$-group. Put*

$$D = \langle [x, y], z^p \mid x, y, z \in P \rangle.$$

*Then*

*(a) Let $H \trianglelefteq P$. Then $P/H$ is elementary Abelian if and only if $D \leq H$. So $D$ is the unique minimal normal subgroup with elementary Abelian quotient.*

*(b) $\Phi(G) = D$ if and only if all maximal subgroups of $G$ are normal.*

*Proof.* (a) $P/H$ is elementary Abelian iff $P/H$ is Abelian and $u^p = 1$ for all $u \in P/H$. Thus iff $[x, y] \in H$ and $z^p \in H$ for all $x, y, z \in P$ and iff $D \leq H$.

(b) Suppose $\Phi(D) = D$ and let $M$ be maximal subgroup of $P$. Then $D = \Phi(D) \leq M$. Since $P/D$ is Abelian, $M/D \trianglelefteq P/D$ and so $M \trianglelefteq P$.

Suppose next that all maximal subgroups of $P$ are normal and let $M$ be a maximal subgroup of $P$. Then $P/M \cong C_p$. Thus $P/M$ is elementary Abelian and so $D \leq M$. This proves that $D \leq \Phi(P)$. Since $P/D$ is elementary Abelian, 1.8.7 shows that $\Phi(P/D) = 1$. Hence by 1.8.5 $\Phi(P)D/D \leq \Phi(P/D) = 1$ and so $\Phi(P) \leq D$. Hence $\Phi(D) = D$. □

**Lemma 1.8.9.** *Let $P$ be a finite $p$-groups and $k$ the minimal size of a generating set of $P$. Then $P/\Phi(P) \cong C_p^k$.*

*Proof.* By 1.8.8, $P/\Phi(P)$ is elementary Abelian and so $P/\Phi(P) \cong C_p^n$ for some $n$. Thus the minimal size of a generating set for $P/\Phi(P)$ is $n$. 1.8.4 implies that $k = n$. □

## 1.9 Finite $p$-groups with cyclic maximal subgroups

**Lemma 1.9.1.** *Let $p$ be a prime, $H = \langle h \rangle$ be a cyclic group of order $p^n$ and $B \leq \mathrm{Aut}(H)$ with $|B| = p$. Then there exists $1 \neq b \in B$ such that one of the following holds:*

1. *$n \geq 2$ and $h^b = h^{1+p^{n-1}}$.*

2. *$p = 2$, $n \geq 3$ and $h^b = h^{-1}$.*

3. *$p = 2$, $n \geq 3$ and $h^b = h^{-(1+p^{n-1})}$.*

*Proof.* Let $h^b = h^s$ for some $1 \leq s < p^n$ and put $l = s - 1$. Then $h^b = h^{1+l} = hh^l$ and $0 \leq l < p^n - 1$. Let $l = p^r m$ with $r \in \mathbb{N}, m \in \mathbb{Z}^+$ and $p \nmid m$. Note that $H/\langle h^p \rangle$ has order $p$ and so $B$ centralizes $H/\langle h^p\}$. This $[h, b] = h^l \in \langle h^p \rangle$, $p \mid l$ and $r \geq 1$. In particular, $n \geq 2$.

**1°.** *If $r = n - 1$ for all $b \in B^\sharp$, then (1) holds for some $b \in B^\sharp$.*

Note that $1 \leq m < p$ and there are $p - 1$ choices for $b$. It follows that $m = 1$ for some choice of $b$ and so (1) holds in this case.

So we may assume from now on that $r < n - 1$ for some $b \in B^\sharp$. Then $r + 2 \leq n$ and $n \geq 3$.

We claim that $h^{b^i} = h^{s^i}$ for all $i \in \mathbb{N}$. This clearly holds for $i = 0$ and if it holds for $i$, then $h^{b^s} = (h^b)^{b^i} = (h^s)^{b^i} = (h^{b^i})^(s) = (h^{(s)^i})^s = h^{s^{1+l}}$.

Since $b$ has order $p$, $h = h^{b^p} = h^{s^p}$ and so $h^{s^p - 1} = 1$ and $p^n \mid s^p - 1$. Thus $p^n$ divides

$$(s^p - 1) = (1 + l)^p - 1 = \sum_{i=1}^{p} \binom{p}{i} l^i = \sum_{i=1}^{p} \binom{p}{i} p^{ri} m^i$$

Since $r+2 \leq n$, also $p^{r+1}$ divides this number. If $i \geq 3$, then since $r \geq 1$, $ri \geq r+r+r \geq r+2$ and so $p^{r+2}$ divides $\binom{p}{i}p^{ri}m^i$. If $i = 1$, then $\binom{p}{i}p^{ri}m^i = pp^r m = p^{r+1}m$ and so $r+2$ does not divide $\binom{p}{i}p^{ri}$. It follows that $p^{r+2}$ does not divide $\binom{p}{2}p^{2r}m^2$ and so $p^2$ does not divide $\binom{p}{2}p^r$. Thus $r = 1$ and $p$ does not divide $\binom{p}{2}$. The latter implies that $p = 2$. We proved that $p = 2$. $r = 2$ implies that $l = 2m$ with $m$ odd. We proved that for any $B$ which does not fulfill (1°), we have $p = 2$ and $h^b = h^{1+2m}$ for some odd $m$ with $0 \leq 2m < 2^n - 1$

Define $\tilde{b}$ in $\mathrm{Aut}(H)$ by $h^{\tilde{b}} = (h^b)^{-1}$. Then $\tilde{b}^2 = 1$. If $\tilde{b} = 1$, then $h^b = h^{-1}$ and (2) holds in this case. So suppose $\tilde{b} \neq 1$. We have

$$h^{\tilde{b}} = (h^b)^{-1} = h^{-(1+2m)} = h^{2^n-1-2m} = h^{1+(2^n-2-2m)} = h^{1+2(2^{n-1}-1-m)}$$

Put $\tilde{m} = 2^{n-1} - 1 - m$. Since $n > 1$ and $m$ is odd, $\tilde{m}$ is even. Since $m$ is even whenever the assumptions of (1°) are not fulfilled, we can apply (1°) to $\tilde{B} = \langle \tilde{b} \rangle$. Since $|\tilde{B}| = p = 2$, this gives $h^{\tilde{b}} = h^{1+p^{n-1}}$ and so $h^b = h^{-(1+p^{n-1})}$. Thus (3) holds. □

**Lemma 1.9.2.** *Let $G$ be a group, $x, y \in G$, $n, m \in \mathbb{Z}$ and $p$ a prime. Put $z = [x, y]$ and suppose that $z$ commutes with $x$ and with $y$. Then*

*(a)* $[x^n, y^m] = [x, y]^{nm} = z^{nm}$.

*(b)* $(yx)^n = y^n x^n z^{\binom{n}{2}}$.

*(c)* $|z|$ *divides* $|x|$ *and* $|y|$.

*(d) If* $|z| = p$ *and* $p$ *is odd, then* $(yx)^p = y^p x^p$.

*(e) If* $|z| = 2$, *then* $(yx)^2 = y^2 x^2 z$ *and* $(yx)^4 = y^4 x^4$.

*(f) If* $x^p = y^p = 1$ *and* $p$ *is odd, then* $(yx)^p = 1$.

*(g) If* $x^2 = y^2 = 1$, *then* $(yx)^2 = z$ *and* $(yx)^4 = 1$.

*Proof.* (a) If $n = 0$ or $n = m = 1$, this is obvious. Suppose (a) holds for $n = 1$ and some $m$. Then

$$[x, y^{m+1}] = [x, y^m y] = [x, y][x, y^m]^y = z(z^m)^y = zz^m = z^{m+1}$$

Moreover,

$$[x, y^{-m}] = [x, (y^m)^{-1}] = ([x, y^m]^{-1})^{y^{-1}} = (z^{-m})^y = z^{-m}$$

So (a) holds for $n = 1$ and $m + 1$, and for $n = 1$ and $-m$. It follows that (a) holds for $n = 1$ and all integers $m$.

Put $\tilde{y} = x$, $\tilde{m} = n$, $\tilde{x} = y^m$ and $\tilde{z} = [\tilde{x}, \tilde{y}] = [y^m, x] = [x, y^m]^{-1} = z^{-m}$. Then $\tilde{z}$ commutes with $\tilde{x}$ and $\tilde{y}$ and so

$$[x^n, y^m] = [\tilde{y}^{\tilde{m}}, \tilde{x}] = ([\tilde{x}, \tilde{y}^{\tilde{m}}]^{-1}] = (\tilde{z}^{\tilde{m}})-1] = ((z^{-m})^n)^{-1} = z^{nm}$$

Thus (a) is proved for all integers $n$, $m$.

(b) This is obvious for $n = 0$ and $n = 1$. Suppose true for $n$. Then

$$(yx)^{n+1} = (yx)^n yx = y^n x^n z^{\binom{n}{2}} yx = y^n (x^n y) x z^{\binom{n}{2}} = y^n (y x^n [x^n, y]) x z^{\binom{n}{2}} = y^n y x^n z^n x z^{\binom{n}{2}} = y^{n+1} x^{n+1} z^{\binom{n}{2}+n} =$$

and

$$(yx)^{-n} = ((yx)^n)^{-1} = (y^n x^n z^{\binom{n}{2}})^{-1} = x^{-n} y^{-n} z^{-\binom{n}{2}} = y^{-n} x^{-n} [x^{-n}, y^{-n}] z^{-\binom{n}{2}} = y^{-n} z^{-n} z^{n^2-\binom{n}{2}} = y^{-n} z^{-n} z^{\binom{-n}{2}}$$

So (b) holds for $n+1$ and $-n$ and so for all integers $n$.

(c) Let $k = |x|$. Then by (a)

$$1 = [1, y] = [x^k, y] = z^k$$

and so $|z|$ divides $k$. Since $z^{-1} = [y, x]$, $|z|$ also divides $|y|$.

(d) Since $p$ is odd, $p \mid \binom{p}{2}$ and so $z^{\binom{p}{2}} = 1$. Thus (d) follows from (b).

(e) Note that $\binom{2}{2} = 1$ and so $z^{\binom{2}{2}} = z$. Also $\binom{4}{2}$ is even and so $z^{\binom{4}{2}} = 1$. Hence (e) follows from (b).

(f) and (g) By (c), $z^p = 1$. Thus (f) follows from (d) and (g) from (e). □

**Definition 1.9.3.** *(a) Let $n \in \mathbb{Z}^+$. Then $D_{2n} := \langle x, y \mid x^n = y^2 = 1, x^y = x^{-1} \rangle$. $D_{2n}$ is called the dihedral group of order $2n$.*

*(b) Let $n \in \mathbb{Z}$ with $n > 1$. Then $QD_{4n} := \langle\langle x, y \mid x^{2n} = y^2 = 1, x^y = x^{-1}x^n \rangle$. $QD_{4n}$ is called the quasi-dihedral group of order $4n$.*

*(c) Let $n \in \mathbb{Z}^+$. Then $Q_{4n} := \langle\langle x, y \mid y^4 = 1, x^n = y^2, x^y = x^{-1} \rangle$. $Q_{4n}$ is called the quasi-quaternion group of order $4n$.*

*(d) Let $n \in \mathbb{N}$ and $p$ a prime. Then $QE_{p^{n+2}} = \langle x, y \mid x^{p^{n+1}} = y^p = 1, x^y = x x^{p^n} \rangle$. $QE_{p^{n+2}}$ is called the quasi-extraspecial group of order $p^{n+2}$.*

Note that $D_2 \cong C_2$, $D_4 \cong QD_4 \cong C_2 \times C_2$, $D_8 \cong QExt(8)$, $QD_8 \cong C_4 \times C_2$, $Q_4 \cong C_4$, and $QExt(p^2) \cong C_p \times C_p$.

**Theorem 1.9.4.** *Let $p$ be a prime, $P$ a finite $p$-group and $H$ a maximal subgroup of $G$. Suppose that $H = \langle h \rangle$ is cyclic of order $p^n$. Then exactly one of the following holds:*

*(a) There exists $b \in P \setminus H$ with $|b| = p$ and $h^p = h$. So $P \cong C_{p^n} \times C_p$.*

*(b) There exists $b \in P \setminus H$ with $|b| = p$ and $h^p = h^{1+p^{n-1}}$. So $P \cong QExt(p^{n+1})$.*

*(c) $p = 2$, $n \geq 3$ and there exists $b \in P \setminus H$ with $|b| = p$ and $h^b = h^{-1}$. So $P \cong D_{2^{n+1}}$.*

*(d)* $p = 2$*,* $n \geq 3$ *and there exists* $b \in P \setminus H$ *with* $|b| = p$ *and* $h^b = h^{-(1+p^{n-1})}$*. So* $P \cong QD_{2^{n+1}}$*.*

*(e) There exists* $b \in P \setminus H$ *with* $b^p = h$*. So* $P \cong C_{p^{n+1}}$*.*

*(f)* $p = 2$*,* $n \geq 2$ *and there exists* $b \in P \setminus H$ *with* $b^2 = h^{p^{n-1}}$ *and* $h^b = h^{-1}$*. So* $P \cong Q_{2^{n+1}}$*.*

*Proof.* Let $b \in P \setminus H$ with $|b|$ minimal. By 1.9.1 we may choose $b$ such that one of the following holds:

(a) $h^b = h$.

(b) $n \geq 2$ and $h^b = h^{1+p^{n-1}}$

(c) $p = 2$, $n \geq 3$ and $h^b = h^{-1}$.

(d) $p = 2$, $n \geq 3$ and $h^b = h^{-(1+p^{n-1}}$.

Suppose first that $|b| = p$. Then one of (a),(b), (c) and (d) holds. So we may assume $|b| > p$ and so

**1°.** $|x| > p$ *for all* $x \in P \setminus H$*.*

Note that $x^p \in H$. If $x^p \notin \langle h^p \rangle$, then $H = \langle x^p \rangle$ and so $P = \langle b \rangle$ and (e) holds. So we may assume that $b^p \in \langle h^p \rangle$ and so

**2°.** $h_0^p b^p = 1$ *for some* $h_0 \in H$

Put $z = [h_0, b]$. If $z = 1$ we get $(h_0 b)^p = h_0^p b^p = 1$, contrary to (1°). Thus

**3°.** $z \neq 1$

In particular, (a) does not hold.

Suppose (b) holds. Then $z \in \langle h^{p^{n-1}} \rangle \leq Z(P)$. If $p$ is odd we conclude that $(h_0 b)^p = h_0^p b^p = 1$, contrary to (1°). Thus $p = 2$ and $((h_0 b)^4 = h_0^4 b^4 = 1$. Thus $|h_0 b| = 4$ and by minimal choice of $|b|$, also $|b| = 4$. Since $h_0^2 = b^{-2}$ we have $|h_0| = |b| = 4$. Observe that $[h^p, b] = 1$ and since $[h_0, b] \neq 1$, $H = \langle h_0 \rangle$. It follow that $h^b = h^{-1}$ and (f) holds.

Suppose that (c) and (d) holds. Put $t = h^{2^{n-1}}$. Then $|t| = 2$ and $t \in \mathbb{Z}(P)$. Note that $h^b = h^{-1}u$ with $u = 1$ or $u = t$. In either case $u^2 = 1$ and $u \in Z(P)$. Hence $(h^2)^b = (h^{-1}u)^2 = (h^2)^{-1}$ and so $b$ inverts $\langle h^2 \rangle$. Since $b^2 \in \langle h^2 \rangle$ and $b$ centralizes $b^2$, this implies $|b^2| = 2$. Thus $b^2 = t$ and $|b| = 4$. In case (c) we conclude that (f) holds. In case (c) we compute

$$(hb)^2 = hbhb = hb^2h^b = hth^{-1}t = t^2 = 1$$

a contradiction to (1°). □

**Lemma 1.9.5.** *Let* $P$ *be a finite* 2*-group,* $H$ *a maximal abelian normal subgroups of* $G$ *and put* $H_4 = \{x \in h \mid x^4 = 1\}$*. Suppose that* *(i) $H$ is cyclic.*

*(ii) If $x \in P \setminus H$ with $x^2 \in H$, then $x$ inverts $H_4$.*

*Then $|G/H| \leq 2$ and $G$ is a cyclic, dihedral, quasi-dihedral or quaternion group.*

*Proof.* Since $H$ is maximal abelian normal subgroup of $P$, $C_P(H) = H$. If $|H| \leq 2$ we conclude that $P = C_P(H) = H$ and the lemma holds in this case. So suppose $|H| \geq 4$ and $H_4 \cong C_4$. Hence $|\mathrm{Aut}(C_4)| = 2$ and $P/C_P(H_4)| \leq 2$.

Let $x \in C_G(H_4)$. Then $x$ does not invert $H_4$ and so (ii) implies that either $x \in H$ or $x^2 \notin H$. In either case $xC_G(G_4)$ does not have order 2 in $C_P(H_4)/H$. It follows that $C_P(H_4)/H$ has no elements of order 2 and so $|C_P(H_4)/H| = 1$ and $H = C_P(H_4)$. Thus $|P/H| \leq 2$. If $P = H$, $P$ is cyclic and the lemma holds. So suppose $H \neq P$. Then $H$ is a maximal subgroup of $P$ and $P$ is not abelian. 1.9.4 now shows that either $P$ is a dihedral, quasi-dihedral or quaternion group or there exists $h \in H$ and $b \in P \setminus H$ with $H = \langle h \rangle$, $|h| = 2^n$, $|b| = 2$ and $h^b = h^{1+2^{n-1}}$. In the latter case $b$ centralizes $\langle h^2 \rangle$. Thus $H_4 \not\leq \langle h^2 \rangle$, $H_4 = H$, $n = 2$ and $P \cong D_8$. □

**Lemma 1.9.6.** *Let $G$ be a finite group.*

*(a) Suppose that $G$ has a unique maximal subgroup $M$ and put $p = |G/M|$. Then $p$ is a prime and $G$ is a cyclic $p$-group.*

*(b) Suppose that $G$ has a unique minimal subgroup $M$ and put $p = |M|$. Then $p$ is a prime and either $G$ is a cyclic $p$-group or $p = 2$ and $G$ is a quaternion group.*

*Proof.* (a) Let $x \in G \setminus M$. Then $\langle x \rangle \not\leq M$ and so $\langle x \rangle$ is not contained in any maximal subgroup of $G$. Since $G$ is finite this gives $\langle x \rangle = G$ and $G$ is cyclic. Let $q$ be a prime with $q \big| |G|$. Then $|G/\langle x^q \rangle| = q$. Thus $\langle x^q \rangle$ is a maximal subgroup of $G$ and so $M = \langle x^q \rangle$ and $p = q$. So $G$ is a $p$-group.

(b) Let $q$ be a prime dividing the order $|G|$ and $x \in G$ with $|x| = q$. Then $M = \langle x \rangle$ and so $p = q$ and $G$ is a $p$-group. Let $H$ be a maximal normal abelian subgroups of $G$. Since $M$ is the unique minimal subgroup of $G$, $H$ is not the direct product of two proper subgroups and so $H$ is cyclic. If $G = H$, $G$ is cyclic and we are done. So suppose $G \neq H$ and let $a \in G \setminus H$ with $a^p \in H$. Put $Q = H\langle a \rangle$. Note that $|Q/H| = p$ and so $H$ is a maximal subgroup of $Q$. Since $M \leq H$, $\langle a \rangle \neq M$ and so $|a| \neq p$. Since $C_G(H) = H$, $Q$ is not abelian and 1.9.4 shows $a$ inverts $H$ and $Q$ is a quaternion group. By 1.9.5, $|G/H| \leq 2$ and so $G = Q$ and $G$ is a quaternion group. □

**Lemma 1.9.7.** *Let $p$ be a prime and $P$ a finite $p$-group all of whose abelian subgroups are cyclic. Then $P$ is cyclic, or $p = 2$ and $P$ is a quaternion group.*

*Proof.* If $P = 1$, $P$ is cyclic. So suppose $P \neq 1$. Then also $Z(P) \neq 1$ and there exists $A \leq Z(P)$ with $|A| = p$. Let $B$ be any minimal subgroups of $P$. Then $|B| = p$ and since $[A, B] = 1$, $AB$ is an abelian subgroup of $P$. So $AB$ is cyclic and thus has a unique subgroup of order $p$. Hence $A = B$ and so $A$ is the unique minimal subgroup of $P$. The lemma now follows from 1.9.6. □

**Lemma 1.9.8.** *Let $p$ be a prime and $P$ a finite $p$-group all of whose abelian normal subgroups are cyclic. Then $P$ is cyclic, or $p = 2$ and $P$ is a dihedral, quasidihedral or quaternion group.*

*Proof.* Let $H$ be a maximal abelian normal subgroups of $P$. Then $H$ is cyclic and $C_P(H) = 1$. If $P = H$ we are done. So we may assume that $H \neq P$. We will show that

**1°.** $p = 2$ *and if* $b \in P \setminus H$ *with* $b^p \in H$*, then* $b$ *inverts* $H_4 := \{x \in H \mid x^4 = 1\}$.

Observe that the lemma will follow from 1.9.5 once we proved (1°).

Let $b \in P \setminus H$ with $b^p \in H$. Put $Q = H\langle b\rangle$. Let $h \in H$ with $H = \langle h\rangle$ and put $|h| = p^n$. Since $P \neq H = C_P(H)$ we have $n \geq 2$ and $b$ does not centralize $h$. Note that $H$ is cyclic maximal subgroups of $Q$ and so we can apply 1.9.4. Since $[h, b] \neq 1$, neither case a nor e hold. In case a nor c and f, $b$ inverts $H$ and so (1°) holds. In case d, $b$ inverts $\langle h^2\rangle$ and since $n \geq 3$, $H_4 \leq \langle h^2\rangle$ and again (1°) holds.

So suppose that $h^b = h^{1+p^{n-1}}$. If $p = 2$ and $n = 2$, $h$ inverts $H$ and (1°) holds. So we may assume that $n \geq 3$ if $p = 2$. We will derive a contradiction in this remaining case. Observe that we may choose $b$ such that $b^p = 1$. Put $z = [h, b] = h^{p^{n-1}}$. Then $|z| = p$ and $\langle z\rangle$ is the only subgroup of order $p$ in $H$. Since $H \trianglelefteq P$, this gives $\langle z\rangle \trianglelefteq P$ and so $z \in Z(P)$. Let $0 \leq i < p^n$. Then by 1.9.2 $[h^i, b] = z^i$ and

$$(bh^i)^p = \begin{cases} b^p h^{ip} = h^{ip} & \text{if } p \text{ is odd} \\ b^p h^{ip} z^i = h^{ip} z^i & \text{if } p = 2 \end{cases}$$

Suppose $p$ does not divide $i$. If $p \neq 2$, $|h| \geq p^2$ and so $h^{ip} \neq 1$. If $p = 2$, then $|h| \geq p^3$, $|h^{pi}| \geq p^2$ and $h^{pi} \neq z^{-i}$. In either case $(bhi)^p \neq 1$. Thus $\Omega(Q) := \{x \in Q \mid x^p = 1\} \leq \langle b > \langle h^p\rangle$. Since $[h^p, b] = z^p = 1$, $\langle b > \langle h^p\rangle$ is Abelian and so also $\Omega(Q)$ is Abelian. Since $| < z\rangle$ and $\langle b\rangle$ are distinct cyclic subgroups of order $p$ in $\Omega(Q)$, $\Omega(Q)$ is not cyclic.

Since $C_P(H) = H$, $P/H$ is isomorphic to a subgroups of $\mathrm{Aut}(H)$. Since $H$ is cyclic, $\mathrm{Aut}(H)$ is Abelian and so $P/H$ is Abelian. Hence $Q/H \trianglelefteq P/H$, $Q \trianglelefteq P$ and so also $\Omega(Q) \trianglelefteq P$, a contradiction since also Abelian normal subgroups of $P$ are cyclic. □

## 1.10 Hypoabelian groups

**Definition 1.10.1.** *Let $G$ be a group.*

*(a)* $G' = [G, G]$, $G'' = (G')')$. *$G$ is called the derived subgroup of $G$.*

*(b) $G$ is called perfect if $G = G'$.*

*(c) Let $\alpha$ be an ordinal. Define $G^{(\alpha)}$ inductively as follows*

$$g^{(\alpha)} = \begin{cases} G & \text{if } \alpha = 0 \\ (G^{\beta})' & \text{if } \alpha = \beta + 1 \\ \bigcap_{\beta<\alpha} G^{(\beta)} & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

*(d)* $d_G$ *is the least ordinal with* $G^{(d_G)} = G^{(d_G+1)}$. $G^{(*)} = G^{(d_G)}$.

*(e)* $(G^{(\alpha)})_{\alpha \leq d_G}$ *is called the derived series of* $G$.

*(f) If* $G^{(*)} = 1$, *then* $G$ *is called hypoabelian of derived length* $d_G$.

*(g) If* $G^{(*)} = 1$ *and* $d_G$ *is finite, then* $G$ *is called solvable of derived length* $d_G$.

**Lemma 1.10.2.** *Let* $G$ *be a group. Then* $G^{(*)}$ *is perfect and* $G^{(*)}$ *contains all perfect subgroups of* $G$.

*Proof.* We have $G^{(*)} = G^{(d_G)} = G^{(d_G+1)} = (G^{(d_G)})' = (G^{(*)})'$. So $G^{(*)}$ is perfect.

Let $H$ be a perfect subgroup of $G$ and $\alpha$ be an ordinal. We will show that $H \leq G^{(\alpha)}$. We may assume inductively that $H \leq G^{(\beta)}$ for all $\beta < \alpha$. If $\alpha = 0$, then $G^{(\alpha)} = G$ and so $H \leq G^{(\alpha)}$. If $\alpha = \beta + 1$ then

$$H = H' = [H, H] \leq [G^{(\beta)}, G^{\beta}] = G^{(\beta+1)} = G^{(\alpha)}$$

and if $\alpha$ is a limit ordinal, then

$$H \leq \bigcup_{\beta < \alpha} G^{(\beta)} = G^{(\alpha)}$$

□

**Corollary 1.10.3.** *Let* $G$ *be a group. Then the following are equivalent*

*(a)* $G$ *is hypoabelian.*

*(b)* $G$ *is no non-trivial perfect subgroup.*

*(c)* $G$ *has no non-trivial normal perfect subgroup.*

*(d)* $G$ *has non non-trivial characteristic perfect subgroup.*

*Proof.* (a) $\Longrightarrow$ (b): If $G$ is hypoabelian, then $G^{(*)} = 1$. Then by 1.10.2 $H = 1$ for all perfect subgroup o$G^{(*)}$. (b) $\Longrightarrow$ (c): and (c) $\Longrightarrow$ (d): are obvious.

(d) $\Longrightarrow$ (a): By 1.10.2 $G^{(*)}$ is a characteristic perfect subgroup of $G$. So $G^{(*)} = 1$ and $G$ is hypoabelian. □

**Lemma 1.10.4.** *Let* $G$ *be a group,* $H \leq G$ *and* $\alpha$ *and* $\beta$ *ordinals. Then*

*(a)* $H^{(\alpha)} \leq G^{(\alpha)}$.

*(b) If* $H \trianglelefteq G$, *then* $G^{(\alpha)}H/H \leq (G/H)^{\alpha}$ *with equality if* $\alpha$ *is finite.*

*(c)* $G^{(\alpha+\beta)} = (G^{(\alpha)})^{(\beta)}$.

*Proof.* (a) Since $H' = [H, H] \leq [G, G] = G'$ this follows by induction on $\alpha$.

(b) By induction on $\alpha$. If $\alpha = 0$, then both sides are equal to $G/H$. If $\alpha = \beta + 1$, then using the induction assumption

$$G^{(\alpha)}H/H = [G^{(\beta)}, G^{(\beta)}]H/H = [G^{(\beta)}H/H, G^{(\beta)}H/H] \leq [(G/H)^{(\beta)}, (G/G)^{(\beta)}] = (G/H)^{(\alpha)}$$

with equality if $\alpha$ and so also $\beta$ is finite.

If $\alpha$ is a limit ordinal, then

$$\begin{array}{llllll} (G^{(\alpha)}H/H & = & \left(\bigcap_{\beta<\alpha} G^{(\beta)}\right) H/H & \leq & \left(\bigcap_{\beta<\alpha} G^{(\beta)}H\right)/H & = \bigcap_{\beta<\alpha} G^{(\beta)}H/H \\ & \leq & \bigcap_{\beta<\alpha}(G/H)^{(\beta)} & = & (G/H)^{(\alpha)} & \end{array}$$

(c) By induction on $\beta$. If $\beta = 0$ both sides are equal to $G^{(\alpha)}$. If $\beta = \gamma + 1$, then

$$G^{(\alpha+\beta)} = G^{(\alpha+(\gamma+1))} = G^{((\alpha+\gamma)+1)} = (G^{(\alpha+\gamma)})' = ((G^{(\alpha)})^{(\gamma)})' = ((G^{\alpha})^{(\gamma+1)} = (G^{\alpha})^{(\beta)}$$

If $\beta$ is a limit ordinal then

$$(G^{(\alpha)})^{(\beta)} = \bigcap_{\gamma<\beta}(G^{(\alpha)})^{(\gamma)} = \bigcap_{\gamma<\beta} G^{(\alpha+\gamma)} = \bigcap_{\alpha\leq\rho<\alpha+\beta} G^{(\rho)} = \bigcap_{\rho<\alpha+\beta} G^{(\rho)} = G^{(\alpha+\beta)}$$

□

**Lemma 1.10.5.** *Let $G$ be group*

*(a) If $G$ is hypoabelian of derived length $\alpha$. Then all subgroups of $G$ are hypoabelian of derived length at most $\alpha$.*

*(b) If $G$ is solvable of derived length $\alpha$ then all quotients of $G$ are solvable of derived length at most $\alpha$.*

*(c) Let $H$ be a normal subgroups of $G$. If $G/H$ is hypoabelian of derived length $\alpha$ and $H$ is hypoabelian of derived length $\beta$, then $G$ is hypoabelian of derived length at most $\alpha + \beta$.*

*(d) Let $H$ be a normal subgroups of $G$. Then $G/H$ is solvable if and only of both $H$ and $G/H$ are solvable.*

*Proof.* (a) If $G^{(\alpha)} = 1$, then by 1.10.4, $H^{(\alpha)} \leq G^{(\alpha)} = 1$.

(b) Suppose $G^{(\alpha)} = 1$ for a finite ordinal $\alpha$. Then by 1.10.4, $(G/H)^{(\alpha)} = G^{(\alpha)}H/H = H/H = 1$.

(c) We have $G^{(\alpha)}H/H \leq (G/H)^{(\alpha)} = 1$ and so $G^{(\alpha)} \leq H$. Thus

$$G^{(\alpha+\beta)} = (G^{(\alpha)})^(\beta) \leq H^{(\beta)} = 1$$

(d) Follows from (b) and (c). □

**Example 1.10.6.** *(a) All free groups are hypoabelian.*

*(b) Every group is the quotient of a hypoabelian group.*

*(c) Quotients of hypoabelian groups are not necessarily hypoabelian.*

*Proof.* (a) Let $F$ be a free group on set $I$. Then $F/F'$ is the free abelian group on $I$. So for $I \neq \emptyset$ we have $F/F' \neq 1$ and so $F \neq F'$. Hence non-trivial free groups are not perfect. Since subgroups of free groups are free groups, $F$ does not have any non-trivial perfect subgroups. Thus by 1.10.3 $F$ is hypoabelian.

(b) Since every groups is the quotient of a free group, this follows from (a).

(c) Since non-trivial perfect groups exist, this follows from (b). □

**Definition 1.10.7.** *Let $G$ be a group and $H \leq G$. A series from $H$ to $G$ is a set $\mathcal{S}$ of subgroups of $G$ such that*

*(i) $G \in \mathcal{S}$.*

*(ii) $H \in \mathcal{S}$ and $H \leq S$ for all $S \in \mathcal{S}$.*

*(iii) If $S, T \in \mathcal{S}$, then $S \leq T$ or $T \leq S$.*

*(iv) If $\mathcal{D}$ is non-empty subset of $\mathcal{S}$, then both $\bigcup \mathcal{D}$ and $\bigcap \mathcal{D}$ are in $\mathcal{S}$.*

*(v) Let $T \in \mathcal{S}$. Define $T^- = \bigcup\{B \in \mathcal{S} \mid B \not\leq T\}$ if $T \neq H$ and $T^- = H$ if $T = H$. Then $T^- \trianglelefteq T$ for all $T \in \mathcal{S}$.*

**Definition 1.10.8.** *Let $G$ be a group, $H \leq G$ and $\mathcal{S}$ a series from $H$ to $G$.*

*(a) A factor of $\mathcal{S}$ is a group $T/T^-$ where $T \in \mathcal{S}$ with $T \neq T^-$.*

*(b) $\mathcal{S}$ is called ascending, if each non-empty subset of $\mathcal{S}$ has a minimal element.*

*(c) $\mathcal{S}$ is called descending, if each non-empty subset of $\mathcal{S}$ has a minimal element.*

*(d) $\mathcal{S}$ is called normal if $T \trianglelefteq G$ for all $T \in \mathcal{S}$.*

*(e) $\mathcal{S}$ is called subnormal if $|\mathcal{S}|$ is finite.*

*(f) $\mathcal{S}$ is called a composition series from $H$ to $G$ if each factor of $\mathcal{S}$ is a simple group.*

*(g) $\mathcal{S}$ is a chief series from $H$ to $G$ if its a normal and for each factor $F$ of $\mathcal{S}$, 1 and $F$ are the only $G$-invariant subgroups of $F$.*

If $\mathcal{S}$ is a subnormal series from $H$ to $G$, then $\mathcal{S} = \{G_0, G_1, \ldots G_n\}$ with

$$H = G_0 \lhd G_1 \lhd G_2 \lhd \ldots G_{n-1} \lhd G_n = G$$

**Definition 1.10.9.** *A subgroups $H$ of $G$ is called a serial (ascending, descending, subnormal) subgroup of $G$ if there exists a (ascending, descending, subnormal) series from $H$ to $G$.*

**Example 1.10.10.** *(a) Let $G$ be a group and $H \leq G$. Then any subnormal series from $H$ to $G$ is of the form $G_i, 0 \leq \leq n$ with*

$$H = G_0 \lhd G_1 \lhd G_2 \lhd \ldots \lhd G_{n_1} \lhd G_n = G$$

*The factors are $G_i/G_{i-1}$, $1 \leq i \leq n$.*

*(b) Let $G = (\mathbb{Z}, +)$ and $H = 0$. For $i \in \mathbb{Z}^+$ let $n_i$ be a positive integer. Put $G_0 = \mathbb{Z}$ and inductively $G_i = n_i G_{i-1} = n_1 n_2 \ldots n_i \mathbb{Z}$. Then $\mathcal{S} = \{G_i \mid i \in \mathbb{N}\} \cup \{0\}$ is a descending series with factors $G_{i-1}/G_i \cong C_{n_i}$. If each $n_i$ is a prime, $\mathcal{S}$ is a composition series. If $p$ is a prime with $n_i = p$ for all $i$, then all factors of $\mathcal{S}$ are isomorphic to $C_p$. Choosing different primes we see that distinct composition series can have non-isomorphic factors.*

*(c) Let $p$ be a prime. Then*

$$1 \leq C_p \leq C_{p^2} \leq \ldots \leq C_{p^k} \leq C_{p^{k+1}} \leq \ldots C_{p^\infty}$$

*is a composition series for $C_{p^\infty}$ with factors isomorphic to $C_p$. Note here that $C_{p^\infty}^- = \bigcup_{k<\omega} C_{p^k} = C_{p^\infty}$.*

*(d) $1 \lhd Alt(n) \lhd \mathrm{Sym}(n)$, is a normal and subnormal series for $\mathrm{Sym}(n)$. If $n \neq 4$, this is also composition series and a chiefseries for $\mathrm{Sym}(n)$.*

*(e)*

$$1 \lhd \langle (12)(34), (13)(24) \rangle \lhd \mathrm{Alt}(4) \lhd \mathrm{Sym}(4)$$

*is a chiefseries but not a compositions series for $\mathrm{Sym}(4)$. The factors are isomorphic to $C_2 \times C_2$, $C_3$ and $C_2$.*

*(f)*

$$1 \lhd \langle (12)(34), (13)(24) \rangle \lhd \mathrm{Alt}(4) \lhd \mathrm{Sym}(4)$$

*is a compositions series for for $\mathrm{Sym}(4)$. Since $\langle (12)(34) \rangle$ is not normal in $\mathrm{Sym}(4)$, this is not a normal series and so also not a chiefseries. The factors are isomorphic to $C_2$, $C_2$, $C_3$ and $C_2$.*

*(g) Let $G$ be a groups. Then $\{1, G\}$ is a series for $G$, called the trivial series.*

*(h) Let $G$ be a simple group. Then the trivial series is a composition series and a chiefseries for $G$. It is the only chiefseries for $G$, but there are example of simple groups which have non-trivial series (and even non-trivial ascending series) Note here that only $G^-$ is guaranteed to be normal in $G$, but $G^-$ might be equal to $G$. But this shows that a simple groups cannot have a non-trivial descending series.*

*(i) Let $I$ be a totally ordered set such that every non-empty subset of $I$ has least upper bound and a greatest lower bound. For $i \in I$ let $i^< = \{k \in I \mid k < i\}$ and let*

$$J = \{j \in I \mid j^< \text{ has maximal element}\}$$

*Suppose that for all $i, l \in I$ with $i < l$ there exists $j \in J$ with $i < j \leq l$.*

*For $j \in J$ let $G_j$ be a nontrivial group. Put $G = \bigoplus_{j \in J} G_j$. For $i \in I$ define*

$$T_i = \bigoplus_{\substack{j \in J \\ j \leq i}} G_j$$

*Then $\{T_i \mid i \in I\}$ is a normal series for $G$ with factors $T_j/T_{j^*} \cong G_j$, where $j \in J$ and $j^*$ is the maximal element of $j^<$.*

Let us prove the assertion in (i). Clearly each $T_i$ is a normal subgroup of $G$. I in particular, $T^- \lhd T$ for all $T \in \mathcal{S}$. Let $i, l \in I$ with $i < l$. Then $T_i \leq T_l$ and so $\mathcal{S}$ is totally ordered with respect to inclusion. Moreover, there exists $j \in J$ with $i < j \leq l$. Then $G_j \leq T_l$, but $G_j \not\leq T_i$. So $T_i \lneq T_l$.

Let $k \in I$. It follow that $T_k < T_i$ if and only if $k < i$. Suppose that $i \in J$. Then $k \leq i^*$ and so $T_k \leq T_{i^*}$. Thus $T_i^- = T_{i^*}$ and $T_i/T_i^- \cong G_i$.

Suppose that $i \notin J$ and let $j \in J$ with $j \leq i$. Then $j \neq i.j < i$ and $G_j \leq T_j < T_i$. Thus $G_j \leq T_i^-$ and so $T_i = \langle G_j \mid j \in J, j \leq i \rangle \leq T_i^-$. Hence $T_i = T_i^-$.

So the factors of $\mathcal{S}$ are the groups, $T_j/T_{j^*}$, $j \in J$.

Let $\mathcal{D}$ be a non-empty subset of $\mathcal{S}$ and put $D = \{i \in I \mid T_i \in \mathcal{D}\}$. By assumption $D$ has a least upper bound $v$ and a greatest lower bound $w$. We will show that $\bigcup D = T_v$ and $\bigcap D = T_w$. Since $w \leq i \leq v$ for all $i \in D$, $T_w \leq T_i \leq T_v$ and so $T_w \leq \bigcap D$ and $\bigcup D \leq T_v$.

Let $j \in J$ with $j \leq v$. If $j \neq v$, then $j < v$ and since $v$ is the least upper bound of $D$, $j$ is not an upper bound. Hence there exists $i \in D$ with $j < i$ and so $G_j \leq T_i \leq \bigcup \mathcal{D}$. If $j = v$, then $v^* < v$ and so $v^* < i$ for some $i \in D$. Then $v^* < i \leq v$, and $v = i \in D$. So $G_j \leq T_v = T_i \leq \bigcup \mathcal{D}$. We proved $G_j \leq \bigcup \mathcal{D}$ for all $j \in J$ with $j \leq v$. Thus $T_v \leq \bigcup \mathcal{D}$ and $T_v = \bigcup \mathcal{D}$.

Let $g \in \bigcup \mathcal{D}$. Let $j \in J$ with $g_j \neq 1$ and $i \in D$. Since $g \in T_i$, $j \leq i$. Since $w$ is the greatest lower bound of $D$, this gives, $j \leq v$ and so $g = \prod_{\substack{j \in J \\ g_j \neq 1}} \in T_w$. Hence $\bigcap \mathcal{D} \leq T_w$ and $\bigcap \mathcal{D} = T_w$.

Let $x$ be the greatest lower bound of $I$. Then $x^- = \emptyset$, $x \notin J$ and $\{j \in J \mid j \leq x\} = \emptyset$. So $T_x = 1$. Let $y$ be the least upper bound of $I$. Then $\{j \in J \mid j \leq x\} = J$ and so $T_y = G$. This completes the proof of (i).

**Lemma 1.10.11.** *Let $G$ be a group, $H$ a subgroup of $G$ and $\mathcal{S}$ a set of subgroups of $G$. Then the following are equivalent:*

*(a) $\mathcal{S}$ is ascending series from $H$ to $G$.*

*(b) There exists an ordinal $\delta$ and subgroups $G_\alpha, \alpha \leq \delta$ of $G$ such that $\mathcal{S} = \{G_\alpha \mid \alpha \leq \delta\}$ and*

*(a) $G_0 = H$ and $G_\delta = G$.*

*(b) $G_\alpha \lhd G_{\alpha+1}$ for all $\alpha < \delta$.*

*(c) $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ for all limit ordinals $\alpha \leq \delta$.*

*(c) There exists an ordinal $\delta$ and subgroups $G_\alpha, \alpha \leq \delta$ of $G$ such that $\mathcal{S} = \{G_\alpha \mid \alpha \leq \delta\}$ and*

*(a) $G_0 = H$ and $G_\delta = G$.*

*(b) $G_\alpha \trianglelefteq G_{\alpha+1}$ for all $\alpha < \delta$.*

*(c) $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$ for all limit ordinals $\alpha \leq \delta$.*

*In (b), the factors of $\mathcal{S}$ are the groups $G_{\alpha+1}/G_\alpha$, $\alpha < \delta$ and in (c) the factors are the groups, $G_{\alpha+1}/G_\alpha$, $\alpha < \delta$, $G_\alpha \neq G_{\alpha+1}$.*

*A similar statement holds for descending series: Replace (b:a) and (c:a) by $G_0 = G$ and $G_\delta = H$, (b:b) by $G_{\alpha+1} \lhd G_\alpha$, (c:b) by $G_{\alpha+1} \trianglelefteq G_\alpha$ and (b:c) and (c:c) by $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$. The factors are now of the form $G_\alpha/G_{\alpha+1}$.*

*Proof.* (a) $\Longrightarrow$ (b): Since $\mathcal{S}$ is a well ordered set there exists an ordinal $\sigma$ and an isomorphism of order sets $f : \sigma \to \mathcal{S}$. Since $\mathcal{S}$ has a maximal element, $\sigma$ has a maximal element $\delta$ and so $\sigma = \{\alpha \mid \alpha < \sigma\} = \{\alpha \mid \alpha \leq \delta\}$. For $\alpha \leq \delta$ define $G_\alpha = f(\alpha)$. $G_0$ is the minimal element of $\mathcal{S}$ and so $G_0 = H$. $G_\delta$ is the maximal element of $\mathcal{S}$ and so $G_\delta = G$. Observe that $\beta < \alpha + 1$ if and only if $\beta \leq \alpha$ and so $G^-_{\alpha+1} = G_\alpha$ and $G_\alpha \trianglelefteq G_{\alpha+1}$. Since $f$ is 1-1, $G_\alpha \neq G_{\alpha+1}$.

Let $\alpha$ be a limit ordinal. Then $G^-_\alpha = G_\beta$ for some $\beta \leq \delta$. Since $G_\beta = G^-_\alpha \leq G_\alpha$, $\beta \leq \alpha$. Also $G_\gamma \leq G_\beta$ for all $\gamma < \alpha$ and so $\gamma < \beta$. Since $\alpha$ is a limit ordinal this gives $\beta = \alpha$ and so

$$G_\alpha = G_{\alpha^-} = \bigcup_{\gamma<\alpha} G_\gamma$$

Thus (b) holds. Moreover, $G_\alpha \neq G^-_\alpha$ if and only if $\alpha = \beta + 1$ for an ordinal $\beta$. So the factors are as claimed.

(b) $\Longrightarrow$ (c): Obvious.

(c) $\Longrightarrow$ (a): By (c:a), $H \in \mathcal{S}$ and $G \in \mathcal{S}$. From (c:b) and (c:b) we have $G_\alpha \leq G_\beta$ whenever $\alpha \leq \beta$ and so $H \leq G_\beta$ for all $\beta \leq \delta$.

Let $\mathcal{D}$ be non-empty subset of $\mathcal{S}$. Let $\Delta = \{\alpha \leq \delta \mid G_\alpha \in \mathcal{D}$. Let $\rho$ be the minimal element if $\Delta$. Then $\bigcap \mathcal{D} = G_\rho \in \mathcal{S}$. Let $\mu = sup\Delta$. If $\beta < \mu$, then $\beta < \alpha$ for some $\alpha \in \Delta$ and so $G_\beta \leq G_\alpha \leq G_\mu$. Thus $G^-_\mu \leq \bigcup \mathcal{D} \leq G_\mu$. If $\mu$ is a limit ordinal, then by (c:c), $G^-_\mu = G_\mu$ and so $\bigcup \mathcal{D} = G_\mu$. If $\mu = 0$, then $\bigcup \mathcal{D} = H$. So suppose $\mu = \beta + 1$ for some ordinal $\alpha$. Then $\beta < \alpha$ for some $\alpha \in \Delta$ and so $\mu = \alpha \in \Delta$. Thus again $\bigcup \mathcal{D} \leq G_\mu$.

Let $H \neq S \in \mathcal{S}$ and pick $\alpha \leq \delta$ minimal with $S = G_\alpha$. Then for $\beta \leq \delta$ we have $S \leq G_\beta$ if and only of $\alpha \leq \beta$ and so $G_\beta < S$ if and only of $\beta < \alpha$. Thus

$$S^- = \bigcup_{\beta<\alpha} G_\beta$$

If $\beta = \alpha + 1$, then $S^- = G_\beta \trianglelefteq G_{\beta+1} = G_\beta$. If $\alpha$ is a limit ordinal we get $S^- = G_\alpha$. So in any case $S^- \trianglelefteq S$ and the factors are as claimed. □

**Lemma 1.10.12.** *Let $G$ be a group. $H$ a subgroup of $G$, $A$ a subset of $G$ and $(G_\alpha)_{\alpha\leq\delta}$ an ascending or descending series from $H$ to $K$. Suppose that one of the follwing holds*

1. *$H \subseteq A \neq G$, $(G_\alpha)_{\alpha\leq\delta}$ is ascending and $\beta$ is the ordinal minimal with respect to $G_\beta \nsubseteq A$.*
2. *$H \nsubseteq A$, $(G_\alpha)_{\alpha\leq\delta}$ is ascending, $A$ is finite and $\beta$ is the ordinal minimal with respect to $A \subseteq G_\beta$.*
3. *$H \nsubseteq A$, $(G_\alpha)_{\alpha\leq\delta}$ is descending, and $\beta$ is the ordinal minimal with respect to $A \nsubseteq G_\beta$.*

*Then $\beta$ is well-defined and there exists a ordinal $\gamma$ with $\beta = \gamma + 1$.*

*Proof.* Suppose (1) holds. Since $A \nsubseteq G = G_\delta$, $\beta$ is well-defined. Since $H = G_0 \subseteq A$, $\beta \neq 0$. Suppose for a contradiction, that $\beta$ is a limit ordinal. Since $G_\gamma \subseteq A$ for all $\gamma < \beta$, we get $G_\beta = \bigcup_{\gamma<\beta} G_\gamma \subseteq A$, a contradiction. Thus $\beta$ is not a limit ordinal and the lemma holds in this case.

Suppose (2) holds. Since $A \leq G = G_\delta$, $\beta$ is well-defined. Since $G_0 = H \nsubseteq A$, $\beta \neq 0$. Suppose that $\beta$ is a limit ordinal. Then $A \subseteq G_\beta = \bigcup_{\gamma<\beta} G_\beta$ and so for each $a \in A$ there exists $\gamma_a < \beta$ with $a \in G_{\gamma_a}$. Since $A$ is finite, $\gamma := \max_{a\in A} \gamma_a$ exists and $\gamma < \beta$. But then $A \subseteq G_\gamma$, contrary to the minimal choice of $\beta$.

Suppose (3) holds. Since $G_\delta = H \nsubseteq A$, $\beta$ is well defined. Since $A \subseteq G = G_0$, $\beta \neq 0$. Suppose $\beta$ is a limit ordinal. Then $A \subseteq G_\gamma$ for all $\gamma < beta$ and so $A \subseteq \bigcap_{\gamma<\beta} G_\gamma = G_\beta$, a contradiction. □

**Lemma 1.10.13.** *Let $G$ be a group, $H, K \leq G$ and $\mathcal{S}$ a series from $H$ to $K$.*

*(a) Put $\mathcal{T} = \{K \cap S \mid S \in \mathcal{S}\}$. Then $\mathcal{T}$ is a series from $K \cap H$ to $K$. Then factors of $\mathcal{R}$ are the groups*

$$K \cap S/K \cap S^- \cong (K \cap S)S^-/S^-$$

*for $S \in \mathcal{S}$ with $K \cap S \neq K \cap S^-$. In particular, every factor of $\mathcal{T}$ is isomorphic to a subgroup of a factor of $\mathcal{S}$.*

*(b) Suppose $K \trianglelefteq G$ and $\mathcal{S}$ is ascending. Put $\mathcal{R} = \{SK/S \mid K \in \mathcal{S}\}$. Then $\mathcal{R}$ is series from $HK/K$ to $G/K$ with factors*

$$SK/K/S^-K/K \cong S/(S \cap K)S^- \cong S/S^-/(S \cap K)S^-/S^-$$

*where $S \in \mathcal{S}$ with $SK \neq S^-K$. In particular, every factor of $\mathcal{R}$ is isomorphic to a quotient of a factor of $\mathcal{S}$.*

*Proof.* (a) Since $H \in \mathcal{S}$, $H \cap K \in \mathcal{T}$ and since $G \in \mathcal{S}$, $K = K \cap G \in \mathcal{T}$. Since $H \leq S$ for all $H \in \mathcal{S}$, $H \cap K \in H \cap S$ for all $H \cap S \in \mathcal{T}$.

Let $\mathcal{D}$ be a non-empty subset of $\mathcal{T}$ and put $\mathcal{E} = \{S \in \mathcal{S} \mid K \cap S \in \mathcal{D}\}$. Then $\mathcal{D} = \{K \cap S \mid S \in \mathcal{E}\}$.

Thus

$$\bigcap \mathcal{D} = \bigcap_{S \in \mathcal{E}} K \cap S = K \cap \bigcap \mathcal{E} \in \mathcal{T} \text{ and } \bigcup \mathcal{D} = \bigcup_{S \in \mathcal{E}} K \cap S = K \cap \bigcup \mathcal{E} \in \mathcal{T}$$

Let $T \in \mathcal{T}$ and let $S = \bigcap\{R \in \mathcal{S} \mid K \cap R = T\}$. Then $K \cap S = T$. Let $R \in \mathcal{S}$. If $S \leq R$, then $T = K \cap S \leq K \cap R$. If $K \cap R \leq T$, then $K \cap (S \cap R) = (K \cap S)\alpha' R = T \cap R = T$ and so $S \leq S \cap R$ be definition of $S$. Thus $S \leq R$. We shows that $S \leq R$ if and only if $T \leq K \cap R$ and so $R < S$ if and only if $K \cap R < T$. Hence

$$T^- = \bigcup\{K \cap R \mid R \in \mathcal{S}, K \cap R < T\} = \bigcup\{K \cap R \mid R \in \mathcal{S}, R < S\} = K \cap S^-$$

Since $S^- \trianglelefteq S$, $T^- = K \cap S^- \trianglelefteq K \cap S = T$. Also $T/T^- = (K \cap S)/(K \cap S^-) = (K \cap S)/(K \cap S) \cap S^- \cong (K \cap S)S^-/S^-)$.

Give any $R$ in $\mathcal{S}$ with $T = K \cap R$ and $K \cap R^- \neq K \cap R$. Then $S \leq R$ by definition of $S$. Since $T_K \cap S \not\leq K \cap R^-$, $S \not\leq R^-$. Thus $S \not< R$ and so $R = S$. So the factors are exactly as claimed.

(b) $\overline{G} = G/K$ and let $\mathcal{S} = \{G_\alpha \mid \alpha \leq \delta\}$ as in 1.10.11. Then $\mathcal{T} = \{\overline{G}_\alpha \mid \alpha \leq \delta\}$, $\overline{G}_0 = \overline{H}$, $\overline{G}_\delta = \overline{G}$, $\overline{G}_\alpha \trianglelefteq \overline{G}_{\alpha+1}$ and if $\alpha$ is a limit ordinal, then

$$\overline{G}_\alpha = \overline{\bigcup_{\beta<\alpha} G_\beta} = (\bigcup_{\beta<\alpha} G_\beta)K/K = \bigcup_{\beta<\alpha} G_\beta K)/K = \bigcup_{\beta<\alpha} \overline{G}_\beta$$

So by 1.10.11 $\overline{S}$ is an ascending series with factors $\overline{G}_{\alpha+1}/\overline{G}_\alpha$ where $\alpha \leq \delta$ with $\overline{G}_{\alpha+1} \neq \overline{G}_\alpha$, that is $G_{\alpha+1}K \neq G_\alpha K$. Since

$$\begin{aligned} G_{\alpha+1}K/G_\alpha K &= G_{\alpha+1}G_\alpha K/G_\alpha K &\cong& \quad G_{\alpha+1}/G_{\alpha+1} \cap G_\alpha K \\ &= G_{\alpha+1}/(G_{\alpha+1} \cap K)G_\alpha &\cong& \quad G_{\alpha+1}/G_\alpha\big/(G_{\alpha+1} \cap K)G_\alpha/G_\alpha \end{aligned}$$

the factors are as claimed. □

**Lemma 1.10.14.** *Let $G$ be a group and $(G_\alpha)_{\alpha<\delta}$ a descending series from $H$ to $G$ with Abelian factors. Then $G^{(\alpha)} \leq G_\alpha$ for all $\alpha \leq \delta$. In particular, $G^{(*)} \leq G^{(\delta)} \leq H$.*

*Proof.* By induction on $\alpha$, $G_0 = G = G^{(0)}$. Suppose $\alpha = \beta + 1$. Since $G_\beta/G_\alpha$ is Abelian,

$$G^{(\alpha)} = (G^{(\beta)})' \leq G'_\beta \leq G_\alpha$$

If $\alpha$ is a limit ordinal, then

$$G^{(\alpha)} = \bigcap_{\beta<\alpha} G^{(\beta)} \leq \bigcap_{\beta<\alpha} G_\beta = G^{(\alpha)}$$

□

**Corollary 1.10.15.** *Let $G$ be a group.*

*(a) $G$ is hyboabelian if and only there exists a descending series with abelian factors for $G$.*

*(b) $d_G$ is the smallest length of a descending series with Abelian factors from $G^{(*)}$ to $G$.*

*Proof.* Note that $G^{(\alpha)}/G^{(\alpha+1)} = G^{(\alpha)}/(G^\alpha)'$ is Abelian and so the derived series is a descending series of length $d_G$ with abelian factors from $G^{(*)}$ to $G$. So there exists a descending series with Abelian factors of length $d_G$ from $G^{(*)}$ to $G$. Also if $G$ is hyper abelian, there exists a descending series with abelian factors for $G$.

Now let Let $(G_\alpha)_{\alpha\leq\delta}$ be a descending series from $H$ to $G$. Then $G^{(*)} \leq H$. If $H = 1$ we conclude that $G$ is hypoabelian. If $H = G^{(*)}$ we get $G_\delta = H = G^{(*)}$ and so $d_G \leq \delta$. □

**Lemma 1.10.16.** *Let $G$ be a group and $A$ and $B$ subgroups of $G$ such that $A$ normalizes $B$ Suppose that $A$ is solvable of derived length $\alpha$ and $B$ hypoabelian of derived length $\beta$. Then $AB$ is hypoabelian of derived length at most $\alpha + \beta$. In particular, if $A$ and $B$ are solvable, so is $AB$.*

*Proof.* Note that $AB/B \cong A/A \cap B$ and so $AB/B$ is solvable of derived length at most $\alpha$. Thus by 1.10.5(c), $AB$ is hypoabelian of derived length at most $\alpha + \beta$. □

**Definition 1.10.17.** *Let $G$ be a group. Then $F(G)$ is the subgroup generated by the nilpotent normal subgroups of $G$ and* Sol$(G)$ *is the groups generated by the solvable normal subgroups of $G$. $F(G)$ is called the Fitting subgroup of $G$.*

**Corollary 1.10.18.** *Let $G$ be a finite group. Then* Sol$(G)$ *is solvable and so* Sol$(G)$ *is t has unique maximal solvable normal subgroup of $G$.*

*Proof.* Since $G$ has only finitely many solvable normal subgroups, 1.10.16 implies that Sol$(G)$ is solvable. □

**Lemma 1.10.19.** *Let $G$ be group and $A$ and $B$ be hypercentral normal subgroups of $G$. Then $AB$ is hypercentral of class at most $(z_B + 1)z_A + z_B$.*

*Proof.* Put $x = z_A$, $y = z_B$. Define $X_\alpha = Z_\alpha(A)$ for $\alpha \leq x$ and $X_{x+1} = G$. Put $Y_\beta = Z_\beta(B)$ for all $\beta \leq y$. Then

$$[X_{\alpha+1}, A] \leq X_\alpha \text{ for all } \alpha \leq x \text{ and } [Y_{\gamma+1}, B] \leq Y_\gamma \text{ for all } \gamma < y$$

For $\alpha \leq x$ and $\beta \leq y$ define $Z_{\alpha,\beta} = X_\alpha(X_{\alpha+1} \cap Z_\beta(A))$. Note that $X_x = A$, $X_{x+1} = G$ and $Y_y = B$ so $Z_{x,y} = A(G \cap B) = AB$. We claim that

$$Z_{\alpha,\beta} \leq Z_{(y+1)\alpha+\beta}(AB) \text{ for all } \alpha \leq x, \beta \leq y$$

The proof of the claim is by induction on $\alpha$ and then by induction on $\beta$. If $\alpha = \beta = 0$, both sides are equal to 1.

Suppose $\alpha = \gamma + 1$ and $\beta = 0$. Then $Z_{\alpha,0} = X_\alpha$, $[X_\alpha, A] \leq X_\gamma \leq Z_{\gamma,y}$ and

$$[X_\alpha, B] \leq X_\alpha \cap B = X_{\gamma+1} \cap Y_y \leq Z_{\gamma,y}$$

and so

$$[Z_{\alpha,0}, AB] \leq Z_{\gamma,y} \leq Z_{(y+1)\gamma+y}(AB)$$

So

$$Z_{\alpha,0} \leq Z_{((y+1)\gamma+y)+1}(AB) = Z_{(y+1)\gamma+(y+1)}(AB) = Z_{(y+1)\alpha+0}(AB)$$

and the claim holds in this case.

Suppose $\alpha$ is a limit ordinal and $\beta = 0$. Then

$$Z_{\alpha,0} = X_\alpha = \bigcup_{\gamma<\alpha} X_\gamma = \bigcup_{\gamma<\alpha} Z_{\gamma,0} \leq \bigcup_{\gamma<\alpha} Z_{(y+1)\gamma}(AB) \leq Z_{(y+1)\alpha+0}$$

Suppose $\beta = \gamma + 1$. Then

$$[Z_{\alpha,\beta}, A] \leq [X_{\alpha+1}, A] \leq X_\alpha \leq Z_{\alpha,\gamma}$$

and

$$[Z_{\alpha,\beta}, B] \leq X_\alpha(X_{\alpha+1} \cap Z_\gamma) = Z_{\alpha,\gamma}.$$

Thus

$$[Z_{\alpha,\beta}, AB] \leq Z_{\alpha,\gamma} \leq Z_{(y+1)\alpha+\gamma}(A)$$

and so

$$Z_{\alpha,\beta} \leq Z_{(y+1)\alpha+\gamma+1}(A) \leq Z_{(y+1)\alpha+\beta}(A).$$

Suppose $\beta$ is a limit ordinal. Then

$$Z_{\alpha,\beta} = X_\alpha(X_{\alpha+1} \cap \bigcup_{\gamma<\beta} Y_\gamma) = \bigcup_{\gamma<\beta} X_\alpha(X_{\alpha+1} \cap Y_\gamma) \leq \bigcup_{\gamma<\beta} Z_{(y+1)\alpha+\gamma}(A) \leq Z_{(y+1)\alpha+\beta}(A).$$

This proves the claim. Hence $AB = Z_{x,y} \leq Z_{(y+1)x+y}(AB)$ and the lemma is proved. □

**Corollary 1.10.20.** *(a) Let $A$ and $B$ be normal nilpotent subgroups of a group $G$. Then $AB$ is nilpotent.*

(b) *Let $G$ be a finite group, then $F(G)$ is nilpotent and so $F(G)$ is unique maximal nilpotent normal subgroup of $G$.*

*Proof.* (a) Note that $(z_B+1)z_A+Z_B$ is finite. So (a) follows from 1.10.19. (b) follow from (a). □

**Remark 1.10.21.** *There exist a group $G$ with a normal ascending series*

$$1 = G_0 < G_1 < \ldots G_2 \ldots G_\omega = G$$

*such that for all $k<\omega$, $G_k$ is nilpotent for each finite $k$ and*

$$\bigcap_{k\leq i<\omega} Z(G_i) = 1$$

*It follows that $Z(G)=1$ and so $G$ is not hypercentral. Since $G$ is the union of its normal nilpotent subgroups, $G=F(G)$. It follows that $F(G)$ is neither nilpotent nor hypercentral. So $G$ has neither a maximal nilpotent normal subgroup, not a maximal hypercentral normal subgroup.*

## 1.11 The Theorem of Schur-Zassenhaus

**Theorem 1.11.1** (Schur-Zassenhaus)**.** *Let $G$ be a finite group and $K$ a normal subgroup of $G$ such that* $\gcd(|K|,|G/K|)=1$. *Then there exists a complement to $K$ in $G$. If in addition, $K$ or $G/K$ is solvable*[1]*, then all such complements are conjugate.*

*Proof.* We will first prove the existence of a complement. Let $H$ be a subgroup of $G$ minimal with respect to $G=HK$. Put $A=H\cap K$. If $H=UA$ for some $U\leq H$, then $G=HK=UAK=UK$ and so $U=H$. Let $S$ be a Sylow $p$-subgroup of $A$. Then $H=N_H(S)A$ and so $S\trianglelefteq H$. Hence $A$ is nilpotent. Put $\overline{H}=H/A'$. Note that $|\overline{A}|$ divides $|K|$ and $|\overline{H}/\overline{A}|=|H/A|=|H/H\cap K|=|HK/K|=|G/K$. So $\gcd(|\overline{H}/\overline{A}|,|\overline{A}|)=1$. Hence by Gaschütz' Theorem, there exist complement $\overline{U}$ to $\overline{A}$ in $\overline{H}$. Let $U$ be the inverse image of $\overline{U}$ in $H$. Then $H=UA$ and $U\cap A=A'$. Thus $H=U$ and $A=A'$. Thus $l_A=0$ and since $A$ is nilpotent, $A=1$. Hence $H$ is a complement to $K$ in $H$.

Let $H_1$ and $H_2$ be complements to $K$ in $G$.

Suppose that $K$ is solvable. Let $\overline{G}=G/K'$. Then $\overline{H_i}$ is a complement to $\overline{K}$ in $\overline{G}$ and so by Gaschütz' Theorem $\overline{H}_1^g=\overline{H}_2$ for some $g\in G$. Then $K'H_1^g=K'H_2$ and $H_1^g$ and $H_2$ are complement to $K'$ in $K'H_2$. By induction on the derived length of $K$, $H_1^g$ and $H_2$ are conjugate in $K'H_2$. Hence $H_1$ and $H_2$ are conjugate in $G$.

Suppose next that $G/K$ is solvable. If $G=K$, $H_1=H_2=1$. So suppose $G\neq K$. Then $G/K\neq(G/K)'$ and there exists a $M$ maximal subgroup $M$ of $G$ with $KG'\leq M$. Then $M\trianglelefteq G$ and so $|G/M|=p$, $p$ a prime. Note that $M\cap H_i$ is a complement to $K$ in $M$ and so

[1]Since $K$ or $G/K$ have odd order, the Feit-Thompson odd order theorem asserts that this assumption is always fulfilled

by induction on $|G|$, $(M \cap H_1)^g = M \cap H_2$ for some $g \in M$. Replacing $H_1$ by $H_1^g$ we may assume that $M \cap H_1 = M \cap H_2$. Put $D = \langle H_1, H_2 \rangle$ and not that $M \cap H_1$ is normal in $D$. Also $|\overline{H_i}| = p$ and since $|D/H_i|$ divides $|G/H_i| = |K|$, $p$ does not divide $\overline{D}/\overline{H})i$. Thus $\overline{H}_i$ is a Sylow $p$-subgroup of $\overline{D}$. Hence $\overline{H_1}^d = \overline{H_2}$ for some $d \in D$ and then $H_1^d = H_2$. □

## 1.12 Varieties

**Definition 1.12.1.** *A class of groups is a class $\mathcal{D}$ such that*

*(i) All members of $\mathcal{D}$ are groups.*

*(ii) $\mathcal{D}$ contains a trivial group.*

*(iii) If $G \in \mathcal{D}$ and $H$ is a groups isomorphic to $G$, then $H \in \mathcal{D}$.*

Examples: The class of all groups, the class of finite groups, the class of abelian groups and the class of solvable groups.

A $\mathcal{D}$-group is a member of $\mathcal{D}$, a $\mathcal{D}$ subgroup of a group $G$ is a $\mathcal{D}$-group $H$ with $H \leq G$. A $\mathcal{D}$-quotient of a group $G$ is group $G/H$ where $H \trianglelefteq G$ and $G/H \in \mathcal{D}$.

**Definition 1.12.2.** *Let $(G_i)_{i \in I}$ be a family of groups. Then a subdirect product of $(G_i)_{i \in I}$ is subgroups $G$ of $\bigtimes_{i \in I} G_i$ such that the projection of $G$ onto each $G_i$ is onto.*

**Lemma 1.12.3.** *Let $H$ be a subdirect product of $(G_i)_{i \in I}$. If $G$ is finite, there exists a finite subset $J$ of $I$ such that $H$ is isomorphic a subdirect product of $(G_j)_{j \in J}$.*

*Proof.* For $J \subseteq I$, let $H_J$ be the projection of $H$ on $\bigtimes_{j \in J} G_j$ and let $K_J$ be the kernel of this projection. Observe that $H_J$ is a subdirect product of $(G_j)_{j \in J}$. Choose $J \subseteq I$ such that $J$ is finite and $K_J$ is minimal. Let $h \in H$ with $h \neq 1$ and pick $i \in I$ with $h_i \neq I$. Put $R = J \cup \{i\}$. Note that $K_R \leq K_J$ and so by minimality of $K_J$, $H_R = H_J$. Note that $h \notin K_R$. Thus $h \notin K_J$ and so $K_J = 1$. Hence $H \cong H_J$. □

**Definition 1.12.4.** *Let $\mathcal{D}$ and $\mathcal{E}$ be a classes of group with $\mathcal{D} \subseteq \mathcal{E}$.*

*(a) We say that $\mathcal{D}$ is* **S***-closed in $\mathcal{E}$, if all $\mathcal{E}$-subgroups of $\mathcal{D}$-groups are $\mathcal{D}$-groups. (That is, if $G \in \mathcal{D}$ and $H \leq G$ with $H \in \mathcal{E}$, then $H \in \mathcal{D}$.)*

*(b) We say that $\mathcal{D}$ is* **Q***-closed in $\mathcal{E}$, if all $\mathcal{E}$-quotients of $\mathcal{D}$-groups are $\mathcal{D}$-groups.*

*(c) We say that $\mathcal{D}$ is* **R***-closed in $\mathcal{E}$, if each $G \in \mathcal{E}$ which is a subdirect produc of $\mathcal{D}$ groups, is a $\mathcal{D}$-group.*

*(d) Let $\mathfrak{A} \subseteq \{\mathbf{S}, \mathbf{Q}, \mathbf{R}\}$. Then $\mathcal{D}$ is called $\mathfrak{A}$-closed in $\mathcal{E}$ if $\mathcal{D}$ is $\mathbf{T}$-closed in $\mathcal{E}$ for all $\mathbf{T} \in \mathfrak{A}$. $\mathcal{D}$ is called $\mathfrak{A}$-closed if $\mathcal{D}$ is $\mathfrak{A}$-closed in the class of all groups.*

**Example 1.12.5.** *(a) The class of abelian groups is $\{\mathbf{S}, \mathbf{Q}, \mathbf{R}\}$-closed.*

*(b) The classes of finite groups is* $\{\mathbf{S},\mathbf{Q}\}$*-closed but not* $\mathbf{R}$*-closed.*

*(c) The class of solvable groups is* $\{\mathbf{S},\mathbf{Q}\}$*-closed but not* $\mathbf{R}$*-closed.*

*(d) The class of nilpotent groups is* $\{\mathbf{S},\mathbf{Q}\}$*-closed but not* $\mathbf{R}$*-closed*

*(e) The classes of finite solvable groups is* $\{\mathbf{S},\mathbf{Q},\mathbf{R}\}$ *closed in the class of finite groups.*

*(f) The class of finite nilpotent groups is* $\{\mathbf{S},\mathbf{Q},\mathbf{R}\}$ *closed in the class of finite groups.*

*(g) For a fixed prime* $p$*, the class of finite* $p$*-groups is* $\{\mathbf{S},\mathbf{Q},\mathbf{R}\}$ *-closed in the class of finite groups.*

For $i \in \mathbb{Z}^+$ let $G_i$ be a finite solvable group of order $i$, a solvable group of derived length $i$ and nilpotent groups of class $i$, respectively. Then $\bigtimes_{i\in I} G_i$ is not finite, solvable and nilpotent respectively. This shows the classe of finite groups, the class of solvable groups and the class of nilpotent groups are not $\mathbf{R}$-closed.

Let $\mathcal{D}$ be the class of finite solvable groups, or the class of finite nilpotent groups or the class of finite $p$-groups. Let $H$ be a subdirect product of $\mathcal{D}$-groups. Suppose that $H$ is finite. Then by 1.12.3, $H$ is the subdirect products of finitely many $\mathcal{D}$-groups. Observe that the direct product of finitely many $\mathcal{D}$-groups is a $\mathcal{D}$-groups and so $H$ is a $\mathcal{D}$-group. Thus $\mathcal{D}$ is a $\mathbf{R}$-closed in the class of finite groups.

**Definition 1.12.6.** *A variety is a pair* $(\mathcal{D},\mathcal{E})$ *of classes of groups such that*

*(a)* $\mathcal{D} \subseteq \mathcal{E}$.

*(b)* $\mathcal{E}$ *is* $\mathbf{SQ}$*-closed.*

*(c)* $\mathcal{D}$ *is* $\mathbf{SQR}$*-closed in* $\mathcal{E}$.

We remark that our use of term 'variety' is non-standard. Usually a variety is class of groups defined in terms of vanishing of a set of words. Birkhoff's theorem asserts that class $\mathcal{D}$ of groups is a variety if and only if $\mathcal{D}$ is $\{\mathbf{S},\mathbf{Q},\mathbf{R}\}$-closed. Note that this holds if and only if $(\mathcal{D}, \text{class of all groups})$ is a variety in our sense.

**Example 1.12.7.** *The following pairs of classes of groups are variety:*

*(a)* $(\textit{class of abelian groups}, \textit{class of all groups})$.

*(b)* $(\textit{class of finite nilpotent groups}, \textit{class of all finite groups})$.

*(c)* $(\textit{class of finite solvable groups}, \textit{class of all finite groups})$.

*(d) For a fixed prime* $p$, $(\textit{class of finite } p\textit{-groups}, \textit{class of all finite groups})$.

**Definition 1.12.8.** *Let* $G$ *be a group and* $\mathcal{D}$ *a class of groups. Then*

$$G^{\mathcal{D}} = \bigcap\{H \trianglelefteq G \mid G/H \in \mathcal{D}\}$$

*$G$ is called $\mathcal{D}$-perfect, if $G = G^{\mathcal{D}}$, that is no-nontrivial quotient of $G$ is a $\mathcal{D}$-group.*

**Lemma 1.12.9.** *Let $\mathcal{D}$ be class of groups, $G$ a group and $H \trianglelefteq G$. Then*

$$G^{\mathcal{D}}H/H \leq (G/H)^{\mathcal{D}}.$$

*Proof.* Put $\overline{G} = G/H$ and let $\overline{R} \trianglelefteq \overline{G}$ such that $\overline{G}/\overline{R} \in \mathcal{D}$. Let $R$ be the inverse image of $\overline{R}$ in $\overline{G}$. Then $G/R \cong \overline{G}/\overline{R}$ and so $G/R \in \mathcal{D}$. Thus $G^{\mathcal{D}} \leq R$ and $\overline{G^{\mathcal{D}}} \leq \overline{R}$. Since this holds for all such $\overline{R}$, $\overline{G^{\mathcal{D}}} \leq (\overline{G})^{\mathcal{D}}$. □

**Lemma 1.12.10.** *Let $\mathcal{D}$ and $\mathcal{E}$ be classes of groups with $\mathcal{D} \subseteq \mathcal{E}$. Suppose that $\mathcal{E}$ is* **Q***-closed. Then $\mathcal{D}$ is* **R***-closed in $\mathcal{E}$ if and only if $G/\bigcap \mathcal{M} \in \mathcal{D}$ whenever $G \in \mathcal{E}$ and $\mathcal{M}$ is set of normal subgroups of $G$ with $G/M \in \mathcal{D}$ for all $M \in \mathcal{M}$.*

*Proof.* $\Longrightarrow$: Let $\mathcal{M}$ be a set of normal subgroups of $G$ with $G/M \in \mathcal{D}$ for all $M \in \mathcal{M}$. Put $H = \bigcap \mathcal{M}$. The the map

$$\begin{aligned} \alpha: \ G/H &\to \bigtimes_{M\in\mathcal{M}} G/M \\ Hg &\to (Mg)_{M\in\mathcal{M}} \end{aligned}$$

is a well defined monomorphism. Thus $\operatorname{Im}\alpha$ is a subdirect product of $\mathcal{D}$-groups. Since $\mathcal{E}$ is closed under quotients, $G/H \in \mathcal{E}$ and so $\operatorname{Im}\alpha$ is a $\mathcal{E}$-group. Since $\mathcal{D}$ is **R**-closed in $\mathcal{E}$ we conclude that $\operatorname{Im}\alpha$ and $G/H$-are $\mathcal{D}$-groups.

$\Longleftarrow$: Suppose $G \in \mathcal{E}$ and $G$ is a subdirect product of a family $(G_i)_{i\in I}$ of $\mathcal{D}$-groups. Let $M_i$ be the kernel of the projection of $G$ on $G_i$. Then $G/M_i \cong G_i$ and so $G/M_i$ is a $\mathcal{D}$-group. Put $\mathcal{M} = \{M_i \mid i \in I\}$ and observe that $\bigcap \mathcal{M} = \bigcap_{i\in I} M_i = 1$. Thus $G \cong G/\bigcap \mathcal{M}$ is $\mathcal{D}$-group. □

**Lemma 1.12.11.** *Let $(\mathcal{D}, \mathcal{E})$ be variety.*

*(a) $\mathcal{D}$ is $\{\mathbf{S}, \mathbf{Q}\}$-closed.*

*(b) Let $G \in \mathcal{E}$ and $H \trianglelefteq G$. Then $G/H \in \mathcal{D}$ if and only $G^{\mathcal{D}} \leq H$. In particular, $G^{\mathcal{D}}$ is the smallest normal subgroup of $G$ whose quotient is a $\mathcal{D}$-group.*

*(c) Let $G \in \mathcal{E}$ and $H \trianglelefteq G$. Then $G^{\mathcal{D}}H/H = (G/H)^{\mathcal{D}}$.*

*Proof.* (a) Let $G \in \mathcal{E}$ and $H$ is a subgroups of $G$ or a quotient of $G$. Since $\mathcal{E}$ is $\{\mathbf{S}, \mathbf{Q}\}$-closed, $H \in \mathcal{E}$. Since $\mathcal{D}$ is $\{\mathbf{S}, \mathbf{Q}\}$-closed in $\mathcal{E}$, $H \in \mathcal{D}$.

(b) Let $\mathcal{M} = \{M \trianglelefteq G \mid G/M \in \mathcal{D}\}$. Then $\bigcap \mathcal{M} = G^{\mathcal{D}}$. Since $\mathcal{D}$ is $\mathcal{R}$-closed in $\mathcal{E}$ and $\mathcal{E}$ is $\mathcal{Q}$-closed, 1.12.10 shows that

**1°.** *$G/G^{\mathcal{D}}$ is a $\mathcal{D}$-group.*

Now let $H$ be a normal subgroup of $G$. We have

$$G/H/G^{\mathcal{D}}H/H \cong G/G^{\mathcal{D}}H \cong G/G^{\mathcal{D}}/HG^{\mathcal{D}}/G^{\mathcal{D}}.$$

The group on the right side is a quotient of a $\mathcal{D}$-group and so a $\mathcal{D}$-group. So also

**2°.** *$G/H/G^{\mathcal{D}}H/H$ is a $\mathcal{D}$-group.*

If $G/H$ is a $\mathcal{D}$ group, $G^{\mathcal{D}} \leq H$ by definition of $G^{\mathcal{D}}$. If $G^{\mathcal{D}} \leq H$. (2°) shows that $G/H$ is a $\mathcal{D}$-group. Thus (b) is proved.

From (2°) and the definition $(G/H)^{\mathcal{D}}$ we have

$$(G/H)^{\mathcal{D}} \leq G^{\mathcal{D}}H/H.$$

By 1.12.9, $G^{\mathcal{D}}H/H \leq (G/H)^{\mathcal{D}}$ and so (c) holds. □

**Definition 1.12.12.** *Let $\mathcal{D}$ be a class of groups and $G$ a group.*

*(a) For an ordinal $\alpha$ define $G^{\mathcal{D}}_\alpha$ inductively via*

$$G^{\mathcal{D}}_\alpha = \begin{cases} G & \text{if } \alpha = 0 \\ (G_\beta)^{\mathcal{D}} & \text{if } \alpha = \beta + 1 \\ \bigcap_{\beta<\alpha} G^{\mathcal{D}}_\beta & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

*$(G^{\mathcal{D}}_\alpha)_\alpha$ is called the lower* D*-series for $G$.*

*(b) $d^{\mathcal{D}}_G$ is the smallest ordinal $\alpha$ with $G^{\mathcal{D}}_\alpha = G^{\mathcal{D}}_{\alpha+1}$.*

*(c) $G^{\mathcal{D}}_* := G^{\mathcal{D}}_{d^{\mathcal{D}}_G}$.*

*(d) $G$ is called a hypo $\mathcal{D}$-group, if there exists a normal descending series $(G_\alpha)_{\alpha\leq\delta}$ for $G$ all of whose factors are $\mathcal{D}$-groups.*

*(e) $G$ is called a hyper $\mathcal{D}$-group, if there exists a normal ascending series $(G_\alpha)_{\alpha\leq\delta}$ for $G$ all of whose factors are $\mathcal{D}$-groups.*

**Lemma 1.12.13.** *Let $(\mathcal{D}, \mathcal{E})$ be a variety and $G \in \mathcal{E}$. Then $(G^{\mathcal{D}}_\alpha)_\alpha$ is normal descending series form $G^{\mathcal{D}}_*$ to $G$ with factors in $\mathcal{D}$.*

*Proof.* By 1.10.11, $(G^{\mathcal{D}}_\alpha)_\alpha$ is descending series form $G^{\mathcal{D}}_*$ to $G$ with factors in $G^{\mathcal{D}}_\alpha/G^{\mathcal{D}}_{\alpha+1}$. By 1.12.11

$$G^{\mathcal{D}}_\alpha/G^{\mathcal{D}}_{\alpha+1} = G^{\mathcal{D}}_\alpha/(G_{\mathcal{D}_\alpha})^{\mathcal{D}}$$

is a $\mathcal{D}$-group. □

**Lemma 1.12.14.** *Let $(\mathcal{D}, \mathcal{E})$ be a variety, $G \in \mathcal{E}$ and $H, K$ and $L$ subgroups of $G$ with $K \leq L$. Let $(L_\alpha)_{\alpha\leq\delta}$ be a descending series from $K$ to $L$ with factors in $\mathcal{D}$. Let $\alpha, \beta$ and $\gamma$ be ordinals with $H^{\mathcal{D}}_\alpha \leq L_\beta$. Then*

$$H^{\mathcal{D}}_{\alpha+\gamma} \leq L_{\beta+\gamma}$$

*(where we define $L_\rho = K$ for all $\rho \geq \delta$.)*

*Proof.* Observe that $(L_{\beta+\gamma})_\gamma$ is descending series from $K$ to $L_\beta$ with factors in $\mathcal{D}$. So replacing $L$ be $L_\beta$ and $(L_\rho)_\rho$ by $(L_{\beta+\gamma})_\gamma$ we may assume that $\beta = 0$. Put $H_\rho = H^{\mathcal{D}}_\rho$. Then $H_\alpha \leq L$ and we need to show that $H_{\alpha+\gamma} \leq L_\gamma$. Since $L_0 = L$ this is true for $\gamma = 0$.

Suppose that $\gamma = \rho + 1$. The $H_{\alpha+\gamma} = H_{(\alpha+\rho)+1} = (H_{\alpha+\rho})^{\mathcal{D}}$. By induction $H_{\alpha+\rho} \leq L_\rho$. Hence

$$H_{\alpha+\rho}/H_{\alpha+\rho} \cap L_\gamma \cong H_{\alpha+\rho}L_\gamma/L_\gamma \leq L_\rho/L_\gamma$$

Since $L_\rho/L_\gamma$ is a $\mathcal{D}$-group, also $H_{\alpha+\rho}/H_{\alpha+\rho} \cap L_\gamma$ is a $\mathcal{D}$-group. Hence

$$H_{\alpha+\gamma} = (H_{\alpha+\rho})^{\mathcal{D}} \leq H_{\alpha+\rho} \cap L_\gamma \leq L_\gamma$$

Suppose $\gamma$ is a limit ordinal. Then also $\alpha + \gamma$ is limit ordinal. So

$$H_{\alpha+\gamma} = \bigcap_{\rho<\alpha+\gamma} H_\rho \leq \bigcap_{\mu<\gamma} H_{\alpha+\mu} \leq \bigcap_{\mu<\gamma} L_\mu = L_\gamma$$

and the lemma is proved. □

**Lemma 1.12.15.** *Let $(\mathcal{D}, \mathcal{E})$ be a variety, $G \in \mathcal{E}$ and $\alpha, \beta$ ordinals.*

*(a) If $\alpha \geq d^{\mathcal{D}}_G$, then $G^{\mathcal{D}}_\alpha = G^{\mathcal{D}}_*$.*

*(b) Let $H \leq G$ $H^{\mathcal{D}}_\alpha \leq G^{\mathcal{D}}_\alpha$.*

*(c) $H^{\mathcal{D}} \leq G^{\mathcal{D}}$.*

*(d) $G^{\mathcal{D}}_{\alpha+\beta} = (G^{\mathcal{D}}_\alpha)^{\mathcal{D}}_\beta$.*

*(e) Let $H \trianglelefteq G$. Then $G^{\mathcal{D}}_\alpha H/H \leq (G/H)^{\mathcal{D}}_\alpha$ with equality if $\alpha$ is finite.*

*Proof.* Put $G_\alpha = G^{\mathcal{D}}_\alpha$ and $\delta = d^{\mathcal{D}}_G$.

(a) We have $G^{\mathcal{D}}_* = G_\delta = G_{\delta+1} = (G_\delta)^{\mathcal{D}}$. In particular, (a) holds for $\alpha = \delta$. So suppose $\alpha > \delta$ and that (a) holds for all $\beta$ with $\delta \leq \beta < \alpha$. If $\alpha = \beta + 1$, then $G_\alpha = (G_\beta)^{\mathcal{D}} = (G_\delta)^{\mathcal{D}} = G_\delta$ and if $\delta$ is a limit ordinal, $G_\alpha = \bigcup_{\delta\leq\beta<\alpha} G_\beta = \bigcup_{\delta\leq\beta<\alpha} G_\delta = G_\delta$.

(b) Since $(G_\alpha)$ is series from $G_\delta$ to $G$ with factors in $\mathcal{D}$ this follows follows from 1.12.14 applied with $\tilde{\alpha} = 0 = \tilde{\beta}$, $\tilde{\gamma} = \alpha$ and $L = G$.

(c) This is the special case $\alpha = 1$ in (b).

(d) We have $(G_\alpha)^{\mathcal{D}}_0 \leq G_\alpha$ and so by 1.12.14 (applied with $H = G_\alpha$ and $L = G$

$$(G_\alpha)^{\mathcal{D}}_{0+\beta} \leq G_{\alpha+\beta}.$$

Also $G_\alpha \leq (G_\alpha)^{\mathcal{D}}_0$ and so by 1.12.14 (applied with $H = G$ and $L = G_\alpha$:

$$G_{\alpha+\beta} \leq (G_\alpha)^{\mathcal{D}}_{0+\beta}$$

(e) If $\alpha = 0$, both sides are equal to $G/H$. If $\alpha = \beta + 1$ then

$$G_\alpha H/H = G_\beta^{\mathcal{D}} H/H = (G_\beta H/H)^{\mathcal{D}} \leq ((G/H)_\beta^{\mathcal{D}})^{\mathcal{D}} = (G/H)_a^{\mathcal{D}}$$

with equality if $\beta$ is finite.

Suppose that $\alpha$ is limit ordinal. Then

$$G_\alpha H/H = \left(\bigcap_{\beta<\alpha} G_\beta\right) H/H \leq \bigcap_{\beta<\alpha}(G_\beta H/H) \leq \bigcap_{\beta<\alpha}(G/H)_\beta^{\mathcal{D}} = (G/H)_\alpha^{\mathcal{D}}$$

□

**Lemma 1.12.16.** *Let $(\mathcal{D}, \mathcal{E})$ be a variety. Let $G \in \mathcal{E}$. The the following are equivalent*

*(a) $G_*^{\mathcal{D}} = 1$.*

*(b) $G$ is a hypo-$\mathcal{D}$-group.*

*(c) There exists a descending series for $G$ all of whose are $\mathcal{D}$-groups.*

*Proof.* (a) $\Longrightarrow$ (b):

Suppose that $G_*^{\mathcal{D}} = 1$. Then $(G_\alpha^{\mathcal{D}})_\alpha$ is a normal descending series from 1 to $G$ all of whose factors are in $\mathcal{D}$.

c Obvious.

a Suppose that $(G_\alpha)_{\alpha\leq\delta}$ is descending series from 1 to $G$ with factors in $\mathcal{D}$. Then by 1.12.14 (applied with $H = L = G$),

$$G_*^{\mathcal{D}} \leq G_\delta^{\mathcal{D}} \leq G_\delta = 1$$

□

**Definition 1.12.17.** *Let $\mathcal{C}$ and $\mathcal{D}$ be classes of groups. Then $\mathcal{CD}$ denotes the class of all groups $G$ such that there exists a normal subgroups $H$ of $G$ with*

$$G/H \in \mathcal{C} \text{ and } H \in \mathcal{D}$$

**Lemma 1.12.18.** *Let $(\mathcal{C}, \mathcal{E})$ and $(\mathcal{D}, \mathcal{E})$ be varieties. Let $G \in \mathcal{E}$. Then $G \in \mathcal{CD}$ if and only if $(G^{\mathcal{C}})^{\mathcal{D}} = 1$.*

*Proof.* Suppose first that $G \in \mathcal{CD}$. Then there exists $H \trianglelefteq G$ with $G/H \in \mathcal{C}$ and $H \in \mathcal{D}$. Thus the definition of $G^{\mathcal{C}}$ and $H^{\mathcal{D}}$ implies $G^{\mathcal{C}} \leq H$ and $H^{\mathcal{D}} \leq 1$. So using 1.12.15,

$$(G^{\mathcal{C}})^{\mathcal{D}} \leq H^{\mathcal{D}} = 1$$

Suppose next that $(G^{\mathcal{C}})^{\mathcal{D}} = 1$. Then by 1.12.11, $G/G^{\mathcal{C}}$ is a $\mathcal{C}$-group and $G^{\mathcal{C}}$ is a $\mathcal{D}$-groups. Hence $G \in \mathcal{CD}$. □

**Lemma 1.12.19.** *Let $\mathcal{D}$ and $\mathcal{E}$ be classes of groups. Suppose $\mathcal{E}$ is* **Q***-closed. Then $G^{\mathcal{D}} = G^{\mathcal{D}\cap\mathcal{E}}$ for all $G \in \mathcal{E}$.*

*Proof.* Let $G \in \mathcal{E}$ and $H \trianglelefteq G$. Since $\mathcal{E}$ is **Q**-closed, $G/H \in \mathcal{E}$. Thus $G/H \in \mathcal{D}$ if and only if $G/H \in \mathcal{D} \cap \mathcal{E}$. The lemma now follows from the definition of $G^{\mathcal{D}}$. □

**Lemma 1.12.20.** *Let $(\mathcal{C}, \mathcal{E})$ and $(\mathcal{D}, \mathcal{E})$ be varieties. Then $\mathcal{C} = \mathcal{D}$ if and only if $G^{\mathcal{C}} = G^{\mathcal{D}}$ for all $G \in \mathcal{E}$.*

*Proof.* Suppose $G^{\mathcal{C}} = G^{\mathcal{D}}$ for all $G \in \mathcal{E}$. Let $G$ be a group. Then by 1.12.11, $G \in \mathcal{C}$ if and only if $G \in \mathcal{E}$ and $G^{\mathcal{C}} = 1$ and so if and only if $G \in \mathcal{D}$. □

**Lemma 1.12.21.** *Let $(\mathcal{C}, \mathcal{E})$ and $(\mathcal{D}, \mathcal{E})$ be varieties. Then $(\mathcal{C}\mathcal{D} \cap \mathcal{E}, \mathcal{E})$ is a variety and $G^{\mathcal{C}\mathcal{D}} = (G^{\mathcal{C}})^{\mathcal{D}}$ for all $G \in \mathcal{E}$.*

*Proof.* Let $G \in \mathcal{C}\mathcal{D} \cap \mathcal{E}$. If $H \leq G$,

$$(H^{\mathcal{C}})^{\mathcal{D}} \leq (G^{\mathcal{C}})^{\mathcal{D}} = 1$$

and so $H \in \mathcal{C}\mathcal{D} \cap \mathcal{E}$ and thus $\mathcal{C}\mathcal{D} \cap \mathcal{E}$ is **S**-closed.

Now let $G \in \mathcal{E}$ and $H \trianglelefteq G$. Note that

$$((G/H)^{\mathcal{C}})^{\mathcal{D}} = (G^{\mathcal{C}}H/H)^{\mathcal{D}} = (G^{\mathcal{C}})^{\mathcal{D}}H/H$$

and so by 1.12.18

**1°.** *$G/H \in \mathcal{C}\mathcal{D}$ if and only if $(G^{\mathcal{C}})^{\mathcal{D}} \leq H$.*

In particular, if $G \in \mathcal{C}\mathcal{D}$, then $G^{\mathcal{C}\mathcal{D}} = 1 \leq H$ and so $G/H \in \mathcal{C}\mathcal{D}$. Thus $\mathcal{C}\mathrm{D} \cap \mathcal{E}$ is **Q**-closed.

Let $\mathcal{M}$ be a set of normal subgroups of $G$ such that $G/M \in \mathcal{C}\mathcal{D}$ for all $M \in \mathcal{M}$. Then by (1°) $(G^{\mathcal{C}})^{\mathcal{D}} \leq M$ for $M \in \mathcal{M}$ and so $(G^{\mathcal{C}})^{\mathcal{D}} \leq \bigcap \mathcal{M}$ and $G/\bigcap \mathcal{M} \in \mathcal{D}$. Hence by 1.12.10, $\mathcal{C}\mathcal{D} \cap \mathcal{E}$ is **R**-closed in $\mathcal{E}$.

Thus $(\mathcal{C}\mathcal{D} \cap \mathcal{E}, \mathcal{E})$ sis a variety. It follows that $G/H \in \mathcal{C}\mathcal{D} \cap \mathcal{E}$ if and only if $G^{\mathcal{C}\mathcal{D}} = G^{\mathcal{C}\mathcal{D}\cap\mathcal{E}} \leq H$. Together with 1.12.18 this shows $G^{\mathcal{C}\mathcal{D}} = (G^{\mathcal{C}})^{\mathcal{D}}$. □

**Definition 1.12.22.** *Let $\mathcal{D}$ and $\mathcal{E}$ be classes of groups with $\mathcal{D} \subseteq \mathcal{E}$. We say that $\mathcal{D}$ is* **P***-closed in $\mathcal{E}$ if $G \in \mathcal{D}$ whenever $G$ is an $\mathcal{E}$-group with a normal subgroups $H$ such that $H$ and $G/H$ are $\mathcal{D}$-groups.*

In the following $(\mathcal{D}, \mathcal{E})$ is **T** closed for means that $\mathcal{D}$ is **T**-closed in $\mathcal{E}$.

**Lemma 1.12.23.** *Let $(\mathcal{D}, \mathcal{E})$ be a variety. Then the following are equivalent:*

*(a) $\mathcal{D}$ is* **P***-closed in $\mathcal{E}$.*

*(b) $\mathcal{D}\mathcal{D} \cap \mathcal{E} = \mathcal{D}$.*

*(c) $G^{\mathcal{D}}$ is $\mathcal{D}$-perfect for all $G \in \mathcal{E}$.*

*Proof.* (a) $\iff$ (b) :

Since $\mathcal{D}$ contains the trivial groups, $\mathcal{D} \subseteq \mathcal{D}\mathcal{D} \cap \mathcal{E}$. By definition, $\mathcal{D}$ is **P**-closed in $\mathcal{E}$ if and only if $\mathcal{D}\mathcal{D} \cap \mathcal{E} \subseteq \mathcal{D}$.

(b) $\iff$ (c) : By 1.12.20 $\mathcal{D}\mathcal{D} \cap \mathcal{E} = \mathcal{D}$ if and only if $\mathcal{G}^{\mathcal{D}\mathcal{D}} = G^{\mathcal{D}\mathcal{D}\cap\mathcal{E}} = G^{\mathcal{D}}$ for all $G \in \mathcal{E}$. By 1.12.21, $\mathcal{G}^{\mathcal{D}\mathcal{D}} = (G^{\mathcal{D}})^{\mathcal{D}}$ and so $\mathcal{D}\mathcal{D} \cap \mathcal{E} = \mathcal{D}$ if and only if $(G^{\mathcal{D}})^{\mathcal{D}} = G^{\mathcal{D}}$ for all $G \in \mathcal{E}$. □

**Definition 1.12.24.** *Let $\mathcal{D}, \mathcal{E}$ be a class of group with $\mathcal{D} \subseteq \mathcal{E}$*

*(a) Let $G$ be a group. Then $G_{\mathcal{D}}$ is subgroup of $G$ generated by all the normal $\mathcal{D}$-subgroup of $G$.*

*(b) We say that $\mathcal{D}$ is **N**-closed in $\mathcal{E}$ if $G \in \mathcal{D}$ whenever $G$ is an $\mathcal{E}$-subgroup generated by normal $\mathcal{D}$-subgroups of $G$.*

*(c) We say that $\mathcal{D}$ is $\mathbf{N}_0$ closed in $\mathcal{E}$ if $G \in \mathcal{D}$, whenever $G$ is an $\mathcal{E}$-subgroup generated by finitely many normal $\mathcal{D}$-subgroups of $G$.*

*(d) We say that $\mathcal{D}$ is $\mathbf{R}_0$ closed in $\mathcal{E}$ if every $\mathcal{E}$-group which is the subdirect product of finitely many $\mathcal{D}$-groups, is an $\mathcal{D}$-group.*

**Lemma 1.12.25.** *Let $\mathcal{D}, \mathcal{E}$ be a classes of group with $\mathcal{D} \subseteq \mathcal{E}$. Suppose that $\mathcal{E}$is $\mathbf{S_n}$-closed and $\mathcal{D}$ is **N**-closed in $\mathcal{E}$. Let $G \in \mathcal{E}$.*

*(a) Let $H$ be a subnormal $\mathcal{D}$-subgroup of $G$. Then $\langle H^G \rangle \in \mathcal{D}$.*

*(b) Let $H$ be a subgroup of $G$ generated by subnormal $\mathcal{D}$-subgroups of $G$. Then $H \in \mathcal{D}$.*

*(c) $G_{\mathcal{D}} \leq \mathcal{D}$.*

*(d) $G_{\mathcal{D}}$ is the subgroup of $G$ generate by all the subnormal $\mathcal{D}$ subgroups of $G$.*

*(e) Let $H$ be subnormal in $G$. Then $H_{\mathcal{D}} \leq G_{\mathcal{D}}$.*

*Proof.* (a) Let $(G_\alpha)_{\alpha \leq \delta}$ be a subnormal series from $H$ to $G$. Put $H_\alpha = \langle H^{G_\alpha} \rangle$. We will show by induction on $\alpha$, that $H_\alpha$ is a $\mathcal{D}$-group. For $\alpha = 0$, $H_0 = H$ is a $\mathcal{D}$-group. So suppose $\alpha > 0$. Since $\alpha$ is finite, $\alpha = \beta + 1$. By induction $H_\beta$ is a normal $\mathcal{D}$-subgroup of $G_\beta$. Let $g \in G_\alpha$. Since $G_\beta \trianglelefteq G_\alpha$, $H_\beta^g$ is a normal $\mathcal{D}$-subgroup of $H_\beta$. Since $\mathcal{D}$ is **N** closed in $\mathcal{E}$, $H_\alpha = \langle H_\beta^g \mid g \in G_\alpha \rangle$ is a normal $\mathcal{D}$-subgroup of $G_\alpha$.

(b) Let $H = \langle \mathcal{H} \rangle$ where $\mathcal{H}$ is a set of subnormal $\mathcal{D}$-subgroups of $G$. Then $H$ is subnormal in $G$ and so $H \in \mathcal{E}$. Let $F \in \mathcal{H}$. By (a), $\langle F^H \rangle$ is a normal $\mathcal{D}$-subgroup of $H$ and since $\mathcal{D}$ is **N**-closed in $\mathcal{E}$, $H = \langle \langle F^H \rangle \mid F \in \mathcal{H} \rangle$ is a $\mathcal{D}$-group.

(c) and (d): Let $D$ be the subgroup of $G$ generated by the subnormal $\mathcal{D}$-subgroups of $G$. Then $G_{\mathcal{D}} \leq D$. By (a), $D \in \mathcal{D}$ and so $G_{\mathcal{D}} = D$ and $G_{\mathcal{D}} \in \mathcal{D}$.

(e) Since $H$ is subnormal in $G$, $H \in \mathcal{E}$. So by (c) $H_{\mathcal{D}}$ is a $\mathcal{D}$-group. Note that $H_{\mathcal{D}}$ is subnormal in $G$ and so by (d), $H_{\mathcal{D}} \leq G_{\mathcal{D}}$. □

**Definition 1.12.26.** *Let $\pi$ be a set of primes and $G$ a group.*

*(a) $\mathcal{G}$ is the class of all groups and $\mathcal{F}$ the class of finite groups.*

*(b) $G$ is called periodic if all elements of $G$ have finite order.*

*(c) $g \in G$ is called a $\pi$-element if $g$ is finite and all prime divisors of $|g|$. are in $\pi$.*

*(d) $G$ is called a $\pi$ group if all elements of $g$ are $\pi$-elements.*

*(e) $O^\pi(G) := G^{\mathcal{G}_\pi}$ and $O_\pi(G) := G_{\mathcal{G}_\pi}$.*

*(f) $\mathcal{G}_\pi$ is the class of all $\pi$ groups, $\mathcal{G}_{\mathrm{Nil}}$ is the class of nilpotent groups and $\mathcal{G}_{\mathrm{Sol}}$ is the class of solvable groups.*

*(g) For any symbol $\mathbf{T}$, $\mathcal{F}_{\mathbf{T}} = \mathcal{G}_{\mathbf{T}} \cap \mathcal{F}$ is the class of finite $\mathcal{G}_{\mathbf{T}}$-groups,*

*(h) $F(G) := G_{\mathcal{G}_{\mathrm{Nil}}}$ and $\mathrm{Sol}(G) = G_{\mathcal{G}_{\mathrm{Sol}}}$.*

**Lemma 1.12.27.** *Let $\mathcal{D}$ be a class of finite groups.*

*(a) $\mathcal{D}$ is $\mathcal{R}$-closed in $\mathcal{F}$ if and only if $\mathcal{D}$ is $\mathcal{R}_0$ closed.*

*(b) $\mathcal{D}$ is $\mathcal{N}$-closed in $\mathcal{F}$ if and only of $\mathcal{D}$ is $\mathcal{N}_0$ closed.*

*(c) $(\mathcal{D}, \mathcal{F})$ is a variety of and only if $\mathcal{D}$ is $\{\mathbf{S}, \mathbf{Q}, \mathbf{R_0}\}$-closed.*

*Proof.* (a) Suppose $\mathcal{D}$ is $\mathcal{N}$-closed and let $G$ be a subdirect product of finitely $\mathcal{D}$-groups. Then $G$ is finite and since $\mathcal{D}$ is $\mathcal{N}$-closed in $\mathcal{F}$, $G \in \mathcal{D}$. So $\mathcal{D}$ is $\mathcal{N}_0$-closed.

Suppose that $\mathcal{D}$ is $\mathcal{N}_0$-closed and let $G$ be a finite subdirect product of $\mathcal{D}$-groups. By 1.12.3, $G$ is isomorphic to a subdirect product of finitely many $\mathcal{D}$-groups and so $G \in \mathcal{D}$. Thus $\mathcal{D}$ is $\mathcal{N}$-closed in $\mathcal{F}$. (b) Very similar to (a).

(c) Since $\mathcal{F}$ is $\{\mathcal{S}, \mathcal{Q}\}$-closed, $\mathcal{D}$ is a $\{\mathcal{S}, \mathcal{Q}\}$-closed in $\mathcal{F}$ if and only of $\mathcal{D}$ is a $\{\mathcal{S}, \mathcal{Q}\}$-closed. Thus (c) follows from (a) and the definition of a variety. □

**Lemma 1.12.28.** *Let $\pi$ be a set of primes.*

*(a) The class of $\pi$-groups is $\{\mathbf{S}, \mathbf{Q}, \mathbf{P}, \mathbf{N}, \mathbf{R}_0\}$-closed.*

*(b) $(\mathcal{G}_\pi, \mathcal{G}_{\mathrm{Per}})$ is a $\{\mathbf{P}, \mathbf{N}\}$ closed variety.*

*(c) $(\mathcal{F}_\pi, \mathcal{F})$ is a $\{\mathbf{P}, \mathbf{N}\}$-closed variety.*

*(d) $(\mathcal{F}_{\mathrm{Nil}}, \mathcal{F})$ is a $\mathbf{N}$-closed variety.*

*(e) $(\mathcal{F}_{\mathrm{Sol}}, \mathcal{F})$ is a $\{\mathbf{P}, \mathbf{N}\}$-closed variety.*

*Proof.* Let $G$ be group and $H$ a normal subgroup of $G$ such that $G/H$ and $H$ are $\pi$-groups. Let $\overline{G} = G/H$, $g \in G$ and $n = |\overline{g}|$ and $m = |g^n|$. Then $|g| = nm$ and so $G$ is a $\pi$-group. Thus $\mathcal{G}_\pi$ is $\mathbf{P}$-closed.

Since $\mathcal{G}_\pi$ is $\{\mathbf{Q}, \mathbf{P}\}$-closed, $\mathcal{G}_\pi$ is $\mathbf{N}_0$-closed. Let $G$ be a group generated by normal $\pi$-subgroups. Let $g \in G$. Then $g$ is contained in the product of finitely many normal $\pi$ $\pi$-groups and so $g$ is a $\pi$-element. So $G$ is $\pi$-group and $\mathcal{G}_\pi$ is $\mathbf{N}$-closed.

Let $G$ be a periodic group and suppose $G$ is the subdirect product of $\pi$-groups. Let $g \in G$. Then $\langle g \rangle$ is finite and so $\langle g \rangle$ is the subdirect product of finitely many $\pi$-groups. Thus $\langle g \rangle$ is a $\pi$-group and so also $G$ is a $\pi$-group.

The remaining assertion are readily verified. □

**Lemma 1.12.29.** *Let $G$ be a group, $H \leq G$ and $\mathcal{S}$ a series from $H$ to $G$. For $F = A/B$ a factor of $\mathcal{S}$, let $\mathcal{S}_F$ be a series for $F$. Let $\mathcal{T}_F = \{X \mid B \leq X \leq A, X/B \in \mathcal{S}_F\}$ and $\mathcal{T} = \bigcup\{\mathcal{T}_F \mid F$ a factor of $\mathcal{S}\} \cup \mathcal{S}$. Then*

*(a) $\mathcal{T}_F$ is a series from $B$ to $A$ with factors isomorphic to the factors of $\mathcal{S}_F$.*

*(b) $\mathcal{T}$ is a series from $H$ to $G$, with factors isomorphic to the factors of $\mathcal{S}_F$, $F$ a factor of $\mathcal{S}$.*

*Proof.* (a) Let $\mathcal{U} = \{X \mid B \leq X \leq A\}$. The map $X \to X/B$ is bijection from $\mathcal{U}$ to the subgroups of $A/B$. We have $X \leq Y$ if and only if $X/B \leq Y/B$. Also $X \trianglelefteq Y$ if and only if $X/B \trianglelefteq Y/B$. $\mathcal{V} \subseteq \mathcal{U}$, then $(\bigcup \mathcal{V})/B = \bigcup_{X \in \mathcal{V}} X/B$ and $(\bigcap \mathcal{V})/B = \bigcap_{X \in \mathcal{V}} X/B$. It now follows easily that $\mathcal{T}_F$ is a series from $B$ to $A$. Also if $X/Y$ is a factor of $\mathcal{T}_F$, then $X/B\big/Y/B$ is a factor of $\mathcal{S}_F$ isomorphic to $X/Y$.

(b) Since $H, G \in \mathcal{S}$ we have $H, G \in \mathcal{T}$. Let $X \in \mathcal{T}$. If $X \in \mathcal{S}$, put $X_- = X_+ = X$. If $X \notin \mathcal{S}$ pick a factor $F = X_+/X_-$ of $\mathcal{S}$ with $X/X_- \in \mathcal{S}_F$; note here that $X_- < X < X_+$ and $X_+$ and $X_-$ are uniquely determined.

Let $X, Y \in \mathcal{S}$ and choose notation such that $X_+ \leq Y_+$. If $X_+ \leq Y_-$, then $X \leq X_+ \leq Y_- \leq Y$ and so $X \leq Y$. So suppose $Y_- < X_+$. Then $Y_- < X_+ \leq Y_+$ and so $Y \notin S$, $F = Y_+/Y_-$ is a factor of $\mathcal{S}$ and $X_+ = Y^+$. Note that either $X = X_+ = Y_+$ or $X \neq X_+$ and $X_- = Y_-$. In either case both $X$ and $Y$ are contained in $\mathcal{T}_F$. Hence either $X \leq Y$ or $Y \leq X$.

Let $\mathcal{D}$ be a non-empty subset of $\mathcal{T}$. Put $C_+ = \bigcap_{D \in \mathcal{D}} D_+$. If $\bigcup \mathcal{D} = C_+$ we have $\bigcup \mathcal{D} \in \mathcal{S} \subseteq \mathcal{T}$. So suppose $\bigcup \mathcal{D} \neq C_+$. Since $\bigcup \mathcal{D} \leq C_+$, there exists $D \in \mathcal{D}$ with $C_+ \nleq D$ and so $D < C_+$. By definition of $D_+$, $D_+ \leq C_+$ and by definition of $C_+$, $C_+ \leq D_+$. Thus $D \notin \mathcal{S}$, $F = C_+/(C_+)^-$ is the factor associated to $D$. Thus $D \in \mathcal{T}_F$ for all $D \in \mathcal{D}$ with $C_+ \nleq D$. From $D \leq C_+ \leq E$ for all $E \in \mathcal{D}$ with $C_+ \leq E$ we conclude that

$$\bigcap \mathcal{D} = \bigcap (\mathcal{D} \cap \mathcal{T}_F) \in \mathcal{T}_F \subseteq \mathcal{T}$$

Similarly put $C_- = \bigcup_{D \in \mathcal{D}} D_-$. If $\bigcup \mathcal{D} = C_-$ we have $\bigcup \mathcal{D} \in \mathcal{S} \subseteq \mathcal{T}$. So suppose $\bigcap \mathcal{D} \neq C_-$. Since $C_- \leq \bigcap \mathcal{D}$, there there exists $D \in \mathcal{D}$ with $D \nleq C_-$ and so $C_- < D$. By definition of $D_-$, $C_- \leq D_-$ and by definition of $C_-$, $D_+ \leq C_+$. Thus $D \notin \mathcal{S}$ and

$F = (C_-)^+/C_-$ is the factor associated to $D$. Thus $D \in \mathcal{T}_F$ for all $D \in \mathcal{D}$ with $D \not\leq C_-$. From $E \leq C_- \leq D$ for all $E \in \mathcal{D}$ with $E \leq C_-$ we conclude that

$$\bigcup \mathcal{D} = \bigcup (\mathcal{D} \cap \mathcal{T}_F) \in \mathcal{T}_F \subseteq \mathcal{T}$$

Now let $T \in \mathcal{T}$ and put $\mathcal{D} = \{X \in \mathcal{T} \mid X < T\}$ and $B = \bigcup \mathcal{D}$. Suppose that $B \neq T$. Observe that $T^- \leq B$ and so $T^- \neq B$. Put $F = T/T-$ and let $D \in \mathcal{D}$. Then either $D \leq T^- \in \mathcal{D} \cap \mathcal{T}_F$ or $T_- < D < T$ and $D \in \mathcal{T}_F$. Thus

$$B = \bigcup \mathcal{D} = \bigcup \mathcal{D} \cap \mathcal{T}_F = \bigcup \{D \in \mathcal{T}_F \mid D < T\}$$

and so by (a), $B \trianglelefteq T$, $T/B$ is a factor of $\mathcal{T}_F$ and $T/B$ is isomorphic to a factor of $\mathcal{S}_F$. □

**Lemma 1.12.30.** *(a) Let $\mathcal{D}$ be an $\{\mathbf{S_n}, \mathbf{Q}\}$-closed class of finite groups and $G$ a finite group. Then $G$ is a hypo-$\mathcal{D}$-group if and only if there exists a chief-series for $G$ all of whose factors are in $\mathcal{D}$.*

*(b) Let $(\mathcal{D}, \mathcal{F})$ be a variety and $G$ a finite group. Then $G$ is a hypo-$\mathcal{D}$-group if and only if there exists a composition-series for $G$ all of whose factors are in $\mathcal{D}$.*

*Proof.* (a) Suppose $\mathcal{S}$ is chief-series for $G$ all of whose factors are in $\mathcal{S}$. Since $G$ is finite $\mathcal{S}$ is a normal descending series and so $G$ is a hypo-$\mathcal{D}$-group.

Suppose that $G$ is a hypo-$\mathcal{D}$-group and let $\mathcal{S}$ be a normal descending series for $G$ with factors in $\mathcal{D}$. Let $F$ be a factor of $G$ and choose a maximal $G$-invariant series $\mathcal{S}_F$. If $T$ is a factor of $\mathcal{S}_F$, then $T = X/Y$ where $X$ and $Y$ are normal subgroups of $F$. Since $F \in \mathcal{D}$ and $\mathcal{D}$ is $\mathbf{S_n}$-closed, $X \in \mathcal{D}$. Since $\mathcal{D}$ is $\mathbf{Q}$-closed, $X/Y \in \mathcal{D}$. So all factors of $\mathcal{S}_F$ are $\mathcal{D}$-groups. Thus by 1.12.29 there exists a series $\mathcal{T}$ for $G$ whose factors are $\mathcal{D}$-groups. Since $\mathcal{S}_F$ is $G$-invariant, $\mathcal{T}$ is a normal series. Since $G$ is finite, $\mathcal{T}$ is descending. the maximality of $\mathcal{S}_F$ shows that $\mathcal{T}$ is a chief-series.

(b) By 1.12.16, $G$ is a hypo-$\mathcal{D}$-groups if and only if there exists some descending series for $G$ all of whose factor are in $\mathcal{D}$. So the same argument as in (a) probes (a) (Just replace 'chief-series' by 'composition series' and remove 'normal' and '$G$-invariant') □

**Lemma 1.12.31.** *Let $\mathcal{D}$ be a class of groups and $G$ a subdirect product of family of $\mathcal{D}$-groups $(G_i)_{i \in I}$.*

*(a) There exists a normal descending series for $G$ with factors $(F_i, i \in I)$, where $F_i$ is isomorphic to a normal subgroups of $G_i$.*

*(b) If $\mathcal{D}$ is $\mathbf{S_n}$-closed, $G$ is a hypo $\mathcal{D}$-group.*

*Proof.* (a) By the well-ordering axiom the exists some well ordering $\prec$ on $I$. Fix $m \in I$. Define a ordering $<$ on $I$ by $i < j$ if either $i, j \in I \setminus \{m\}$ with $i \prec j$, or $i \in I \setminus \{m\}$ and $j = m$. Then $<$ is a well ordering on $I$ with maximal element $m$. So we may assume that $I = \{\alpha \mid \alpha \leq \delta\}$ for some ordinal $\delta$.

Let $\gamma \leq 2\delta + 1$. Define the normal subgroup $T_\gamma$ as follow: By A.1.11, $\gamma = 2\alpha + \rho$ for some uniquely determined ordinals $\alpha, \rho$ with $\rho < 2$. Then $\rho = 0$ or $\rho = 1$. Moreover, since $\gamma \leq 2\delta$ we have $\alpha \leq \delta$.

If $\rho = 0$ define

$$T_\gamma = T_{2\alpha} = \{g \mid g_i = 1 \text{ for all } i < \alpha\};$$

and if $\rho = 1$ define

$$T_\gamma = T_{2\alpha+1} = \{g \mid g_i = 1 \text{ for all } i \leq \alpha\}.$$

Observe that $T_\gamma$ is a normal subgroups of $G$. We have $T_0 = G$ and $T_{2\delta+1} = 1$. Define $\pi_\alpha : G \to G_\alpha, g \to g_\alpha$. Then $\pi_\alpha$ is an epimorphism. Since $T_{2\alpha} \trianglelefteq G$,

$$\pi_\alpha(T_{2\alpha}) \trianglelefteq \pi_\alpha(G) = G_\alpha$$

Observe that $T_{2\alpha+1} = T_{2\alpha} \cap \ker \pi_\alpha$ and so $T_{2\alpha+1} \trianglelefteq T_{2\alpha}$ and $F_\alpha := T_{2\alpha}/T_{2\alpha+1}$ is isomorphic $\pi_\alpha(T_{2\alpha})$.

By A.1.13, $(2\alpha + 1) + 1 = 2\alpha + 2 = 2(\alpha + 1)$. Also $i < \alpha + 1$ if and only if $i \leq \alpha$. Thus $T_{(2\alpha+1)+1} = T_{2\alpha+1}$.

Suppose that $\gamma$ is a limit ordinal. Then $\rho = 0$ and $\alpha$ is a limit ordinal. Let $\tilde{\gamma}$ be an ordinal and let $\tilde{\gamma} = 2\tilde{\alpha} + \tilde{\rho}$ with $\rho = \{0, 1\}$. By A.1.11, $\tilde{\gamma} < \gamma = 2\alpha$ if and only if $\tilde{\alpha} < \alpha$. Since $T_{2\tilde{\alpha}} \leq T_{2\tilde{\alpha}+1}$ we get

$$\bigcap_{\tilde{\gamma}<\gamma} T_{\tilde{\gamma}} = \bigcap_{\tilde{\alpha}<\alpha} T_{2\tilde{\alpha}+1} = T_{2\alpha} = T_\gamma$$

Thus $(T_\gamma)_{\gamma \leq 2\delta+1}$ is a normal descending series with factors $F_\alpha, \alpha \leq \delta$.

(b) If $\mathcal{D}$ is $\mathbf{S_n}$ closed, each $F_\alpha$ is a $\mathcal{D}$-group. So (b) holds. □

## 1.13 π-separable groups

**Definition 1.13.1.** *Let $\pi$ be a set of primes and $G$ a group.*

*$\pi'$ is the set of primes not in $\pi$.*

*Let $n$ be an integer. Then $\pi(n)$ is the set of prime divisors of $n$. $n_\pi$ is supremum of all the divisor $m$ of $n$ with $\pi(m) \subseteq \pi$. $n$ is coprime to $\pi$ if $n_\pi = 1$, (that is $pi(n) \subseteq \pi'$.*

*$G$ is called $\pi$-separable if $G$ is a periodic hypo-$(\mathcal{G}_\pi \cup \mathcal{G}_{\pi'})$-group.*

*$G$ is called $\pi$-solvable if $G$ is a periodic hypo-$((\mathcal{G}_\pi \cap \mathcal{G}_{\mathrm{Sol}}) \cup \mathcal{G}_{\pi'})$-group.*

**Lemma 1.13.2.** *Let $G$ be a periodic group and $\pi$ a set of primes.*

*(a) $G$ is $\pi$-separable if and only if $G$ is hypo-$\mathcal{G}_\pi \mathcal{G}_{\pi'}$-group.*

*(b) $G$ is $\pi$-solvable if and only if only if $G$ is hypo-$(\mathcal{G}_\pi \cap \mathcal{G}_{\mathrm{Sol}})\mathcal{G}_{\pi'}$-group.*

*Proof.* (a) Since every $\pi$ and every $\pi'$-group is a $\mathcal{G}_\pi\mathcal{G}_{\pi'}$-group, the forward direct holds.

Now let $\mathcal{S}$ be a descending series for $G$ with factors in $\mathcal{G}_\pi\mathcal{G}_{\pi'}$. Let $F$ be a one of the factors. Then $1 \leq O^\pi(F) \leq F$ is a series for $F$ with two factors, one is a $\pi$-group and the other a $\pi'$-group. Thus 1.12.29 shows that $G$ is $\pi$-separable.

(b) Use a similar argument as in (a). □

**Lemma 1.13.3.** *Let $\pi$ be a set of primes and $G$ a group. Then $O_\pi(G)$ is a $\pi$-group and contains all subnormal $\pi$-subgroups of $G$.*

*Proof.* Since $\mathcal{G}_\pi$ is **N**-closed, this follows from 1.12.25. □

**Lemma 1.13.4.** *Let $\pi$ be a set of primes and $G$ a finite $\pi$-separable group with $O_{\pi'}(G) = 1$. Then $C_G(O_\pi(G)) \leq O_\pi(G)$.*

*Proof.* Put $D = C_G(O_\pi(G))$ and $C = O_\pi(D)$. Then $C \leq O_\pi(G)$ and so $[C, D] = 1$ and $C \leq Z(D)$. Put $\overline{D} = D/C$ and let $E$ be the inverse image of $O_{\pi'}(\overline{D})$ in $D$. Then $E/C$ is a $\pi'$-group and $C$ is normal abelian $\pi'$-subgroup of $E$. By Gaschütz Theorem, there exists a complement $K$ to $C$ in $K$. Note that $K \cong E/C$ is a $\pi'$-group. Since $C \leq Z(E)$, $K$ is normalized by $CK = E$. Since $E \trianglelefteq G$ we get $K \trianglelefteq\trianglelefteq G$ and so $K \leq O_{\pi'}(G) = 1$. Hence $E = C$ and so $O_{\pi'}(\overline{D}) = 1$. Since $C = O_\pi(D)$, also $O_\pi(\overline{D}) = 1$. Since $\overline{D}$ is $\pi$-separable this gives $\overline{D} = 1$. Thus $D = C \leq O_\pi(G)$. □

**Definition 1.13.5.** *Let $G$ be a group and $\pi$ a set of prime. of $G$.*

*(a) A Sylow $\pi$-subgroup of $G$ is a maximal $\pi$-subgroup of $G$.* $\mathrm{Syl}_\pi(G)$ *is the set of Sylow $\pi$-subgroups of $G$.*

*(b) A $\pi$-subgroup $H$ of $G$ is called a Hall $\pi$-subgroup of $G$ if, for all $p \in \pi$, $H$ contains a Sylow $p$-subgroup of $G$.*

*(c) We say that the Sylow $\pi$-Theorem holds for $G$ if any two Sylow $\pi$-subgroups of $G$ are conjugate in $G$.*

**Lemma 1.13.6.** *Let $G$ be a group and $\pi$ a set of primes. Then every $\pi$-subgroups of $G$ is contained in a Sylow $\pi$-subgroup of $G$. In particular, $G$ has a Sylow $\pi$-subgroup.*

*Proof.* Let $\mathcal{S}$ be a set of $\pi$-subgroups of $G$ which is totaly ordered with respect to inclusion. Then $\bigcup \mathcal{D}$ is a $\pi$-subgroup of $G$. So the lemma follows from Zorn's lemma. □

**Lemma 1.13.7.** *Let $G$ be a finite group, $\pi$ a set of primes and $H$ a subgroup of $G$. Then $G$ is a Hall $\pi$-subgroup of $G$ if and only if $H$ is $\pi$-group and $|G/H|$ is coprime to $\pi$.*

*Proof.* Let $p \in \pi$ and $S$ a Sylow $p$-subgroup of $H$. Then $|S| = |H|_p$ and so $S \in \mathrm{Syl}_p(G)$ if and only if $|H|_p = |G|_p$, that is if and only if $p$ does not divide $|G/H|$. □

**Lemma 1.13.8.** *Let $G$ be a group and $\pi$ a set of primes.*

*(a) If the Sylow $\pi$-theorem holds in $G$, then all Sylow $\pi$-subgroups of $G$ are Hall $\pi$-subgroups.*

*(b) If $G$ is finite, then all Hall $\pi$-subgroups of $G$ are Sylow $\pi$-subgroups.*

*Proof.* (a) Let $H$ be Sylow $\pi$-subgroups of $G$, $p \in \pi$ and $S$ a Sylow $p$-subgroup of $G$. Then $S$ is contained in a Sylow $\pi$-subgroup $R$ of $G$. By assumption $R^g = H$ for some $g \in G$ and so $S^g$ is a Sylow $p$-subgroups of $G$ contained in $H$. Thus $H$ is a Hall $\pi$-subgroup of $G$. (b) This follows since $|G/H|$ is coprime to $\pi$ for all Hall $\pi$-subgroups $H$ of $G$. □

**Example 1.13.9.** *Sylow $p'$-subgroups of* $\mathrm{Sym}(p)$.

Let $p$ be a prime, $G = \mathrm{Sym}(p)$ and $I = \{1, 2 \ldots, p\}$. Let $H \leq G$. $H$ acts transitively on $I$ if and only if $p \mid |H|$. Thus $H$ is a $p'$ group if and only if $H$ normalizes a proper subset $J$ if $I$. If $H$ is a Sylow $p'$-subgroups we get $H = N_G(J) \cong \mathrm{Sym}(J) \times \mathrm{Sym}(I \setminus J)$. Such an $H$ is Hall $p'$-subgroup if and only if $p = |G/H| = \binom{p}{|J|}$ and so if and only if $|J| = 1$ or $|I \setminus J| = 1$. So the Sylow $p'$ subgroups of $\mathrm{Sym}(p)$ are $\mathrm{Sym}(k) \times \mathrm{Sym}(p-k)$ and the Hall $p'$-subgroups are $\mathrm{Sym}(p-1)$.

**Example 1.13.10.** *Sylow and Hall subgroups of* $\mathrm{Sym}(5)$

The Sylow $\{2,3\}$-subgroups of $\mathrm{Sym}(5)$ are $\mathrm{Sym}(3) \times \mathrm{Sym}(2)$ and $\mathrm{Sym}(4)$, with the latter being a Hall-$\{2,3\}$-subgroup.

Let $q \in \{2,3\}$ and $H$ a Sylow $\{q,5\}$ subgroup of $G = \mathrm{Sym}(5)$. Suppose $5 \mid |H|$. $G$ has six Sylow 5-subgroups, $H$ has at most six Sylow 5-subgroups. Since $6 \nmid |H|$. $H$ has a unique Sylow 5-subgroup $S$. Thus $H \leq N_G(S) \cong \mathrm{Frob}_{20}$. If $q = 2$ we get $G = N_G(S) \cong \mathrm{Frob}_{20}$ and if $q = 3$ we have $G = S \cong C_5$.

Suppose $5 \nmid |H|$. Then $H$ is a $q$-groups and so $H$ is a Sylow $q$-subgroups. For $q = 2$ we get $H \cong D_8$ and for $q = 3$, $H \cong C_3$.

**Example 1.13.11.** *Hall subgroups in $GL_3(2)$.*

Let $V$ be a 3-dimensional vector-space over $\mathbb{F}_2$ and $G = \mathrm{GL}_{\mathbb{F}_2}(V)$. Let $i \in \{1,2\}$ and $\mathcal{P}_i$ the set of $i$-dimensional subspace of $V$. Let $V_i \in \mathcal{P}_i$ and put $H_i = N_G(V_i)$. Then $|\mathcal{P}_i| = \frac{2^3-1}{2-1} = 7$ and $|H_i| = 3 \cdot 2 \cdot 4 = 24 = 2^3 \cdot 3$. So $H_1$ and $H_2$ are Hall $7'$-subgroups of $G$. But $H_1$ and $H_2$ are not-conjugate to in $G$ and so the Sylow $7'$-Theorem does not hold.

Let $H$ be $\{3,7\}$ subgroup of $G$. Then $|H| = 1, 3, 7$ or $21$. Suppose the latter. By Sylow Theorem, $G$ has 8 Sylow 7-subgroups and $H$ has a unique Sylow 7-subgroups $S$. Hence $|N_G(S)| = 21$ and $H = N_G(S)$. Hence $H$ is a Hall $\{3,7\}$ subgroups and all Sylow $\{3,7\}$ subgroups are conjugate. So the Sylow $\{3,7\}$-theorem holds.

**Lemma 1.13.12.** *Let $G$ be a group and $\pi$ a set of primes. Let $\mathcal{S}$ be non-empty $G$-invariant subset of* $\mathrm{Syl}_\pi(G)$. *Then $O_\pi(G) = \bigcap \mathcal{S} = \bigcap \mathrm{Syl}_\pi(G)$.*

*Proof.* Clearly $\bigcap \mathcal{S}$ is a normal $\pi$-subgroup of $G$ and so $\bigcap \mathcal{S} \leq O_\pi(G))$. Also $\bigcap \mathrm{Syl}_p(G) \leq \bigcap \mathcal{S}$. Let $H \in \mathrm{Syl}_p(G)$ and $N$ a normal $\pi$-subgroup of $G$. Since $\mathcal{G}_\pi$ is $\{\mathbf{Q}, \mathbf{P}\}$-closed, $NH$ is a $\pi$-group and so $N \leq H$ by maximality of $H$. Thus $O_\pi(G) \leq \bigcap \mathrm{Syl}_\pi(G)$. □

Observe that the preceding lemma provides a new proof that $O_\pi(G)$ is a $\pi$-group.

**Lemma 1.13.13.** *Let $\pi$ be a set of primes and $G$ a finite $\pi$-separable group. Then $G$ has a Hall $\pi$-subgroup.*

*Proof.* If $G = 1$, $G$ is a Hall $\pi$-subgroup. So suppose $G \neq 1$ and let $A = O_\pi(G)$, $\overline{G} = G/A$ and $\overline{B} = O_{\pi'}(\overline{G}))$. Since $G$ is $\pi$-seperable, $B \neq 1$ and so by induction $G/B$ has a Hall $\pi$-subgroup $K/B$. Since $\overline{K}/\overline{B} \cong K/B$ is a $\pi$-group and $\overline{B}$ is a $\pi'$-group, there exists a complement $\overline{H}$ to $\overline{B}$ in $\overline{K}$. Since $A$ and $\overline{H}$ are $\pi$-group, $H$ is a $\pi$-groups. Also

$$|G/H| = |G/K||K/H| = |\overline{G}/\overline{H}||\overline{K}/\overline{H}| = |\overline{G}/\overline{H}||\overline{B}|.$$

and so $|G/H|$ is coprime to $\pi$. □

**Lemma 1.13.14.** *Let $\pi$ be a set of primes. Then every $\pi$-subgroup of a $\pi$-solvable group is hypo-abelian.*

*Proof.* Let $G$ be a $\pi$-subgroup of a $\pi$-solvable group. Then $G$ has a descending series all of whose factors are in $\mathcal{G}_{\mathrm{Sol}} \cup \mathcal{G}_{\pi'}$. Since only the trivial group is a $\pi$- and a $\pi'$-group, all factors are solvable. So each factor has a finite series with Abelian factors and so by 1.12.29, $G$ is hypo-abelian. □

**Lemma 1.13.15.** *Let $\pi$ be set of primes and $G$ a finite group. Suppose that $G$ is $\pi$-solvable or $\pi'$-solvable. Then the Sylow $\pi$-theorem holds in $G$.*

*Proof.* By 1.13.13 $G$ has a Hall $\pi$-subgroup $H$. Let $S$ be Sylow $\pi$-subgroup of $G$. We will show that $S^g \leq H$ for some $g \in G$. By 1.13.12 $O_\pi(G) \leq S \cap H$ and replacing $G$ by $G/O_\pi(G)$ we may assume that $O_\pi(G) = 1$. Let $B = O_{\pi'}(G))$. Since $G$ is $\pi$-separable, $B \neq 1$. Observe that $HB/B$ is a Hall-subgroups of $G/B$ and $SB/B$ is contained in a Sylow $\pi$-subgroups of $G/B$. So by induction $S^h \leq HB$ for some $h \in G$ and we may assume that $S \leq HB$. Since $G$ is $\pi$- or $\pi'$-solvable, 1.13.14 implies that either $B$ or $H$ is solvable. Also both $S$ and $H \cap BS$ are complements to $B$ in $BS$ and so by 1.2.2, $S^g = H \cap BS$ for some $g \in G$. So $S^g \leq H$ and thus $S^{kh} = H$ by maximality of $S$. □

**Lemma 1.13.16.** *Let $G$ be a finite group and $A$ and $B$ subgroups of $G$ with $G = BA$.*

*(a) If $N$ is an $A$-invariant subgroup of $B$, then $\langle N^G \rangle \leq B$.*

*(b) If $\pi$ is set of primes with $|G/B|_\pi = 1$, then $\langle O_\pi(A)^G \rangle \leq B$.*

*Proof.* (a) $\langle N^G \rangle = \langle N^{AB} \rangle = \langle N^B \rangle \leq B$.

(b) Observe that

$$|O_\pi(A)/O_\pi(A) \cap B| = |O_\pi(A)B/B|$$

is a $\pi$-number and a $\pi'$-number and so $|O_\pi(A)B/B| = 1$. Thus $O_\pi(A) \leq B$ and (a) follows from (b). □

**Lemma 1.13.17.** *Let $G$ be a finite group, $\pi$ and $\mu$ sets of primes with $\pi' \cap \mu' = \emptyset$ and $A$,$B$ and $C$ subgroups of $G$. Let $\mathcal{D}$ be $\{\mathbf{N_0}, \mathbf{P}, \mathbf{Q}, \mathbf{S_n}\}$-closed class of finite groups. Suppose that* *(a)* $G = AB = AC$.

*(b)* $|G/B|_\pi = 1 = |G/C|_\mu$.

*(c)* $B$ *and* $C$ *are* $\mathcal{D}$*-groups.*

*(d)* $A$ *is a hypo-*$\mathcal{G}_\pi \cup \mathcal{G}_\mu$*-group.*

*Then* $G$ *is* $\mathcal{D}$*-group.*

*Proof.* Replacing $G$ by $G/G_{\mathcal{D}}$ we may assume that $G_{\mathcal{D}} = 1$. By 1.13.16 $\langle O_\pi(A)^G \rangle \leq B$. Since $B$ is a $\mathcal{D}$-group, and $\mathcal{D}$ is $\mathbf{S_n}$ closed we conclude that $\langle O_\pi(A)^G \rangle$ is a normal $\mathcal{D}$-subgroup of $G$. But $G_{\mathcal{D}} = 1$ and so $O_\pi(A) = 1$. By symmetry, $O_\mu(A) = 1$ and since $A$ is a hypo-$\mathcal{G}_\pi \cup \mathcal{G}_\mu$-group, $A = 1$. Thus $G = B$ is a $\mathcal{D}$-group. □

**Lemma 1.13.18.** *Let* $G$ *be group* $\pi$ *and* $\mu$ *sets of primes with* $\pi' \cap \mu' = \emptyset$ *and suppose* $A$ *and* $B$ *are subgroups of finite index with*

$$|G/A|_\pi = 1 = |G/B|_\mu$$

*Then* $G = AB$ *and*

$$|G/A \cap B| = |G/A| \cdot |G/B|$$

*Proof.* Note that $|G/AB|$ divides $|G/A|$ and $|G/B$, so $|G/AB|$ is both a $\pi'$ and a $\mu'$ number. Hence $|G/AB| = 1$ and $G = AB = BA$. Thus

$$|G/A \cap B| = |AB/A \cap B| = |A/A \cap B||B/A \cap B| = |AB/B||BA/A| = |G/B||G/A|$$

□

**Corollary 1.13.19.** *Let* $G$ *be a finite groups,* $\pi$ *and* $\mu$ *sets of primes with* $\pi' \cap \mu' = 1$. *Suppose* $H_\pi$ *and* $H_\mu$ *are Hall* $\pi$*- and* $\mu$ *subgroups, respectively. Then* $H_\pi H_\mu = H$ *and* $H_\pi \cap H_\mu$ *is an Hall* $\pi \cap \mu$*-subgroup of* $G$.

*Proof.* By 1.13.18 $G = H_\pi H_\mu$ and $|G/H_\pi \cap H_\mu| = |G/H_\pi| \cdot |G/\mu|$. The latter number is coprime to $\pi \cap \mu$. Also $H_\pi \cap H_\mu$ is a $\pi \cap \mu$-groups and so $H_\pi \cap H_\mu$ is Hall $\pi \cap \mu$-subgroup of $G$. □

**Corollary 1.13.20.** *Let* $G$ *be a finite groups,* $\pi_i, 1 \leq i \leq 3$ *be sets of primes and* $H_i$ *a* $\pi_i$*-Hall-subgroups of* $G$. *Suppose that* $\pi_i' \cap \pi_j' = \emptyset$ *for all* $1 \leq i < j \leq 3$ *and that* $H_1, H_2$ *and* $H_3$ *are solvable. Then* $G$ *is solvable.*

*Proof.* By 1.13.19 $G = H_1 H_2 = H_1 H_3$. Since $H_1$ is solvable, all composition factor of $H_1$ are $p$-groups. Since $\pi_1 \cup \pi_2$ is the set of all primes, $H_1$ is a hypo $\mathcal{G}_{\pi_1} \cup \mathcal{G}_{\pi_2}$-groups. So by 1.13.17, $G$ is solvable. □

**Definition 1.13.21.** *Let $G$ be group and $\mathbb{P}$ the set of all primes. A Sylow-system of $G$ is familiy $(S_p)_{p\in\mathcal{P}}$ such that for each $p \in \mathbb{P}$, $S_p$ is a Sylow $p$-subgroup of $G$ and for all $p, q \in \mathbb{P}$, $S_pS_q = S_qS_p$.*

**Lemma 1.13.22.** *Let $G$ be a finite group. Then the following are equivalent:*

*(a) For each set of primes $\pi$, $G$ has a Hall $\pi$-subgroup.*

*(b) For each prime $p$, $G$ has a Hall $p'$-subgroups.*

*(c) $G$ has a Sylow-system.*

*Proof.* (a) $\Longrightarrow$ (b): Obvious.

(b) $\Longrightarrow$ (c): For a prime $p$ pick a Hall $p'$-subgroup, $H_{p'}$ of $G$. So $\pi$ a set of primes, put $H_\pi = \bigcap_{p\in\pi'} H_{p'}$. Then by 1.13.19, $H_\pi$ is a Hall $\pi$-subgroup of $G$. Let $p$ and $q$ be primes. Put $H_p = H_{\{p\}}$. Then $H_p$ and $H_q$ are Sylow $p$ and $q$-subgroups of $H_{\{p,q\}}$ and hence $H + pH_q = H\{p, q\} = H_qH_p$. Thus $(H_p)_{p\in\mathbb{P}}$ is a commuting Sylow system.

(c) $\Longrightarrow$ (a): Let $(S_p)_{p\in\mathbb{P}}$ be a Sylow system. For $\pi$ a set of primes, define $H_\pi = \prod_{p\in\pi} S_p$. Suppose inductively that $H_\pi$ is a $\pi$-subgroup of $G$ and let $q$ be prime. Since $S_qSp$ for all $p \in \pi$, $H_\pi S_q = S_q H_\pi$. Hence $H_\pi S_q$ is a $\pi\cup\{q\}$-subgroups of $G$. So $H_\pi$ is a $\pi$-subgroup of $G$. Since it contains a Sylow $p$-subgroup for each $p \in \pi$, it is a Hall $\pi$-subgroup.

□

**Lemma 1.13.23.** *Let $G$ be a finite group with a Sylow-system $(S_p)_{p\in\mathbb{P}}$. Suppose that $S_pS_q$ is solvable for all $p, q \in \mathbb{P}$. Then $G$ is solvable.*

*Proof.* If $|G|$ has at most two primes divisor, $G = S_pS_q$ for some $p, q \in \mathbb{P}$ and the lemma holds. So suppose $p_1, p_2, p_3$ are three distinct primes dividing $|G|$. Define $H_i = \prod_{q\in p_i'} S_q$. Then $H_i$ is Hall $p_i'$-subgroups. By induction on the number of primes divisors, each $H_i$ is solvable. So by 1.13.20, $G$ is solvable. □

## 1.14 Join of Subnormal Subgroups

**Lemma 1.14.1.** *Let $G$ be a groups, $A$ a normal subgroup of $G$ and $B$ a subnormal subgroups of $G$. Then $AB$ is subnormal in $G$.*

*Proof.* Just observe that $BA/A$ is subnormal in $G/A$. □

**Lemma 1.14.2.** *Let $G$ be a group and $\mathcal{M}$ be a $G$-invariant set of subnormal subgroups of $G$. Suppose that*

*(i) If $A, B \in \mathcal{M}$ with $\langle A, B\rangle$ subnormal in $G$, then $\langle A, B\rangle \in \mathcal{M}$.*

*(ii) Every non-empty subset of $\mathcal{M}$ has a maximal element.*

*Then*

*(a)* $\langle A, B\rangle \trianglelefteq\trianglelefteq G$ *for all* $A \in \mathcal{M}$ *and all subnormal subgroups* $B$ *of* $G$*.*

*(b)* $\langle \mathcal{N}\rangle \in \mathcal{M}$ *for all* $\mathcal{N} \subseteq \mathcal{M}$*.*

*Proof.* (a) Suppose false. By (ii) we can choose a counter example, $(A, B)$ with $A$ maximal. Since $A$ is subnormal in $G$ we can choose a subnormal series

$$A = A_0 \lhd A_1 \lhd A_2 \lhd \ldots \lhd A_{n-1} \lhd A_n = G$$

Since $\langle A, B\rangle$ is not subnormal in $G$, we can $i$ minimal such that there exists $D \trianglelefteq\trianglelefteq A_i$ such that $\langle A, D\rangle$ is not subnormal in $G$. Then $i > 0$.

Suppose that $D$ does not normalizes $A$. Then $A \neq A^g$ for some $g \in D$. Since $A \leq A_{i-1} \trianglelefteq A_i$ and $D \leq A_i$, $A^g \leq A_{i-1}$ and so by minimal choice of $i$, $\langle A, A^g\rangle$ is subnormal in $G$. By (ii), $\langle A, A^g\rangle \in \mathcal{M}$. Then $\langle A, D\rangle = \langle\langle A, A^g\rangle, D\rangle$ is subnormal in $G$ by maximal choice of $A$, a contradiction.

Thus $D$ normalizes $A$. Hence $A \lhd \langle A_1, D\rangle$ and by 1.14.1, $\langle A, D\rangle \trianglelefteq\trianglelefteq \langle A_1, D\rangle$. By maximal choice of $A$, $\langle A_1, D\rangle$ is subnormal in $G$. Thus $\langle A, D\rangle$ is subnormal in $G$, a contradiction. So (a) holds.

(b) By (a), (i) and induction, $\langle \mathcal{P}\rangle \in \mathcal{M}$ for all finite subsets $\mathcal{P}$ of $\mathcal{N}$. Hence by (ii) we can choose a finite subset $\mathcal{P}$ of $\mathcal{N}$ with $\langle \mathcal{P}\rangle$ maximal. But the $\langle cN\rangle = \langle \mathcal{P}\rangle$ and the lemma is proved. □

We remark the preceding lemma is false without the maximal condition.

**Definition 1.14.3.** *Let* $G$ *be a group. We say that* $G$ *fulfills max-subnormal if every non-empty set of subnormal subgroups of* $G$ *has a maximal element.*

**Corollary 1.14.4.** *If* $G$ *is a group and fulfills max-subnormal, then every set of subnormal subgroups of* $G$ *generates a subnormal subgroup of* $G$*.*

*Proof.* Just apply 1.14.2 to the set $\mathcal{M}$ of subnormal subgroups of $G$. □

## 1.15 Near-components

**Definition 1.15.1.** *Let* $G$ *be a group.*

*(a)* $M(G)$ *is the subgroup of* $G$ *generated by the proper normal subgroups of* $G$*.*

*(b)* $G$ *is called nearly-simple if* $G$ *is perfect and* $G \neq M(G)$*.*

*(c) A near-component of* $G$ *is a subnormal, nearly-simple subgroup of* $G$*.*

*(d)* $G$ *is called quasi-simple, if* $G$ *is a non-trivial perfect group with* $G/Z(G)$ *simple.*

*(e) A component of* $G$ *is a subnormal, quasi-simple subgroup of* $G$*.*

**Lemma 1.15.2.** *Let* $G$ *be groups.*

*(a) $G/M(G)$ is simple.*

*(b) Every quasi-simple group is a nearly-simple.*

*(c) Every component of $G$ is a near-component of $G$.*

*Proof.* (a) Let $N/M(G)$ be normal subgroups of $G/M(G))$. By definition of $M(G)$ either $N \leq M(G)$ or $N = G$. Thus $N/M(G) = 1$ or $N/M(G) = G/M(G)$.

(b) Suppose $G$ is quasi-simple and let $N \trianglelefteq G$ with $N \not\leq Z(G)$. Since $G/Z(G)$ is simple, $G = NZ(G)$ and so $G = G' = [G, NZ(G)][G, N] \leq N$. Thus $N = G$ and so $M(G) \leq Z(G)$. Since $G = G' \neq 1$, $G \neq Z(G)$ and so also $M(G) \neq G$. Thus $G$ is nearly-simple.

(c) Follows from (b). □

**Lemma 1.15.3.** *Let $G$ be a group and $H$, $K$ and $E$ subgroups of $G$ with $E \leq K$. Then $E$ is a supplement to $H$ in $G$ if and only $K$ is a supplement to $H$ in $G$ and $E$ is a supplement to $H \cap K$ in $K$.*

*Proof.* If $E(H \cap K) = K$ and $KH = G$, then $G = KH = E(H \cap K)H = EH$. And if $G = EH$, then $K = K \cap EH = E(K \cap H)$ and $G = EH \leq KH \leq G$. □

**Lemma 1.15.4.** *Let $G$ be a finite group and $N \triangleleft G$ with $G/N$ perfect.*

*(a) There exists a unique minimal subnormal supplement $R$ to $N$ in $G$.*

*(b) $R$ is perfect.*

*(c) Suppose in addition that $G/N$ is simple, then $M(R) = R \cap N$ and $R$ is the unique near-component of $G$ with $R \not\leq N$.*

*Proof.* (a) Let $S_1$ and $S_2$ be minimal subnormal supplement to $N$ in $G$. If $S_1 = G$ we get $S_1 \leq S_2$ and so $S_1 = S_2$. Thus we may assume that $S_i \neq G$ and so there exists $G_i \triangleleft G$ with $S_i \leq G_i$. Then $G = G_iN$. Since $G/N$ is perfect,

$$G = [G, G]N = [G_1N, G_2N]N = [G_1, G_2]N$$

Thus also $G_0 = [G_1, G_2]$ is a normal supplement to $N$ in $G$. Note that $G_0 \leq G_1 \cap G_2$ and that for $0 \leq i \leq 2$. $G_i/G_i \cap N \cong G_iN/N = G/N$ is perfect. By induction there exists a unique minimal subnormal supplement $R_i$ to $N \cap G_i$ in $G_i$. Since $S_i \leq G_i \geq R_0$, 1.15.3 now shows that $S_1 = R_1 = R_0 = R_2 = S_2$. So (a) holds.

(b) As above $[R, R]$ is supplement to $N$ in $G$ and so $R = [R, R]$ by minimality of $R$.

(c) Let $K$ be any subnormal subgroup of $K$ with $K \not\leq N$. Then $1 \neq KN/N \trianglelefteq\trianglelefteq G/N$ and since $G/N$ is simple, $KN = G$.

If $K \leq R$, this implies $K = R$. So any proper normal subgroups of $R$ is contained in $R \cap N$. Thus $M(R) = R \cap N \neq R$. By (b) $R$ is perfect and so $R$ is nearly-simple.

Let $K$ be any near-component of $G$ with $K \not\leq N$. Then $GN = K$ and so $R \leq K$ and thus $K = (K \cap N)R$. Since $K \cap N \leq M(K) < K$, $R \not\leq M(K)$ and so $R = K$. □

**Lemma 1.15.5.** *Let $G$ be a finite group, $K$ a near-component of $G$ and $N$ a subnormal subgroup of $N$. Then one of the following holds:*

1. $K \leq N$.

2. *$N$ normalizes $K$ and $[K, N] \leq M(K)$.*

*Proof.* If $N = G$, (1) holds. So suppose $N \neq K$ and let $H$ be a maximal normal subgroup of $G$ with $N \leq H$. If $K \leq H$, then (1) or (2) holds by induction on $|G|$.

So suppose $K \not\leq H$. Then by 1.15.4, $K$ is the unique minimal subnormal supplement to $N$ in $G$ and $K \cap H = M(K)$. Thus $K \trianglelefteq G$ and so $[K, N] \leq [K, H] \leq K \cap H = M(K)$ and (2) holds. □

**Lemma 1.15.6.** *Let $K$ and $L$ be near components of a finite groups $G$. Then exactly one of the following holds.*

$$K = L, \quad K \leq M(L), \quad L \leq M(K), \quad [K, L] \leq M(L) \cap M(K)$$

*Proof.* We will first show that one of the four statements hold. Suppose $K \leq L$. By 1.15.4 $L$ is the only near-component of $L$ not contained in $M(L)$ and so $K \leq M(L)$ or $K = L$. So suppose $K \not\leq L$. By symmetry we may also assume $L \not\leq K$. Then by 1.15.5 $[K, L] \leq M(K)$ and by symmetry, $[K, L] \leq M(L)$.

So one of the four statements hold. If $K \leq L$, then $K = [K, K] \leq [K, L]$ and so $[K, L] \not\leq M(K)$. It follows that at most one of the four statements can hold. □

**Lemma 1.15.7.** *Let $G$ be a finite group and $K$ a component of $G$.*

*(a)* *Let $N \trianglelefteq\trianglelefteq G$. Then $K \leq N$ or $[K, N] = 1$.*

*(b)* *Let $L$ be a component of $G$. Then $K = L$ or $[K, L] = 1$.*

*(c)* $[K, \mathrm{Sol}(G)] = [K, \mathrm{F}(G)] = 1$.

*Proof.* (a) Suppose $K \not\leq N$. Then by 1.15.5 $[K, N] \leq M(K) = Z(K)$. Thus

$$[N, K, K] \leq [Z(K), K] = 1 \text{ and } [K, N, K] = [N, K, K] = 1.$$

So by the Three Subgroup Lemma, $[K, K, N] = 1$. Since $K$ is perfect, $K = [K, K]$ and so $[K, N] = 1$.

(b) Suppose $[K, L] \neq 1$. Then by (a), $K \leq L$. By symmetry, $L \leq K$ and so $L = K$.

(c) Since $1 \neq K = K'$, $K$ is not solvable. Thus $K \not\leq \mathrm{Sol}(G)$ and so by (a), $[K, \mathrm{Sol}(G)] = 1$. Since $\mathrm{F}(G) \leq \mathrm{Sol}(G)$ also $[K, \mathrm{F}(G)] = 1$. □

**Definition 1.15.8.** *Let $G$ be a group.*

*(a)* *$G$ is called nearly-Abelian if $H \leq Z(G)$ for all proper normal subgroups $H$ of $G$.*

*(b)* $\mathrm{F}^*(G)$ *is the subgroup generated by the nearly-Abelian subnormal subgroups.* $\mathrm{F}^*(G)$ *is called the generalized Fitting subgroup of* $G$*.*

*(c)* $\mathrm{E}(G)$ *is the subgroup generated by the components of* $G$*.*

**Lemma 1.15.9.** *Let* $G$ *be a finite group.*

*(a)* $C_G(\mathrm{F}^*(G)) \leq \mathrm{F}^*(G)$*.*

*(b)* $\mathrm{F}^*(G) = \mathrm{F}(G)\mathrm{E}(G)$*.*

*(c)* $[\mathrm{F}(G), \mathrm{E}(G)] = 1$*.*

*Proof.* (a) follows from Homework 4.

(b) By Homework 4, a group is nearly-Abelian if and only if it is Abelian or quasi-simple. The subgroup of $G$ generated by the quasi-simple subnormal subgroups is $\mathrm{E}(G)$. Let $F$ be the group generated by the Abelian subnormal subgroups of $G$. Since $\mathrm{F}(G)$ contains all nilpotent subnormal subgroups of $G$, $F \leq \mathrm{F}(G)$. Since $\mathrm{F}(G)$ is nilpotent, each subgroup of $\mathrm{F}(G)$ is subnormal in $\mathrm{F}(G)$ and so also in $G$. Since any group is generated by its cyclic subgroups, and so also by it Abelian subgroups, $\mathrm{F}(G) \leq F$. Thus $F = \mathrm{F}(G)$ and $\mathrm{F}^*(G) = \mathrm{F}(G)\mathrm{E}(G)$.

(c) By 1.15.7, $[\mathrm{F}(G), K] = 1$ for any components $K$ of $G$. Thus (c) holds.

□

**Lemma 1.15.10.** *Let* $G$ *be a finite group and* $\mathcal{K}$ *a totally unordered set of near-components of* $G$*, that is* $K \not\leq L$ *for all* $K \neq L \in \mathcal{K}$*. Put* $E = \langle \mathcal{K} \rangle$*,* $M = \langle M(K) \mid K \in \mathcal{K} \rangle$ *and* $\overline{E} = E/M$*. Let* $K \in \mathcal{K}$ *and put* $K^\perp = \langle L \in \mathcal{K} \mid L \neq K \rangle M(K1)$*. Then*

*(a)* $E$ *is a perfect subnormal subgroup* $G$*.*

*(b)* $K \trianglelefteq E$*.*

*(c)* $K \cap M = K \cap K^\perp = M(K)$*,* $E = KK^\perp$ *and so* $\overline{K} \cong E/K^\perp \cong K/M(K)$*.*

*(d)* $K^\perp = C_E(K/M(K))$ *and* $K^\perp$ *is the unique maximal normal subgroup of* $E$ *with* $K \not\leq K^\perp$*.*

*(e)* $\overline{M} = \bigoplus_{K \in \mathcal{K}} \overline{K} \cong \bigoplus_{K \in \mathcal{K}} K/M(K)$*.*

*(f) Let* $R$ *be a near-component of* $G$*. Then either* $R \in \mathcal{K}$ *or* $R \leq M$ *and there exists* $K \in \mathcal{K}$ *with* $R \leq M(K)$*. In particular,* $\mathcal{K}$ *is the set of maximal near-components of* $E$ *and it is also the set of the near-components of* $E$ *which are not contained in* $M$*.*

*(g) The map* $K \to K^\perp$ *is a bijection between* $\mathcal{K}$ *and the set of maximal normal subgroups of* $G$*.*

*(h)* $M$ *is the intersection of the maximal normal subgroups of* $E$*.*

*Proof.* By 1.14.2 $E$ is subnormal in $G$. Any group generated by perfect subgroups is perfect and so (a) holds.

Let $K, L \in \mathcal{K}$ with $K \neq L$. Since $\mathcal{K}$ is totally unordered neither $K \not\leq L$ nor $L \not\leq K$. Thus by 1.15.6,

$$[K, L] \leq M(K) \cap M(L) \leq M$$

In particular, $L$ normalizes $K$ and so (b) holds. Moreover, $[K, K^\perp] \leq M(K) \leq M$. Since $[K, K] = K \not\leq M(K)$, $K \not\leq K^\perp$ and so $K \cap K^\perp \leq M(K) \leq M$. Thus $K \cap K^\perp = M(K)$. Note that $M(K) \leq M \leq K^\perp$ and so also $K \cap M = M(K)$. Furthermore, $E = KK^\perp$ and thus (c) is proved.

Since $[K, K^\perp] \leq M(K)$, $K^\perp \leq C_E(K/M(K))$. Since $E/K^\perp \cong K/M(K)$ is simple, $K^\perp$ is a maximal normal subgroups of $E$. Thus $K^\perp = C_E(K/M(K))$. Let $N$ be any normal subgroup of $E$ with $K \not\leq N$. By 1.15.5 $[K, N] \leq M(K)$ and $N \leq C_E(K/M(K)$. Hence (d) is proved.

We have $KM \cap K^\perp = (K \cap K^\perp)M = M$ and so $\overline{E} = \overline{K} \times \overline{K}^\perp$. Since

$$(*) \qquad \overline{K^\perp} = \langle \overline{L} \mid L \in \mathcal{K}, L \neq K \rangle$$

we conclude that (e) holds.

From (*) and (e),

$$(**) \qquad \bigcap_{K \in \mathcal{K}} K^\perp = M$$

Let $R$ be a near-component of $E$. If $R \not\leq M$, then $R \not\leq K^\perp$ for some $K \in \mathcal{K}$. Since $E/K^\perp$ is perfect and simple, $K \not\leq K^\perp$ we have $R = K$ by 1.15.4. Suppose $R \leq M$. Since $R = R' \leq [R, \langle \mathcal{K} \rangle]$, there exists $K \in \mathcal{K}$ with $[R, K] \not\leq M(R)$. Since $R \leq M$ we have $K \not\leq R$ and so by 1.15.6 $R \leq M(K)$. Thus (f) holds.

Let $N$ be a maximal normal subgroups of $E$. Since $E$ is perfect, $E/N$ is perfect and so by 1.15.4 there exists a unique near-component $R$ of $E$ with $R \not\leq N$. By (f), $R \leq K$ for some $K \in \mathcal{K}$. Then $K \not\leq N$ and so by 1.15.4 $R = K$. Thus by (d), $N = K^\perp$. So the map $N \to R$, is inverse to the map $K \to K^\perp$ and (g) is proved.

(h) follows from (g) and (**). □

**Remark 1.15.11.** *Let $G$ be a finite group and $H \trianglelefteq\trianglelefteq G$.*

*(a) The set of maximal near-components of $H$ is totally unordered.*

*(b) The set of minimal near-components of $H$ is totally unordered.*

*(c) If $K$ is a near-component of $G$, then $K^G$ is totally unordered. In particular, $K \trianglelefteq \langle K^G \rangle$.*

*(d) The set of components of $G$ is totally unordered.*

**Corollary 1.15.12.** *Let $G$ be a finite group. Then map $\mathcal{K} \to \langle \mathcal{K} \rangle$ is a bijection between the totally-unordered sets of near-components of $G$ and the perfect subnormal subgroups of $G$. The inverse is given by $H \to \mathcal{K}^*(H)$, where $\mathcal{K}^*(H)$ is the set of maximal near components of $H$.*

*Proof.* $H$ be a perfect subnormal subgroup of $G$ and $D$ be the subgroup of $H$ generated by all the near-components of $G$ which are contained in $H$. Observe that $D = \langle \mathcal{K}^*(H) \rangle$.

Suppose for a contradiction that $D \neq H$. Then there exist a maximal normal subgroup $N$ of $H$ with $D \leq N$. By 1.15.4 there exists unique near-component $K$ if $H$ with $K \not\leq N$. Since $H \trianglelefteq\trianglelefteq G$, $K \trianglelefteq\trianglelefteq H$ and so $K$ is near-component of $G$. Thus $K \leq D \leq N$, a contradiction.

Hence $H = \langle \mathcal{K}^*(H) \rangle$ and the Corollary follows from 1.15.10 □

**Lemma 1.15.13.** *Let $G$ be a finite group and $\mathcal{S}$ a composition series for $G$. For a a near component $K$, choose $S_K \in \mathcal{S}$ minimal with $K \leq S_K$. Then $K/M(K) \cong S_K/S_K^-$ and the map $K \to S_K/S_K^-$ is a bijection between the set of near-components of $G$ and the non-Abelian factors of $\mathcal{S}$.*

*Proof.* Note that $K$ is a near-component of $S_K$ with $K \not\leq S_K^-$. Since $K$ is perfect, $S_K/S_K^-$ is a non-Abelian simple groups and so perfect. It follows that $K$ is the minimal normal supplement to $S_K^-$ in $S_K$. In particular, $K$ is uniquely determined by $S_K$, $S_K/S_K^- \cong K/M(K)$ and our map is injective.

Let $S \in \mathcal{S}$ such that $S/S^-$ is non-Abelian. Then the minimal subnormal supplement $K$ to $S^-$ in $S$ is a near-component of $G$ with $K \not\leq S$ and $K \not\leq S^-$. It follows that $S = S_K$ and so our map is surjective. □

## 1.16 Subnormal subgroups

**Definition 1.16.1.** *(a) Let $\mathcal{M}$ be a set of sets and $A$ a set. Then $\mathcal{M}_A = \{B \in \mathcal{M} \mid B \subseteq A\}$.*

*(b) Let $\mathcal{M}$ be a partial ordered set. Then we say that* $\max -\mathcal{M}$ *holds, if every non-empty subset of $\mathcal{M}$ has a maximal element.*

**Lemma 1.16.2.** *Let $G$ be a group and $\mathcal{M}$ a $G$-invariant set of subgroups of $G$. Let $\mathcal{L} \subsetneq \mathcal{M}$ and suppose that $\langle \mathcal{L} \rangle \trianglelefteq\trianglelefteq G$. Then $\mathcal{L} \subsetneq \mathcal{M}_{N_G(\langle \mathcal{L} \rangle)}$.*

*Proof.* Since $\langle \mathcal{L} \rangle \trianglelefteq\trianglelefteq G$ there exists a subnormal series

$$\langle \mathcal{L} \rangle = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots G_{n-1} \trianglelefteq G_n = G$$

from $\langle \mathcal{L} \rangle$ to $G$. Since $\mathcal{L} \subsetneq \mathcal{M} = \mathcal{M}_G$ we can choose $i$ minimal with $\mathcal{L} \subsetneq \mathcal{M}_{G_i}$.

We claim that $G_i \leq N_G(G_0)$. If $i = 0$, $G_i = G_0 \leq N_G(G_0)$. So suppose that $i > 0$. By minimality of $i$, $\mathcal{M}_{G_{i-1}} = \mathcal{L}$. Since $\mathcal{M}$ and $G_{i-1}$ are $G_i$-invariant, also $\mathcal{M}_{G_{i-1}}$ is $G_i$ invariant. Thus $G_i$ normalizes $\langle \mathcal{M}_{G_{i-1}} \rangle = \langle \mathcal{L} \rangle = G_0$ and the claim is proved.

Hence $\mathcal{L} \subsetneq \mathcal{M}_{G_i} \subseteq \mathcal{M}_{N_G(G_0)} = \mathcal{M}_{N_G(\langle \mathcal{L} \rangle)}$ and the lemma is proved. □

**Lemma 1.16.3.** *Let $G$ be a group, $\mathcal{A}$ a $G$ invariant set of subgroups of $G$. For $H \leq G$ define*

$$\mathcal{A}_H^{\mathrm{sn}} = \{A \in \mathcal{A} \mid A \trianglelefteq\trianglelefteq H\}$$

*Let $\mathcal{B}$ be a non-empty set of subgroups of $G$. Put*

$$\mathcal{D} = \{\langle \mathcal{A}_B^{\mathrm{sn}} \cap \mathcal{A}_{\tilde{B}}^{\mathrm{sn}} \rangle \mid B, \tilde{B} \in \mathcal{B}\}.$$

*Suppose that*

*(i)* $B = \langle \mathcal{A}_B^{\mathrm{sn}} \rangle$ *for all* $B \in \mathcal{B}$.

*(ii)* $\langle \mathcal{P} \rangle \trianglelefteq\trianglelefteq B$ *for all* $B \in \mathcal{B}$ *and* $\mathcal{P} \subseteq \mathcal{A}_B^{\mathrm{sn}}$.

*(iii) For all $D \in \mathcal{D}$ and all $A_1, A_2 \in \mathcal{A}_{N_G(D)}$ with $A_i \trianglelefteq\trianglelefteq A_iD$ for $i = 1, 2$, there exists a maximal element $B$ of $\mathcal{B}$ with $A_iD \trianglelefteq\trianglelefteq B$ for $i = 1, 2$.*

*(iv)* $\max -\mathcal{D}$*-holds.*

*Then* $\langle \mathcal{B} \rangle \in \mathcal{B}$.

*Proof.* Let $B \in \mathcal{B}$. Then $B = \langle \mathcal{A}_B^{\mathrm{sn}} \rangle = \langle \mathcal{A}_B^{\mathrm{sn}} \cap \mathcal{A}_B^{\mathrm{sn}} \rangle \in \mathcal{D}$. Hence by (iv) $\{B^* \in \mathcal{B} \mid B \leq B^*\}$ has a maximal element. So every element of $\mathcal{B}$ is contained in maximal element of $\mathcal{B}$. Hence $\langle \mathcal{B} \rangle \in \mathcal{B}$ if and only if $\mathcal{B}$ has a unique maximal element.

Suppose the lemma is false. By (iv) we can choose maximal elements $B_1, B_2$ of $\mathcal{B}$ such that

$$D := \langle \mathcal{A}_{B_1}^{\mathrm{sn}} \cap \mathcal{A}_{B_2}^{\mathrm{sn}} \rangle$$

is maximal with respect to $B_1 \neq B_2$. Let $i \in \{1, 2\}$. By (ii) and the definition of $D$, $D \trianglelefteq\trianglelefteq B_i$. Thus $\mathcal{A}_D^{\mathrm{sn}} \subseteq \mathcal{A}_{B_i}^{\mathrm{sn}}$. Since $B_1$ and $B_2$ are maximal in $\mathcal{B}$, $B_1 \not\leq B_2$ and $B_2 \not\leq B_1$. Since $D \leq B_1 \cap B_2$ we have $B_1 \neq D \neq B_2$. Note that $D = \langle \mathcal{A}_D^{\mathrm{sn}} \rangle$ and by (i) $B_i = \langle \mathcal{A}_{B_i}^{\mathrm{sn}} \rangle$. So $D \lneq B_i$ implies, $\mathcal{A}_D^{\mathrm{sn}} \subsetneq \mathcal{A}_{B_i}^{\mathrm{sn}}$. . Observe that $\langle \mathcal{A}_D^{\mathrm{sn}} \rangle = D \trianglelefteq\trianglelefteq B_i$ and so by 1.16.2 there exists $A_i \in \mathcal{A}_{B_i}^{\mathrm{sn}}$ with $A_i \leq N_G(D)$ and $A_i \notin \mathcal{A}_D^{\mathrm{sn}}$. Then $A_i \not\leq D$ and $A_i \trianglelefteq\trianglelefteq A_iD$. Thus by (iii) there exists a maximal element $B_3 \in \mathcal{B}$ such that $A_iD \trianglelefteq\trianglelefteq B_3$ for $i = 1, 2$. Since both $A_i$ and $D$ are subnormal in $A_iD$ we conclude that

$$\{A_1, A_2\} \cup \mathcal{A}_D^{\mathrm{sn}} \subseteq \mathcal{A}_{B_3}^{\mathrm{sn}}.$$

Put $E = \langle \mathcal{A}_{B_1}^{\mathrm{sn}} \cap \mathcal{A}_{B_3}^{\mathrm{sn}} \rangle$. Then $\langle A_1, D \rangle = \langle A_1, \mathcal{A}_D^{\mathrm{sn}} \rangle \leq E$ and so $\lneq E$. Since $A_2$ is subnormal in $B_2$ and $A_2 \not\leq D$, $A_2$ is not subnormal in $B_1$. Since $A_2 \trianglelefteq\trianglelefteq B_3$ this implies $B_1 \neq B_3$, a contradiction to the maximality of $D$. □

**Lemma 1.16.4.** *Let $G$ be a group and $\mathcal{A}$ and $\mathcal{E}$ non-empty $G$-invariant sets of subgroups of $G$. Suppose that:*

*(i)* $\mathcal{A}_E = \mathcal{A}_E^{\mathrm{sn}}$ *for all* $E \in \mathcal{E}$.

*(ii) Let* $A_1, A_2 \in \mathcal{A}$. *Then* $\langle A_1, A_2 \rangle \in \mathcal{E}$ *or* $A_1 \trianglelefteq\trianglelefteq \langle A_1, A_2 \rangle$.

*(iii)* $\langle \mathcal{A}_H^{\mathrm{sn}} \rangle \in \mathcal{E}$ *for all* $H \leq G$.

*(iv)* $\max -\mathcal{E}$*-holds.*

*Then* $\langle \mathcal{P} \rangle \in \mathcal{E}$ *and* $\langle \mathcal{P} \rangle \trianglelefteq\trianglelefteq G$ *for all* $\mathcal{P} \subseteq \mathcal{E}$.

*Proof.* Put $\mathcal{B} = \{\langle \mathcal{A}_E \rangle \mid E \in \mathcal{E}\}$ and $\mathcal{C} = \{\langle \mathcal{A}_H^{\mathrm{sn}} \rangle \mid H \leq G\}$. We will show that $\mathcal{A}$ and $\mathcal{B}$ fulfill that assumptions of 1.16.3.

**1°.** $\mathcal{C} = \mathcal{B} \subseteq \mathcal{E}$ *and* $B = \langle \mathcal{A}_B^{\mathrm{sn}} \rangle$ *for all* $B \in \mathcal{B}$.

By (iv) $\mathcal{C} \subseteq \mathcal{E}$.
Let $E \in \mathcal{E}$. Then by (i) $B = \langle \mathcal{A}_E \rangle = \langle \mathcal{A}_E^{\mathrm{sn}} \rangle \in \mathcal{C} \subseteq \mathcal{E}$.
Let $H \leq G$ and put $E = \langle \mathcal{A}_H^{\mathrm{sn}} \rangle$. Then $E \in \mathcal{E}$ and

$$E = \langle \mathcal{A}_H^{\mathrm{sn}} \rangle \leq \langle \mathcal{A}_E^{\mathrm{sn}} \rangle = \langle \mathcal{A}_E \rangle \leq E$$

So $E = \langle \mathcal{A}_E \rangle \in \mathcal{B}$ and (1°) is proved.

**2°.** *Define* $\mathcal{D}$ *as in 1.16.3. Then* $\mathcal{D} \subseteq \mathcal{E}$ *and* $\max -\mathcal{D}$*-holds.*

Let $D \in \mathcal{D}$. Then for some $B, \tilde{B} \in \mathcal{B}$, $D = \langle \mathcal{A}_B^{\mathrm{sn}} \cap \mathcal{A}_{\tilde{B}}^{\mathrm{sn}} \rangle \leq \langle \mathcal{A}_D^{\mathrm{sn}} \rangle \leq D$ and so $D = \langle \mathcal{A}_D^{\mathrm{sn}} \rangle \in \mathcal{C} = \mathcal{B} \subseteq \mathcal{E}$. Together with (iv) this gives (2°).

**3°.** *Let* $E \in \mathcal{E}$ *and* $\mathcal{P} \subseteq \mathcal{A}_E$. *Then* $\langle \mathcal{P} \rangle \in \mathcal{B}$ *and* $\langle \mathcal{P} \rangle \trianglelefteq\trianglelefteq E$. *In particular,* $B \trianglelefteq\trianglelefteq E$ *for all* $B \in \mathcal{B}_E$.

Put $\mathcal{M} = \mathcal{B}_E^{\mathrm{sn}}$. Then $\mathcal{M} \subseteq \mathcal{E}$ and $\max -\mathcal{M}$-holds. Let $X, Y \in \mathcal{M}$ with $\langle X, Y \rangle \trianglelefteq\trianglelefteq E$. Then $\langle X, Y \rangle \in \mathcal{C} = \mathcal{B}$ and so $\langle X, Y \rangle \in \mathcal{M}$. So we can apply 1.14.2 and conclude that $\langle \mathcal{Q} \rangle \in \mathcal{M}$ for all $\mathcal{Q} \subseteq \mathcal{M}$. Hence $\langle \mathcal{Q} \rangle \in \mathcal{B}$ and $\langle \mathcal{Q} \rangle \trianglelefteq\trianglelefteq E$. By (i) $\mathcal{A}_E \subseteq \mathcal{M}$ and so (3°) holds.

**4°.** $A_1 \trianglelefteq\trianglelefteq \langle A_1, A_2 \rangle$ *for all* $A_1, A_2 \in \mathcal{A}$.

If $\langle A_1, A_2 \rangle \in \mathcal{E}$ this follows from (i). (4°) now follows from (ii).

**5°.** *Let* $D \in \mathcal{D}$ *and* $A_1, A_2 \in \mathcal{A}_{N_G(D)}$ *with* $A_i \trianglelefteq\trianglelefteq A_i D$ *for* $i = 1$ *and* $2$. *Then there exists a maximal element* $B$ *of* $\mathcal{B}$ *with* $A_i D \trianglelefteq\trianglelefteq B$ *for* $i = 1$ *and* $2$.

Put $B_0 = \langle A_1, A_2 \rangle B$. By (4°) $A_i \trianglelefteq\trianglelefteq \langle A_1, A_2 \rangle$. Thus $A_i D \trianglelefteq\trianglelefteq \langle A_1, A_2 \rangle D = B_0$. Since $A_i \trianglelefteq\trianglelefteq A_i D$ this gives $A_i \trianglelefteq\trianglelefteq B_0$. Therefore

$$B_0 = \langle A_1, A_2, D \rangle = \langle A_1, A_2, \mathcal{A}_D^{\mathrm{sn}} \rangle \leq \langle \mathcal{A}_{B_0}^{\mathrm{sn}} \rangle \leq B_0$$

and so $B_0 = \langle \mathcal{A}_{B_0}^{\mathrm{sn}} \rangle \in \mathcal{C} = \mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{E}$, (iv) implies there exists a maximal element $B$ of $\mathcal{B}$ with $B_0 \leq B$. By (3°), $A_i D \trianglelefteq\trianglelefteq B$ and so (5°) holds.

We verified the assumptions of 1.16.3 and so $\langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle \in \mathcal{B} \subseteq \mathcal{E}$. Let $\mathcal{P} \subseteq \mathcal{A}$. Then by (3°) applied with $E = \langle \mathcal{A} \rangle$, $\langle \mathcal{P} \rangle \in \mathcal{B}$ and $\langle \mathcal{P} \rangle \trianglelefteq\trianglelefteq \langle \mathcal{A} \rangle \trianglelefteq D$. So the lemma is proved. □

**Lemma 1.16.5.** *Let* $G$ *a finite group and* $\mathcal{A}$ *a* $G$*-invariant set of subgroups of* $G$. *If* $A \trianglelefteq\trianglelefteq \langle A, B \rangle$ *for all* $A, B \in \mathcal{A}$, *then* $A \trianglelefteq\trianglelefteq G$ *for all* $A \in \mathcal{A}$.

*Proof.* We may assume that $G$ acts transitively on $\mathcal{A}$ and $A \neq G$ for $A \in \mathcal{A}$. Let $\mathcal{E}$ be the set of proper subgroups of $G$. By induction on $|G|$ we may assume that $A \trianglelefteq\trianglelefteq E$ for all $E \in \mathcal{E}$ and all $A \in \mathcal{A}$ with $A \leq E \in \mathcal{E}$. So $\mathcal{A}_E = \mathcal{A}_E^{\mathrm{sn}}$ for all $E \in \mathcal{E}$. Let $H \leq G$. If $\langle \mathcal{A}_H^{\mathrm{sn}} \rangle \notin \mathcal{E}$, then $H = G$ and $\mathcal{A}_G^{\mathrm{sn}} \neq \emptyset$. Thus $A \trianglelefteq\trianglelefteq G$ for some $A \in \mathcal{A}$ and since $G$ acts transitively on $\mathcal{A}$, the theorem holds. So we may assume that $\langle \mathcal{A}_H^{\mathrm{sn}} \rangle \in \mathcal{E}$ for all $H \leq G$. It follows that the assumptions 1.16.4 are fulfilled and again $A \trianglelefteq\trianglelefteq G$ for all $A \in \mathcal{A}$. □

**Lemma 1.16.6.** *Let $G$ be a group.*

*(a) Suppose max-normal-nil holds in $G$, that is every non-empty set normal nilpotent subgroups of $G$ has a maximal element. Then $\mathrm{F}(G)$ is nilpotent.*

*(b) Suppose max-subnormal-nil holds in $G$. Then every every nilpotent subnormal subgroups of $G$ is contained in $\mathrm{F}(G)$.*

*Proof.* (a) Since max-normal-nil holds in $G$, every nilpotent normal subgroup of $G$ is contained in a maximal normal-nilpotent subgroup of $G$. The product of any two maximal normal nilpotent subgroups is a normal nilpotent subgroup and so $G$ has a unique maximal nilpotent normal subgroup.

(b) Let $N = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \ldots \trianglelefteq G_n = G$ be a subnormal series from $N$ to $G$. By induction on $n$, we may assume that $N \leq \mathrm{F}(G_{n_1})$. By (a) $\mathrm{F}(G_{n-1})$ is a normal nilpotent subgroup of $G$ and so $\mathrm{F}(G_{n-1}) \leq \mathrm{F}(G)$. □

**Lemma 1.16.7.** *Suppose max-nil holds in the group $G$ and let $\mathcal{A}$ be a $G$-invariant set of subgroups of $G$. If $\langle A, B \rangle$ is nilpotent for all $A, B \in \mathcal{A}$, then $\langle \mathcal{A} \rangle$ is nilpotent.*

*Proof.* Let $\mathcal{E}$ be the set of nilpotent subgroups $B$ of $G$. Subgroups of nilpotent groups are subnormal and so $\mathcal{A}_E = \mathcal{A}_E^{\mathrm{sn}}$ for all $E \in \mathcal{E}$. max-nil holds in $G$ and so $\max -\mathcal{E}$-hold. By assumption $\langle A, B \rangle \in \mathcal{E}$ for all $A, B \in \mathcal{A}$. Also $A = \langle A, A \rangle$ is nilpotent for all $A \in \mathcal{A}$ and thus by 1.16.6 $\langle \mathcal{A}_H^{\mathrm{sn}} \rangle$ is nilpotent for all $H \leq G$. Thus the assumption of 1.16.4 are fulfilled and so $\langle \mathcal{A} \rangle \in \mathcal{E}$, that is $\langle \mathcal{A} \rangle$ is nilpotent. □

# Appendix A

# Set Theory

## A.1 Ordinals

**Definition A.1.1.** *A well ordering on set $S$ is a relation $<$ such that*

*(i) If $a, b \in S$ then exactly one of $a < b$, $a = b$ and $b < a$ holds.*

*(ii) If $a, b, c \in S$ with $a < b$ and $b < c$, then $a < c$.*

*(iii) If $T$ is a non-empty subset of $S$, then there exists $t \in T$ with $t < r$ for all $r \in T$ with $r \neq t$.*

**Definition A.1.2.** *An ordinal is a set $S$ such that*

*(i) Each element of $S$ is a subset of $S$.*

*(ii) $\in$ is a well-ordering on $S$.*

**Example A.1.3.** *The following sets are ordinals:*

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$$

**Lemma A.1.4.** *Let $\alpha$ be a ordinal.*

*(a) Define $\alpha + 1 = \alpha \cup \{\alpha\}$. Then $\alpha + 1$ is an ordinal.*

*(b) Every element of ordinal is an ordinal.*

*(c) Let $\beta$ be an ordinal, then exactly on of $\beta \in \alpha, \alpha = \beta$ and $\beta \in \alpha$ holds.*

*(d) Let $\beta, \gamma$ be ordinals with $\alpha \in \beta \in \gamma$. Then $\alpha \in \gamma$.*

*(e) Let $\beta$ an ordinal. Then $\alpha \in \beta$ if and only if $\alpha \subsetneq \beta$*

*(f) Let $A$ be a non-empty set of ordinals, then $\bigcap A$ is an ordinal. Moreover, $\bigcap A \in A$ and so $\bigcap A$ is the minimal element of $A$.*

*(g) Let $A$ be a set of ordinals. Then $\bigcup A$ is an ordinal.*

*Proof.* (a) Let $x \in \alpha + 1$. Then $x \in \alpha$ or $x = \alpha$. If $x \subseteq \alpha$ and so also $x \subseteq \alpha + 1$. Then $x = \alpha$, then again $x \subseteq \alpha$. So every element of $\alpha + 1$ is a subset of $\alpha$. Now let $y$ by any non-empty subset of $\alpha + 1$. If $y = \{\alpha\}$, then $\alpha$ is a minimal element of $y$. If $y \neq \{\alpha\}$, then $y \setminus \{\alpha\}$ is a subset of $\alpha$ and so has minimal element $m$ with respect to $\in$. Then $m \in \alpha$ and so $m$ is also a minimal element of $y$. Since $z \in \alpha$ for all $z \in \alpha + 1$ with $z \neq \alpha$ is readily verified that '$\in$' is a total ordering in $\alpha + 1$.

(b) Let $\beta \in \alpha$ and $\gamma \in \beta$. Since $\beta$ is subset of $\alpha$, $\gamma$ is an element and so also a subset of $\alpha$. If $\delta \in \gamma$, we conclude that $\delta \in \alpha$. Since $\delta \in \gamma$ and $\gamma \in \beta$ and '$\in$' is a transitive relation on $\alpha$ have that $\delta \in \beta$. Thus $\gamma$ is a subset of $\beta$. Since '$\in$' is a well-ordering on $\alpha$ and $\beta$ is a subset of $\alpha$, '$\in$' is also a well-ordering on $\alpha$.

(c) Let $\gamma \in \alpha$. By induction (on the elements of $\alpha + 1$) we may assume that $\gamma \in \beta$, $\gamma = \beta$ or $\beta \in \gamma$. If $\gamma = \beta$, then $\beta \in \alpha$. If $\beta \in \gamma$ then $\beta \in \alpha$, since $\gamma$ is a subset of $\alpha$. So we may assume that $\gamma \in \beta$ for all $\gamma \in \alpha$. Thus $\alpha \subseteq \beta$. We also may assume that $\alpha \neq \beta$ and so there exist $\delta$ minimal in $\beta$ with $\delta \notin \alpha$. Let $\eta \in \delta$. Then $\eta \in \beta$ and so $\eta \in \alpha$ by minimality of $\delta$. Thus $\delta \subseteq \alpha$. Since $\delta \notin \alpha$ and $\gamma$ is both and element of $\alpha$ and a subset of $\alpha$, $\delta \neq \gamma$ and $\delta \notin \gamma$. As both $\delta$ and $\gamma$ are in $\beta$ and '$\in$' is an ordering on $\beta$ we conclude that $\gamma \in \delta$. Thus $\alpha \subseteq \delta$ and so $\alpha = \delta \in \beta$.

(d) This follows since $\beta$ is a subset of $\gamma$

(e) If $\alpha \in \beta$, then $\alpha \subseteq \beta$. Since $\in$ is ordering on $A$ and $\alpha = \alpha$ we have $\alpha \notin \alpha$ and so $\alpha \neq \beta$ and $\alpha \subsetneq \beta$.

Suppose now that $\alpha \subsetneq \beta$. Then $\alpha \neq \beta$. If $\beta \in \alpha$, then $\beta \subseteq \alpha$ and so $\alpha = \beta$. So $\beta \notin \alpha$ and by (c), $\alpha \in \beta$.

(f) Any subset of a well-ordered set is well-ordered. So $\bigcap A$ is well-ordered with respect to '$\in$. Let $x \in \bigcap A$. Then $x \in a$ for all $a \in A$ and so $x \subseteq a$ for all $a \in A$. Hence $x \subseteq \bigcup A$. Thus $\bigcup A$ is an ordinal. If $\bigcap A \neq a$ for all $a \in A$, then $\bigcap A \subsetneq a$ and by (e), $\bigcap A \in a$ for all $a \in A$. Hence $\bigcap A \in \bigcap A$, a contradiction to (e).

(g) Let $x_1, x_2, x_3 \in \bigcup A$. Then $x_i \in a_i$ for some $a_i \in A$. Then $x_i \subseteq a_i$ and so $x_i \subseteq A$. By (c) and (d) there exists $a \in \{a_1, a_2, a_3\}$ with $a_i \leq a$ for all $a$. Thus $x_1, x_2, x_3 \in a$. Since '$\in$' is an ordering on $a$ we conclude that '$\in$' is also an ordering on $\bigcup A$. Let $d$ be a non-empty subset of $\bigcup A$ and define $B = \{a \in A \mid d \cap a \neq \emptyset\}t$. By (f), $B$ has a minimal element $b$. Then $b \cap d$ has a minimal element $m$ and $m$ is also a minimal element of $d$. Thus '$\in$ is a well-ordering in $\bigcup A$. □

**Definition A.1.5.** *Let $\alpha$ and $\beta$ be ordinals. Define the ordinal $\alpha + \beta$ inductively via*

$$\alpha + \beta = \begin{cases} \alpha & \text{if } \alpha = 0 \\ (\alpha + \gamma) + 1 & \text{if } \beta = \gamma + 1 \text{ for some ordinal } \gamma \\ \sup_{\gamma < \beta} \alpha + \gamma & \text{otherwise} \end{cases}$$

Let $\alpha_0$ be the smallest limit ordinal. Then $1 + \alpha_0 = \alpha_0 \neq \alpha_0 + 1$. So the addition on ordinals is not commutative.

**Lemma A.1.6.** *Let $\alpha$, $\beta$ and $\gamma$ be ordinals. Then*

*(a) $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.*

*(b) $\alpha + \beta = \alpha + \gamma$ if and only if $\beta = \gamma$.*

*Proof.* Suppose first that $\beta < \gamma$. If $\gamma = \delta + 1$ for some ordinal $\delta$, then $\beta \leq \delta$ and so by induction $\alpha+\beta \leq \alpha+\delta$. Since $(\alpha+\gamma) = (\alpha+(\delta)+1$ and $\alpha+\gamma < (\alpha+\gamma)+1$ we conclude that $\alpha + \beta < \alpha + \gamma$. So suppose $\gamma$ is a limit ordinal. Then $\beta + 1 < \gamma$ and so $\alpha + (\beta + 1) \leq \alpha + \gamma$ by the definition of addition. Since $\alpha + \beta < \alpha + (\beta + 1)$ we again have $\alpha + \beta < \alpha + \gamma$.

In general exactly one of

$$\beta < \gamma, \qquad \beta = \gamma, \qquad \text{and } \gamma < \beta$$

In this cases we conclude

$$\alpha + \beta < \alpha + \gamma, \qquad \alpha + \beta = \alpha + \gamma, \qquad \text{and } \alpha + \gamma < \alpha + \beta,$$

respectively and so (a) and (b) holds. □

**Lemma A.1.7.** *(a) Let $\alpha$ and $\beta$ be ordinals with $\alpha \leq \beta$, then there exists a unique ordinal $\delta$ with $\alpha + \delta = \beta$.*

*(b) Let $\alpha$ and $\beta$ be ordinals. Then*

$$\{\alpha + \gamma \mid \gamma < \beta\} = \{\rho \mid \alpha \leq \rho < \alpha + \beta\}$$

*Proof.* (a) The uniqueness follows from A.1.6(b). So it suffices to find an ordinal $\delta$ with $\alpha + \delta = \beta$. If $\beta = \alpha$, we can choose $\delta = 0$. Inductively if $\alpha \leq \gamma < \beta$ let $\gamma^*$ be the unique ordinal with $\alpha + \gamma^* = \beta$. If $\beta = \gamma + 1$ for some ordinal $\gamma$, then

$$\alpha + (\gamma^* + 1) = (\alpha + \gamma^*) + 1 = \gamma + 1 = \beta$$

and we can choose $\delta = \gamma^* + 1$.

So suppose $\beta$ is limit ordinal and put $\delta = \sup_{\alpha\leq\gamma<\beta} \gamma^*\}$. Note that $\Gamma := \{\gamma \mid \alpha \leq \gamma < \beta\}$ has no maximal element and so by A.1.6(a), also $\{\gamma^* \mid \gamma \in \Gamma\}$ has no maximal element. Thus $\delta$ is a limit ordinal. Let $\mu$ be an ordinal with $\mu < \delta$. Then $\mu \leq \gamma^*$ for some ordinal $\gamma \in \Gamma$. Thus $\rho := \alpha + \mu \leq \alpha + \gamma^* < \beta$. It follows that $\alpha + \mu = \rho = \alpha + \rho^*$ and so $mu = \rho^*$. Thus $\{\mu \mid \mu < \beta^*\} = \{\rho^* \mid \rho \in \Gamma$. Hence

$$\alpha + \delta = \sup_{\mu<\delta} \alpha + \mu = \sup_{\rho\in\Gamma^*} \alpha + \rho^* = \sup_{\rho\in\Gamma} \rho = \beta$$

where the last inequality holds, since $\beta$ is a limit ordinal.

(b) If $\gamma < \beta$, then by A.1.6, $\alpha + \gamma < \alpha + \beta$.

Conversely, if $\alpha \leq \rho < \alpha + \beta$, then by (a), $\rho = \alpha + \gamma$ for some ordinal $\gamma$. Since $\alpha + \gamma = \rho < \alpha + \beta$, A.1.6 gives $\rho < \beta$. □

**Lemma A.1.8.** *Let $\alpha, \beta, \gamma$ be ordinals. Then*

$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$$

*Proof.* If $\gamma = 0$, both sides are equal to $\alpha+\beta$. So suppose $\gamma \neq 0$ and that $(\alpha+\beta)+\delta = \alpha+(\beta+\delta)$ for all ordinals $\delta < \gamma$.

Suppose that $\beta = \delta+1$ for some ordinal $\delta$. Then

$$\begin{aligned}(\alpha+\beta)+\gamma &= (\alpha+\beta)+(\delta+1)\\ &= ((\alpha+\beta)+\delta)+1\\ &= (\alpha+(\beta+\delta))+1\\ &= \alpha+((\beta+\delta)+1)\\ &= \alpha+(\beta+(\delta+1))\\ &= \alpha+(\beta+\gamma)\end{aligned}$$

Suppose next that $\gamma$ is a limit ordinal. Then

$$\begin{aligned}(\alpha+\beta)+\gamma &= \sup_{\delta<\gamma}(\alpha+\beta)+\delta\\ &= \sup_{\delta<\gamma}\alpha+(\beta+\delta)\\ &= \sup_{\beta\leq\rho<\beta+\gamma}\alpha+\rho\\ &= \sup_{\rho<\beta+\gamma}\alpha+\rho\\ &= \alpha+(\beta+\gamma)\end{aligned}$$

□

**Definition A.1.9.** *Let $\alpha$ and $\beta$ be ordinals. Then the ordinal $\alpha\beta$ is inductively defined as follows*

$$\alpha\beta = \begin{cases} 0 & \text{if } \beta = 0\\ \alpha\gamma+\alpha & \text{if } \beta = \gamma+1\\ \sup_{\gamma<\beta}\alpha\gamma & \text{if } \beta \text{ is a limit ordinal}\end{cases}$$

Observe that $0\alpha = 0$, $1\alpha = \alpha = \alpha 1$ and $\alpha 2 = \alpha+\alpha$. But $2\alpha = \alpha \neq \alpha+\alpha = 1\alpha+1\alpha$ for any infinite ordinal. So multiplication of ordinals is not commutative and not left distributative.

**Lemma A.1.10.** *Let $\alpha$, $\sigma$, $\tilde{\sigma}$, $\rho, \tilde{\rho}$ be ordinals with $\alpha \neq 0$, $\rho < \alpha$, and $\tilde{\rho} < \alpha$. Then $\alpha\sigma+\rho < \alpha\tilde{\sigma}+\tilde{\rho}$ if and only of $\sigma < \tilde{\sigma}$ or $\sigma = \tilde{\sigma}$ and $\rho < \tilde{\rho}$. Let $\alpha, \beta$ and $\gamma$ be ordinals with $\alpha \neq 0$ and $\beta < \gamma$. Then $\alpha\beta < \alpha\gamma$.*

*Proof.* $\Longrightarrow$: If $\beta \leq \gamma$ the definition of the multiplication of ordinals shows that $\alpha\beta \leq \alpha\gamma$.

Suppose $\sigma < \tilde{\sigma}$, then $\sigma + 1 \leq \tilde{\sigma}$ and so

$$\alpha\sigma + \rho < \alpha\sigma + \alpha = \alpha(\sigma + 1) \leq \alpha\tilde{\sigma} \leq \alpha\tilde{\sigma} + \tilde{\rho}.$$

Suppose $\sigma = \tilde{\sigma}$ and $\rho < \tilde{\rho}$. Then

$$\alpha\sigma + \rho = \alpha\tilde{\sigma} + \rho < \alpha\tilde{\sigma} + \tilde{\rho}.$$

$\Longleftarrow$: Suppose that $\alpha\sigma + \rho < \tilde{<}\alpha\tilde{\sigma} + \tilde{\rho}$. By the forward direction with the roles of $(\sigma, \rho)$ and $(\tilde{\sigma}, \tilde{\rho})$ transposed we neither have $\tilde{\sigma} < \sigma$ nor $\tilde{\sigma} = \sigma$ and $\tilde{\rho} < \rho$. Since $(\rho, \sigma) \neq (\tilde{\rho}, \tilde{\sigma})$ we conclude that either $\sigma < \tilde{s}igma$ or $\sigma - \tilde{\sigma}$ and $\rho < \tilde{\rho}$. ☐

**Lemma A.1.11.** *Let $\alpha$ and $\beta$ be ordinals with $\alpha \neq 0$. Then there exists unique ordinals $\sigma, \rho$ with $\beta = \alpha\sigma + \rho$ and $\rho < \beta$. Moreover, if $\tilde{\sigma}$ and $\tilde{\gamma}$ are ordinals with $\tilde{\rho} < \alpha$. Then $\alpha\sigma + \rho < \alpha\tilde{\sigma} + \tilde{\rho}$ if and only of $\sigma < \tilde{\sigma}$ or $\sigma = \tilde{\sigma}$ and $\rho < \tilde{\rho}$.*

*Proof.* Note that the uniqueness assertion follows from A.1.10. So we just need to proof the existence of $\sigma$ and $\rho$. We use induction on $\beta$. If $\beta = 0$, choose $\sigma = \rho = 0$.

If $\beta = \gamma + 1$, let $\gamma = \alpha\hat{\sigma} + \hat{\rho}$ with $\hat{\rho} < \alpha$. If $\hat{\rho} + 1 < \alpha$ we can choose $\sigma = \hat{\sigma}$ and $\rho = \hat{\rho} + 1$. If $\hat{\rho} + 1 = \alpha$, then

$$\beta = \gamma + 1 = \alpha\hat{\sigma} + \hat{\rho} + 1 = \alpha\hat{\sigma} + \alpha = \alpha(\hat{\sigma} + 1)$$

So we can choose $\sigma = \hat{\sigma} + 1$ and $\rho = 0$.

Suppose now that $\beta$ is a limit ordinal.

For $\delta < \beta$, let $\delta = \alpha\sigma_\delta + \rho_\delta$ with $\rho_\delta < \alpha$. Put

$$\hat{\sigma} = \sup_{\delta < \beta} \sigma_\delta.$$

Suppose that $\hat{\sigma} \neq \sigma_\delta$ for all $\delta < \beta$. Then by A.1.10, $\delta < \alpha\hat{r}ho$ for all $\delta < \beta$ and so $\beta \leq \alpha\hat{\sigma}$. Also $\hat{\sigma}$ is a limit ordinal and if $\epsilon < \hat{\sigma}$, then $\epsilon < \sigma_\delta$ for some $\delta < \beta$. Then $\alpha\epsilon \leq \alpha\sigma_\delta + \rho_\delta = \delta < \beta$ and so by definition of $\alpha\hat{\sigma}$,

$$\alpha\hat{\sigma} = \sup_{\epsilon < \hat{\sigma}} \alpha\epsilon \leq \beta$$

Thus $\beta = \alpha\hat{\sigma}$ and we can choose $\sigma = \hat{\sigma}$ and $\rho = 0$.

Suppose that $\hat{\sigma} = \sigma_\delta$ for some $\delta < \beta$ and let $\Delta = \{\delta < \beta \mid \sigma_\delta = \hat{\sigma}\}$. Put

$$\hat{\rho} = \sup_{\delta \in \Delta} \sigma_\delta.$$

By A.1.10, if $\delta < \beta$ with $\sigma_\delta < \hat{\sigma}$, then $\delta = \alpha\sigma_\delta < \alpha\hat{\sigma} \leq +\alpha\hat{\sigma} + \hat{\rho}$. It follows that

$$\beta = \sup_{\delta < \beta} \delta = \sup_{\delta \in \Delta} \delta = \sup_{\delta \in \Delta} \alpha\hat{\sigma} + \rho_\delta = \alpha\hat{\sigma} + \sup_{\delta \in \Delta} \rho_\delta = \alpha\hat{\sigma} + \rho_\delta$$

Since $\rho_\delta < \alpha$ for all $\delta \in \Delta$, $\hat{\rho} \leq \alpha$. If $\hat{\rho} < \alpha$, choose $\sigma = \hat{\sigma}$ and $\hat{\rho} = \rho$. If $\rho = \alpha$ choose $\sigma = \hat{\sigma} + 1$ and $\rho = 0$. □

**Lemma A.1.12.** *Let $\alpha$ be an ordinal and $\Delta$ a non-empty set of ordinals. Then*

$$\alpha \left( \sup_{\delta \in \Delta} \delta \right) = \sup_{\delta \in \Delta} \alpha\delta$$

*Proof.* Let $\beta = \sup_{\delta \in \Delta} \delta$ and $\gamma = \sup_{\delta \in \Delta} \alpha\delta$. Let $\delta \in \Delta$. Then $\delta \leq \beta$. and so also $\alpha\delta \leq \alpha\beta$. Hence $\gamma \leq \alpha\beta$.

If $\alpha = 0$, both $\gamma$ and $\alpha\beta$ are equal to zero. So suppose $\alpha \neq 0$. Then by A.1.11 we have $\gamma = \alpha\sigma + \rho$ for some ordinals $\sigma$, $\rho$ with $\rho < \alpha$. Since $\alpha\delta \leq \gamma$ we get from A.1.10 that $\delta \leq \sigma$ and so $\beta \leq \sigma$. Thus $\alpha\beta \leq \alpha\sigma + \rho = \gamma$ and so $\gamma = \alpha\beta$. □

**Lemma A.1.13.** *Let $\alpha$, $\beta$ and $\gamma$ be ordinals. Then*

*(a)* $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

*(b)* $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

*Proof.* (a) If $\gamma = 0$ both sides are equal to $\alpha\beta$. Suppose $\gamma = \delta + 1$. Then

$$\begin{aligned} \alpha(\beta + \gamma) &= \alpha(\beta + (\delta + 1)) = \alpha((\beta + \delta) + 1) = \alpha(\beta + \delta) + \alpha = (\alpha\beta + \alpha\delta) + \alpha \\ &= \alpha\beta + (\alpha\delta + \alpha) = \alpha\beta + \alpha(\delta + 1) = \alpha\beta + \alpha\gamma. \end{aligned}$$

Suppose that $\gamma$ is a limit ordinal. Then also $\beta + \gamma$ is limit ordinal. So

$$\alpha(\beta + \gamma) = \sup_{\delta < \beta + \gamma} \alpha\delta = \sup_{\epsilon < \gamma} \alpha(\beta + \epsilon) = \sup_{\epsilon < \gamma} \alpha\beta + \alpha\epsilon = \alpha\beta + \sup_{\epsilon < \gamma} \alpha\epsilon = \alpha\beta + \alpha\gamma$$

(b) For $\gamma = 0$ both sides are equal to 0. Suppose $\gamma = \delta + 1$. Then

$$(\alpha\beta)\gamma = (\alpha\beta)(\delta + 1) = (\alpha\beta)\delta + \alpha\beta = \alpha(\beta\delta) + \alpha\beta = \alpha(\beta\delta + \beta) = \alpha(\beta(\delta + 1)) = \alpha(\beta\gamma)$$

Suppose $\gamma$ is a limit ordinal. Hence using A.1.12

$$(\alpha\beta)\gamma = \sup_{\delta < \gamma} (\alpha\beta)\delta = \sup_{\delta < \gamma} \alpha(\beta\delta) = \alpha \left( \sup_{\delta < \gamma} \beta\delta \right) = \alpha(\beta\gamma)$$

□

**Lemma A.1.14.** *Let $\alpha, \beta$ be ordinals with $\alpha \neq 0 \neq \beta$. Define an ordering on $\alpha \times \beta$ by $(\rho, \sigma) < (\tilde{\rho}, \tilde{\sigma})$ if $\sigma < \tilde{\sigma}$ or $\sigma = \tilde{\sigma}$ and $\rho < \tilde{\rho}$. Define*

$$\begin{array}{rrcl} f: & \alpha \times \beta & \to & \alpha\beta \\ & (\rho, \sigma) & \to & \alpha\sigma + \rho \end{array}$$

*Then $f$ is an isomorphism of order sets.*

*Proof.* This follows immediately from A.1.10 and A.1.11 □

# Bibliography

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