Locally Finite Simple Groups of 1-Type

Stefaan Delcroix
University of Wyoming
Laramie, WY 82071
delcroix@uwyo.edu

Ulrich Meierfrankenfeld
Michigan State University
East Lansing, MI 48824
meier@math.msu.edu

Abstract

A locally finite, simple group G is called of 1-type if every Kegel cover for G has a factor which is an alternating group. In this paper we study the finite subgroups of locally finite simple groups of 1-type. We also introduce the concept of "block-diagonal embeddings" for groups of alternating type. We show that the groups of 1-type are exactly the groups which have an alternating Kegel cover with block diagonal embeddings.

1 Introduction

Let G be a group. G is locally finite if every finite subset of G lies in a finite subgroup of G. G is finitary if there exist a field $\mathbb{K}$ and a faithful $\mathbb{K}G$-module V so that $V/C_V(g)$ is finite dimensional for all $g \in G$.

If H is a group and Ω is an H-set, we denote by H^Ω the image of H in $\mathrm{Sym}(\Omega)$. So $H^\Omega \cong H/C_H(\Omega)$.

Let G be an infinite, locally finite, simple group. Let $\mathcal{A}$ be the set of pairs (H, Ω) so that H is a finite subgroup of G, Ω is an H-set, $|\Omega| \geq 7$ and $H^\Omega = \mathrm{Alt}(\Omega)$.

We say that G is of alternating type if G is non-finitary and if for each finite subgroup F of G there exists $(H, \Omega) \in \mathcal{A}$ such that $F \leq H$ and F acts faithfully on Ω.

Let G be of alternating type and $F \leq G$ finite. We say that F is non-regular if there exists a finite subgroup $F^* \leq G$ with $F \leq F^*$ and so that for all $(H, \Omega) \in \mathcal{A}$ with $F^* \leq H$, F has no regular orbit on Ω.

G is called of non-regular alternating type if G is of alternating type and G has a non-regular finite subgroup.

Our first theorem (proven in section **??**) describes the normal closure of a non-regular subgroup in (large enough) finite over-groups.

Theorem 1.1 *Let G be a locally finite simple group of alternating type and F a finite non-regular subgroup of G. Then there exists a finite subgroup $F \leq F^* \leq G$ such that for all finite $F^* \leq L \leq G$*

(a) *There exist normal subgroups $R_1, \dots, R_n$ of $\langle F^L \rangle$ such that*

$$\langle F^L \rangle = R_1 R_2 \dots R_n$$

and

$$R_i \cong (K_i \wr_{\Omega_i} \mathrm{Alt}(\Omega_i))'$$

for some finite group K_i and some finite set Ω_i.

(b) *For $i = 1, \dots, n$ let B_i be the base group of R_i and choose notation so that $[R_i, F] \not\leq B_i$ if and only if $i \leq m$. Then*

$$R_1 \dots R_m = R_1 \times R_2 \times \dots \times R_m.$$

□

Recall that a Kegel cover for G is a set $\mathcal{K}$ such that

(a) Each $K \in \mathcal{K}$ is a pair (H, M), where H is a finite subgroup of G and M is maximal normal subgroup of H.

(b) For each finite subgroup F of G there exists $(H, M) \in \mathcal{K}$ with $F \leq H$ and $F \cap M = 1$.

The groups H/M, $(H, M) \in \mathcal{K}$, are called the factors of $\mathcal{K}$. $\mathcal{K}$ is alternating if all the factors of $\mathcal{K}$ are alternating groups. If $\mathcal{K}$ is an alternating Kegel cover, we view $\mathcal{K}$ as a subset of $\mathcal{A}$. Indeed, if $(H, M) \in \mathcal{K}$ with $H/M \cong \mathrm{Alt}(\Omega)$, then H acts on Ω (with $M = C_H(\Omega)$) and $(H, \Omega) \in \mathcal{A}$. This also reveals that a non-finitary locally finite simple group G is of alternating type if and only if G has an alternating Kegel cover.

Our next theorem (proven in section **??**) shows that non-regular subgroups can be detected from a given alternating Kegel cover.

Theorem 1.2 *Let G be a locally finite, simple group of alternating type and F a finite subgroup of G. Then F is non-regular if and only if there exists an alternating Kegel cover $\mathcal{K}$ and a non-negative integer t such that for all $(H, \Omega) \in \mathcal{K}$ with $F \leq H$, F has at most t regular orbits on Ω.* □

The preceding theorem, together with [**?**, Proposition 1.33], shows that the groups Brian Hartley called Mf-groups of 'visual diagonal alternating type' [**?**, Definition 1.31], are in fact of non-regular alternating type. Hence (see section **??** for the details) some of the non-absolutely simple, locally finite simple groups constructed in [**?**, Section 6] are of non-regular alternating type :

Theorem 1.3 *There exist non-absolutely simple, locally finite, simple groups of non-regular alternating type.* □

Let G be of alternating type

Let $F \leq G$ be finite. Let $\mathcal{A}_{reg}(F)$ be the set of all $(H, \Omega) \in \mathcal{A}$ so that $F \leq H$ and F has a regular orbit on Ω. We say that F is regular if $\mathcal{A}_{reg}(F)$ is a Kegel cover for G. Note that the definition of a Kegel cover implies that F is non-regular if and only if F is not regular. G is of regular alternating type if G is locally regular, that is if every finite subgroup of G is regular.

We say that G is of ∞-type if G has the following property :

> Let $\mathcal{S}$ be any class of finite simple groups such that every finite group can be embedded into a member of $\mathcal{S}$. Then there exists a Kegel cover for G all of whose factors are isomorphic to a member of $\mathcal{S}$.

We say that G is of 1-type if every Kegel cover for G has a factor which is an alternating group.

The next theorem (proven in section **??**) shows the relationship between groups of 1-, ∞-, regular- and non-regular type.

Theorem 1.4 *Let G be a locally finite, simple group of alternating type.*

(a) *G is of non-regular alternating type if and only if G is of 1-type.*

(b) *G is of regular alternating type if and only if G is of ∞-type.*

□

Let p be a prime and G a non-finitary, locally finite, simple group. G is of p-type if every Kegel cover for G has a factor which is a classical group in characteristic p. From Theorem **??** and [**?**, Theorem A] we have

Theorem 1.5 *Let G be a locally finite, simple group. Then exactly one of the following holds:*

1. *G is finitary.*
2. *G is of 1-type.*
3. *G is of p-type for a unique prime p.*
4. *G is of ∞-type.* □

In [?] "pseudo natural orbits" have been introduced. They are used in [?, Theorem 3.4] to devide alternating Kegel covers into two classes which Brian Hartley [?, Defintion 2.8] called RA- and DA- type. Unfortunately these two types are not disjoint. For example suppose $\{(G_i, \Omega_i) \mid i = 1, 2 \ldots\}$ is a Kegel cover so that $G_i = \mathrm{Alt}(\Omega_i)$, $G_i \leq G_{i+1}$ and G_i acts semiregulary on Ω_i, then this Kegel cover is both of RA and DA type. This comes from the fact that a regular orbit also is a pseudo natural orbit. In this paper we define "block natural orbits" which avoid this problem:

Let $(H, \Omega) \in \mathcal{A}$. By [?, Lemma 2.8], there exists a unique minimal (sub)normal supplement R to $C_H(\Omega)$ in H. Let Λ be an H-set. An orbit Σ for H on Λ is called Ω-essential if $C_H(\Sigma) \leq C_H(\Omega)$. That is if and only if R acts non-trivially on Σ. Σ is called Ω-natural if Σ is isomorphic to Ω as an H-set. Σ is called Ω-block-natural if there exists an H-invariant partition Δ of Σ so that Δ is Ω-natural and such that $N_H(D) = C_H(D)C_H(\Omega)$ for all $D \in \Delta$. In this case, Δ is just the set of orbits of $\mathrm{C}_H(\Omega)$ on Σ. Indeed, since H is transitive on Σ, $N_H(D)$ is transitive on D. Hence $C_H(\Omega)$ is transitive on D. We remark that, since $N_H(D)/C_H(\Omega) \cong \mathrm{Alt}(|\Omega| - 1)$ is simple, the condition $N_H(D) = C_H(D)C_H(\Omega)$ is equivalent to $C_H(D) \not\leq C_H(\Omega)$. Λ is called Ω-block-diagonal if all the Ω-essential orbits are Ω-block-natural.

Theorems **??** and **??** reveal that groups of 1-type are loosely speaking the groups of alternating type with "block-diagonal" embeddings.

Some of the results in this paper first appeared in [?] and some of the arguments have been developed in [?].

2 The Set-up

Proposition 2.1 (Hall's Finitary Lemma) *A locally finite simple group G which has a sectional cover composed of alternating groups and classical groups of unbounded dimension in which the natural degrees of the element $g \neq 1$ are bounded, has a faithful representation as a finitary linear group.*

Proof: This is [?, Corollary 3.13]. □

The reader might consult [?] for the definition of a sectional cover. For our purposes it is enough to know that every Kegel cover is a sectional cover. If H/M is a classical group or an alternating group, $\mathrm{pdeg}_{H/M}(g)$ denotes the natural degree of g in H/M. So if $H/M = \mathrm{Alt}(\Omega)$, then $\mathrm{pdeg}_{H/M}(g) = \deg_\Omega(g)$ is the number of elements in Ω not fixed by g; if H/M is a classical group defined over a $\mathbb{K}$-space V, then $\mathrm{pdeg}_H(g)$ is the minimum of all $\dim_{\mathbb{K}} V/W$, where W is a $\mathbb{K}$-subspace of V on which g acts projectively trivially. If $g \notin H$, we put $\mathrm{pdeg}_{H/M}(g) = 0$.

Corollary 2.2 *Let G be a non-finitary, locally finite, simple group and F a finite subgroup of G. Let $\mathcal{K}$ be a Kegel cover for G all of whose factors are alternating or classical groups.*

Let s be a positive integer. Then

$$\mathcal{K}(F,s) := \{(H,M) \in \mathcal{K} \mid F \leq H \text{ and } \operatorname{pdeg}_{H/M}(f) \geq s,\ \forall 1 \neq f \in F\}$$

is a Kegel cover for G.

Proof: For $1 \neq f \in F$, let $\mathcal{K}_f = \{(H,M) \in \mathcal{K} \mid \operatorname{pdeg}_{H/M}(f) \leq s\}$. Suppose that $\mathcal{K}_f$ is Kegel cover for G. Then by Hall's Finitary Lemma applied to the sectional cover $\mathcal{K}_f$, G is finitary, a contradiction. So $\mathcal{K}_f$ is not a Kegel cover for G. Since

$$\mathcal{K} = \mathcal{K}(F,s) \cup \bigcup_{1 \neq f \in F} \mathcal{K}_f,$$

the Coloring Argument [**?**, Lemma 3.3] implies that $\mathcal{K}(F,s)$ is a Kegel cover for G. □

Let $\mathcal{D}$ be a subset of $\mathcal{A}$, F a finite subgroup of G and s a positive integer. Define

$$\mathcal{D}(F,s) = \{(H,\Omega) \in \mathcal{D} \mid F \leq H \text{ and } \deg_\Omega(f) \geq s, \forall 1 \neq f \in F\}$$

and

$$\mathcal{D}(F) = \mathcal{D}(F,1).$$

Lemma 2.3 *Let G be a locally finite, simple group of alternating type, $F \leq G$ finite and s a positive integer. Then there exists a finite $F \leq F^* \leq G$ so that $\mathcal{A}(F^*) \subseteq \mathcal{A}(F,s)$.*

Proof: Let l be the function from [**?**, Lemma 2.5]. By **??**, we can choose $(F^*, \Lambda) \in \mathcal{A}(F, l(s))$. Let $(H,\Omega) \in \mathcal{A}(F^*)$. Then by [**?**, Lemma 2.5], $\deg_\Omega(f) \geq s$ for all elements f of prime order in F. Since every non-trivial cyclic group contains an element of prime order, $\deg_\Omega(f) \geq s$ for all $1 \neq f \in F$. □

For the remainder of the paper, let G be locally finite, simple group of alternating type.

The following result forms the technical basis for the investigations in this paper.

Lemma 2.4 *There exists an increasing function $f_{reg} : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $f_{reg}(n) \geq 9n^2$ and so that the following statement holds :*

Let F be a finite subgroup of G and $(H,\Omega) \in \mathcal{A}(F, f_{reg}(|F|))$. Suppose that H is transitive and Ω-essential on a set Λ. Then one of the following holds :

1. *F has a regular orbit on Λ.*
2. *There exist $1 \leq t \leq |F| - 2$ and an H-invariant partition Δ of Λ so that the action of H on Δ is isomorphic to the action of H on the subsets of Ω of size t.*

Proof: See [?, Lemma 2.14]. □

For $A \in \mathcal{A}$, we define H_A and Ω_A by $A = (H_A, \Omega_A)$. Let $\mathcal{D} \subseteq \mathcal{A}$. We say that $\mathcal{D}$ is Kegel cover for G if $\{(H, C_H(\Omega)) \mid (H, \Omega) \in \mathcal{D}\}$ is Kegel cover for G.

Let $\mathcal{D} \subseteq \mathcal{A}$ be a Kegel cover for G. Let F be a finite subgroup of G. Put

$$\mathcal{D}^*(F) = \mathcal{D}(F, f_{reg}(|F|).$$

We say that F is $\mathcal{D}$-regular if there exists $D \in \mathcal{D}^*(F)$ so that F has a regular orbit on Ω_D. In other words, F is $\mathcal{D}$-regular if and only if $\mathcal{D}^*(F) \cap \mathcal{A}_{reg}(F) \neq \emptyset$. We will prove in Theorem **??** that F is $\mathcal{D}$-regular if and only if F is regular.

3 Block-Diagonality in Groups of Non-Regular Alternating Type

We continue to use the notation introduced in the previuos section. In particular, G is a locally finite, simple group of alternating type.

Proposition 3.1 *Let F be a finite subgroup of G, $(H, \Omega) \in \mathcal{A}^*(F) \cap \mathcal{A}_{reg}(F)$ and Σ an H-set. Then F has a regular orbit on each Ω-essential orbit for H on Σ. In particular, $\mathcal{A}(H) \subseteq \mathcal{A}_{reg}(F)$.*

Proof: Let Λ be an Ω-essential orbit for H on Σ. We need to show that F has a regular orbit on Λ. So we may assume that (2) in **??** holds. Since F has a regular orbit on Ω, there exists $\omega \in \Omega$ with $C_F(\omega) = 1$. Since $|\Omega| \geq f_{reg}(|F|) \geq 9|F|^2 \geq 2|F|$, there exists a subset U of Ω of size t with $U \cap \omega^F = \{\omega\}$. Then $N_F(U) \leq C_F(\omega) = 1$ and F has a regular orbit on Δ. Hence F has a regular orbit on Λ. □

Proposition 3.2 *Let F be a finite subgroup of G, $(H, \Omega) \in \mathcal{A}^*(F)$ and $A \in \mathcal{A}$ with $H \leq H_A$. Suppose that Λ is an Ω-essential orbit for H on Ω_A and $\omega \in \Omega$ such that*

(i) *F has no regular orbit on Λ.*

(ii) *There does not exist $\lambda \in \Lambda$ with $C_F(\lambda) \lneq C_F(\omega)$.*

Then

(a) *There exists an H-invariant partition Δ of Λ so that $\Delta \cong \Omega$ as an H-set.*

(b) *If $\breve{\omega}$ is the element in Δ corresponding to ω, then $\mathrm{C}_F(\omega) = N_F(\breve{\omega}) = C_F(\breve{\omega}) = C_F(\lambda)$ for all $\lambda \in \breve{\omega}$.*

Proof: Note that all the assumptions of **??** are fullfilled. By (i), **??**(1) does not hold. So we can choose t and Δ as in **??**(2).

Let $1 \neq f \in C_F(\omega)$. Suppose that $t \neq 1$. Since $\deg_\Omega(f) \geq 9|F|^2 \geq 2|F|$, there exists $\rho \in \Omega$ with $\rho \notin \omega^F$ and $\rho \neq \rho^f$. Since $t \leq |F| - 2$, there exists a subset U of Ω of size t with $\rho \in U, \rho^f \notin U$ and $U \cap \omega^F = \{\omega\}$. Then $N_F(U) \leq C_F(\omega)$ and $f \notin N_G(U)$. Thus $N_F(U) \lneq C_F(\omega)$.

Let $\delta \in \Delta$ so that δ corresponds to U. Note that δ is a subset of Λ and pick $\lambda \in \delta$. Then

$$C_F(\lambda) \leq N_F(\delta) = N_F(U) \lneq C_F(\omega),$$

a contradiction to the assumptions.

Thus $t = 1$ and (a) holds. For (b), pick $\lambda \in \breve{\omega}$ and note that

$$C_F(\lambda) \leq N_F(\breve{\omega}) = C_F(\omega).$$

This implies that $C_F(\lambda) = C_F(\omega) = N_F(\breve{\omega}) = C_F(\breve{\omega})$. So (b) holds. □

Theorem 3.3 *Let G be a locally finite, simple group of alternating type and F a finite subgroup of G. Then the following are equivalent :*

1. *F is not $\mathcal{A}$-regular.*
2. *F is not $\mathcal{D}$-regular for some alternating Kegel cover $\mathcal{D}$ for G.*
3. *There exists an alternating Kegel cover $\mathcal{D}$ for G and a non-negative integer t such that for all $A \in \mathcal{D}(F)$, F has at most t regular orbits on Ω_A.*
4. *F is non-regular.*

Proof: Clearly (1) implies (2).

Suppose (2) holds. By **??**, $\mathcal{D}^*(F)$ is a Kegel cover for G. Hence (3) holds with $t = 0$ and $\mathcal{D}^*(F)$ in place of $\mathcal{D}$.

Suppose (3) holds but (4) does not. Then F is regular and so $\mathcal{A}_{reg}(F)$ is a Kegel cover for G. By **??**, there exists $(H, \Omega) \in \mathcal{A}_{reg}(F) \cap \mathcal{A}^*(F)$. Let $A \in \mathcal{D}(H)$. By assumption, F has at most t regular orbits on Ω_A. Hence by **??**, H has at most t Ω-essential orbits on Ω_A. Let R be the minimal normal supplement to $C_H(\Omega)$ in H. As each Ω-essential H-orbit has size at most $|H|$ and since R acts trivially on the non-Ω-essential orbits, $\deg_{\Omega_A}(x) \leq t|H|$ for all $1 \neq x \in R$. Hence by Hall's Finitary Lemma **??**, G is finitary, a contradiction. So (3) implies (4).

Suppose finally that (4) holds but (1) does not. Then there exists $(H, \Omega) \in \mathcal{A}_{reg}(F) \cap \mathcal{A}^*(F)$. By **??**, $\mathcal{A}(H) \subseteq \mathcal{A}_{reg}(F)$. As $\mathcal{A}(H)$ is a Kegel cover for G, so is $\mathcal{A}_{reg}(F)$. So F is regular, a contradiction. □

Let F be a finite, non-regular subgroup of G. Let $\mathcal{M}_F$ be the set of all $E \leq F$ so that $E = C_F(\omega)$ for some $(H, \Omega) \in \mathcal{A}^*(F)$ and $\omega \in \Omega$. Note that $E \neq 1$ for all $E \in \mathcal{M}_F$. Let

$\mathcal{M}_F^*$ be the set of minimal elements of $\mathcal{M}_F$. We say that ω is F-extreme if $C_F(\omega) \in \mathcal{M}_F^*$. Let $\mathcal{B}_F$ be the set of $(H, \Omega) \in \mathcal{A}^*(F)$ so that there exists an F-extreme $\omega \in \Omega$. Let $\mathcal{B}$ be the union of the $\mathcal{B}_F$'s as F runs through the non-regular finite subgroups of G.

Theorem 3.4 *Let G be a locally finite, simple group of alternating type and F a finite non-regular subgroup of G. Then the following holds :*

(a) *Let $(H, \Omega) \in \mathcal{B}_F$ and $A \in \mathcal{A}^*(F)$ with $H \leq H_A$ and $C_H(\Omega_A) \leq C_H(\Omega)$. Then $A \in \mathcal{B}_F$ and H is Ω-block-diagonal on Ω_A.*

(b) *Let $(H, \Omega) \in \mathcal{B}_F$. Then $\mathcal{A}(H) \cap \mathcal{A}^*(F) \subseteq \mathcal{B}_F$.*

(c) *Let $A, B \in \mathcal{B}_F$ with $H_A \leq H_B$. Then H_A is Ω_A-block-diagonal on Ω_B.*

(d) *Both $\mathcal{B}_F$ and $\mathcal{B}$ are Kegel covers for G.*

Proof: (a) Since $\mathrm{C}_H(\Omega_A) \leq C_H(\Omega)$, there exists an Ω-essential orbit Λ for H on Ω_A. Let Λ be any Ω-essential orbit for H on Ω_A. Let $\omega \in \Omega$ be F-extreme. Let Δ be the H-invariant partition of Λ, given by **??**(a). Let $\breve{\omega}$ be the element of Δ, corresponding to ω and $\lambda \in \breve{\omega}$. By **??**(b),

$$C_F(\omega) = C_F(\breve{\omega}) = C_F(\lambda).$$

In particular, $C_F(\lambda) = C_F(\omega) \in \mathcal{M}_F^*$, λ is F-extreme and $A \in \mathcal{B}_F$.

Since $C_F(\breve{\omega}) = C_F(\omega) \neq 1$ and F is faithful on Ω, $C_F(\breve{\omega}) \not\leq C_H(\Omega)$. Hence $C_H(\breve{\omega}) \not\leq C_H(\Omega)$ and Λ is Ω-block-natural. So Ω_A is Ω-block-diagonal.

(b) Follows from (a).

(c) If $C_{H_A}(\Omega_B) \not\leq C_{H_A}(\Omega_A)$, H_A has no Ω_A-essential orbits on Ω_B. So (c) holds in this case. If $C_{H_A}(\Omega_B) \leq C_{H_A}(\Omega_A)$, we can apply (a) and again (c) holds.

(d) Let $(H, \Omega) \in \mathcal{B}_F$. By **??**, $\mathcal{A}(H) \cap \mathcal{A}^*(F)$ is a Kegel cover for G. By (b), $\mathcal{A}(H) \cap \mathcal{A}^*(F) \subseteq \mathcal{B}_F \subseteq \mathcal{B}$. So (d) holds. □

4 Groups Acting Block-Diagonally on a Set

Let $(H, \Omega) \in \mathcal{A}$ and $\Sigma \subseteq \Omega$. If $|\Omega \setminus \Sigma| \geq 5$, let R_Σ be the minimal normal supplement to $C_H(\Omega)$ in $C_H(\Sigma)$; otherwise put $R_\Sigma = 1$. Put $R = R_\emptyset$.

Suppose that H is Ω-block-diagonal on a set Λ. Let Λ^* be the union of the Ω-essential orbits for H on Λ. As every orbit for H on Λ^* is Ω-block-natural, there exists an H-invariant partition Δ of Λ^* so that Ω and Δ are isomorphic as H-sets. Note that this isomorphism is unique. Let $\tilde{\Sigma}$ denote the image of Σ in Δ under this H-isomorphism. Each $D \in \tilde{\Sigma}$ is a subset of Λ^*. Let $\hat{\Sigma} = \bigcup \tilde{\Sigma}$ be the union of these subsets. So $\hat{\Sigma} \subseteq \Lambda^*$ and $N_H(\hat{\Sigma}) = N_H(\tilde{\Sigma}) = N_H(\Sigma)$. Define $H_{\hat{\Sigma}} = C_H(\hat{\Sigma})$ and $H_\Sigma = C_H(\Sigma)$. Then $H_{\hat{\Sigma}} \leq H_\Sigma$. Note that $C_H(\Omega) = H_\Omega \leq H_\Sigma$ but $H_\Omega \not\leq H_{\hat{\Sigma}}$, unless H_Ω acts trivially on Λ^* or $\Sigma = \emptyset$.

Lemma 4.1 *Let $(H, \Omega) \in \mathcal{A}$. Suppose that H is Ω-block-diagonal on a set Λ. Let $\Sigma \subseteq \Omega$.*

(a) *If* $|\Omega \setminus \Sigma| \geq 5$, *then*

$$H_\Sigma = H_{\hat{\Sigma}} H_\Omega = (H_{\hat{\Sigma}} \cap R) H_\Omega \text{ and } R_\Sigma \leq H_{\hat{\Sigma}} \cap R.$$

(b) *Both* $\bigcap_{\omega \in \Omega} R_\omega$ *and* $R \cap H_{\hat{\Omega}}$ *act trivially on* Λ.

(c) *Let* $\Sigma_1, \Sigma_2 \subseteq \Omega$ *with* $\Omega = \Sigma_1 \cup \Sigma_2$. *If* H *is faithful on* Λ, *then*

$$[R_{\Sigma_1}, R_{\Sigma_2}] = [H_{\hat{\Sigma}_1} \cap R, H_{\hat{\Sigma}_2} \cap R] = [H_{\hat{\Sigma}_1}, H_{\hat{\Sigma}_2}] \cap R = 1.$$

Proof: (a) Let $\omega \in \Omega$. Let Ξ be an orbit for H on Λ such that R acts non-trivial on Ξ. Then Ξ is Ω-essential for H and as Λ is Ω-block-diagonal, Ξ is Ω-block-natural. Hence

$$H_\omega = C_H(\Xi \cap \hat{\omega}) H_\Omega.$$

Thus $R_\omega \leq C_H(\Xi \cap \hat{\omega})$. As Ξ was an arbitrary Ω-essential orbit for H on Λ, $R_\omega \leq H_{\hat{\omega}}$. Thus

$$H_\omega = R_\omega H_\Omega = H_{\hat{\omega}} H_\Omega.$$

Let $\omega \in \Sigma$. We compute

$$H_\Sigma = H_\omega \cap H_\Sigma = (H_{\hat{\omega}} H_\Omega) \cap H_\Sigma = (H_{\hat{\omega}} \cap H_\Sigma) H_\Omega.$$

By minimality of R_Σ, $R_\Sigma \leq H_{\hat{\omega}} \cap H_\Sigma \leq H_{\hat{\omega}}$. As this is true for all $\omega \in \Sigma$ and $H_{\hat{\Sigma}} = \bigcap_{\omega \in \Sigma} H_{\hat{\omega}}$, $R_\Sigma \leq H_{\hat{\Sigma}}$. As $H_\Sigma = R_\Sigma H_\Omega$, $H_\Sigma = H_{\hat{\Sigma}} H_\Omega$. Finally, $H = R H_\Omega$ implies $H_\Sigma = (R \cap H_\Sigma) H_\Omega$ and so $R_\Sigma \leq R \cap H_\Sigma$. Hence $R_\Sigma \leq H_{\hat{\Sigma}} \cap R$ and $H_\Sigma = R_\Sigma H_\Omega = (H_{\hat{\Sigma}} \cap R) H_\Omega$. This proves (a).

(b) Note that $\hat{\Omega} = \Lambda^*$ and so $H_{\hat{\Omega}}$ acts trivially on Λ^*. Since R acts trivially on each of the non-Ω-essential orbits for H on Λ, R acts trivially on $\Lambda \setminus \Lambda^*$. So $H_{\hat{\Omega}} \cap R$ acts trivially on Λ. By (a), $\bigcap_{\omega \in \Omega} R_\omega \leq \bigcap_{\omega \in \Omega} (H_{\hat{\omega}} \cap R) = H_{\hat{\Omega}} \cap R$. So (b) holds.

(c) Note that $H_{\hat{\Sigma}_1} \leq H_{\Sigma_1} \leq H_{\Omega \setminus \Sigma_2} \leq N_H(\Sigma_2) \leq N_H(H_{\hat{\Sigma}_2})$. Similarly, $H_{\hat{\Sigma}_2} \leq N_H(H_{\hat{\Sigma}_1})$. So

$$[H_{\hat{\Sigma}_1}, H_{\hat{\Sigma}_2}] \leq H_{\hat{\Sigma}_1} \cap H_{\hat{\Sigma}_2} \leq H_{\widehat{\Sigma_1 \cup \Sigma_2}} = H_{\hat{\Omega}}.$$

Using (a) we get that

$$[R_{\Sigma_1}, R_{\Sigma_2}] \leq [H_{\hat{\Sigma}_1} \cap R, H_{\hat{\Sigma}_2} \cap R] \leq [H_{\hat{\Sigma}_1}, H_{\hat{\Sigma}_2}] \cap R \leq H_{\hat{\Omega}} \cap R.$$

Since H is faithful on Λ, (b) implies $H_{\hat{\Omega}} \cap R = 1$. Thus (c) holds. □

Theorem 4.2 *Let* G *be a locally finite, simple group of alternating type and* $\mathcal{D}$ *an alternating Kegel cover for* G. *Suppose that there exists* $(H, \Omega) \in \mathcal{A}$ *such that for all* $A \in \mathcal{D}(H)$, H *is* Ω*-block-diagonal on* Ω_A. *Then* H *is non-regular and* G *is of non-regular alternating type.*

Proof: By **??**, $\mathcal{D}^*(H)$ is a Kegel cover for G. By **??**(2), it suffices to show that for all $A \in \mathcal{D}^*(H)$, H does not have a regular orbit on Ω_A . Pick $A \in \mathcal{D}^*(H)$ and $\lambda \in \Omega_A$. Put $\Lambda = \lambda^H$. Suppose that Λ is regular. If Λ is not Ω-essential, then $1 \neq R \leq C_H(\lambda)$, a contradiction. Hence Λ is Ω-essential and so $\lambda \in \hat{\omega}$ for some $\omega \in \Omega$. By Lemma **??**(a), $1 \neq R_\omega \leq H_{\hat{\omega}} \leq C_H(\lambda)$, a contradiction. So Λ is not regular. □

Theorem 4.3 *Let* $(H, \Omega) \in \mathcal{A}$ *and suppose that* H *is faithful and* Ω*-block-diagonal on some set. Let* R *be the minimal normal supplement to* H_Ω *in* H*. Let* $\omega \in \Omega$ *and put* $K = C_R(\omega)/R_\omega$*. Then*

$$R \cong (K \wr_\Omega \mathrm{Alt}(\Omega))'.$$

Proof: To simplify notation, we assume that $\Omega = \{1, \ldots, n\}$ and $\omega = 1$. For $i \in \Omega$, pick $r_i \in R$ with $1^{r_i} = i$. Let $\pi : R \to \mathrm{Alt}(\Omega)$ be the homomorphism arising from the action of R on Ω. Note that $r_i g r_{i^g}^{-1} \in C_R(1)$ for all $g \in R$ and all $i \in \Omega$. Hence we obtain a map

$$\phi : R \to K \wr_\Omega \mathrm{Alt}(\Omega) : g \to ((r_i g r_{i^g}^{-1} R_1)_{i \in \Omega}, \pi(g)).$$

We will first show that ϕ is a homomorphism. Indeed let $g, h \in R$. Then

$$\begin{aligned} \phi(g)\phi(h) &= ((r_i g r_{i^g}^{-1} R_1)_{i\in\Omega}, \pi(g))((r_i h r_{i^h}^{-1} R_1)_{i\in\Omega}, \pi(h)) \\ &= ((r_i g r_{i^g}^{-1} r_{i^g} h r_{i^{gh}}^{-1} R_1)_{i\in\Omega}, \pi(g)\pi(h)) \\ &= ((r_i g h r_{i^{gh}}^{-1} R_1)_{i\in\Omega}, \pi(gh)) \\ &= \phi(gh). \end{aligned}$$

If $\phi(g) = 1$, then $\pi(g) = 1$ and so

$$\phi(g) = ((r_i g r_i^{-1} R_1)_{i\in\Omega}, 1).$$

Thus $r_i g r_i^{-1} \in R_1$ and $g \in R_1^{r_i} = R_i$ for all $i \in \Omega$. By **??**(b), $g = 1$. So ϕ is one-to-one.

Put $D = \phi(R)$ and $S = K \wr_\Omega \mathrm{Alt}(\Omega)$. It remains to show that $D = S'$. For $i \in \Omega$ let $D_i = \phi(R_i)$. Also let B be the base group of S. Then $B = \bigoplus_{i=1}^n B_i$ with $B_i \cong K$ for all $i \in \Omega$. Note that $B \cap D = \phi(C_R(\Omega))$. Since $C_H(1) = R_1 C_H(\Omega)$ and $R_1 \leq R$ we have $C_R(1) = R_1 C_R(\Omega)$. As $K = C_R(1)/R_1$, we conclude that $B \cap D$ projects onto B_i for all $i \in \Omega$. For $i \in \Omega$, put $Q_i = B_i \cap D$. As $B \cap D$ normalizes Q_i, $Q_i \trianglelefteq B_i$ for all $i \in \Omega$. Let $Q = \bigoplus_{i\in\Omega} Q_i$. Then $Q \trianglelefteq BD = S$.

Put $E = \langle B \cap D_i \mid i \in \Omega \rangle$ and $D^* = \langle D_i \mid i \in \Omega \rangle$. Note that both E and D^* are normal in D and $D = D^*(B \cap D)$. By the definition of R, R has no proper normal supplement to $C_R(\Omega)$. Thus $D = D^*$.

Let $i \in \Omega$. Then $[C_R(\Omega), R_i] \leq C_R(\Omega) \cap R_i$ and so $[B \cap D, D_i] \leq B \cap D_i$. As $D = D^*$, we get $[B \cap D, D] \leq E$ and D/E is a perfect central extension of $D/B \cap D \cong \mathrm{Alt}(\Omega)$. Note that $D_1/D_1 \cap B \cong \mathrm{Alt}(\Omega \setminus \{1\})$. Since $D_1 \not\leq E$ and $D_1 \cap E = D_1 \cap B$ we conclude $D_1E/E \cong D_1/D_1 \cap E = D_1/D_1 \cap B \cong \mathrm{Alt}(\Omega \setminus \{1\})$. From the Schur multiplier of $\mathrm{Alt}(\Omega)$ [?] and since $|\Omega| \geq 7$ we see that no non-trivial,perfect, central extension of $\mathrm{Alt}(\Omega)$ has a subgroup isomorphic to $\mathrm{Alt}(\Omega \setminus \{1\})$. Thus $B \cap D = E$. Let $i \neq j \in \Omega$. As $[B \cap D, D_i] \leq B \cap D_i$, D_i normalizes $(B \cap D_j)(B \cap D_i)$. Since D_i acts transitively on $\Omega \setminus \{i\}$, we get $B \cap D_k \leq (B \cap D_j)(B \cap D_i)$ for all $k \in \Omega$. Thus

$$E = (B \cap D_i)(B \cap D_j), \quad \forall i \neq j \in \Omega.$$

Put

$$X = [B \cap D, B \cap D_2, \ldots, B \cap D_n].$$

Since $B \cap D_j$ projects trivially onto B_j for all $2 \leq j \leq n$, $X \leq B_1$ and so $X \leq Q_1$. Since $B \cap D = E = (B \cap D_j)(B \cap D_1)$ for all $2 \leq j \leq n$,

$$X(B \cap D_1) = \underbrace{[B \cap D, \ldots, B \cap D]}_{n \text{ terms}}(B \cap D_1).$$

Projecting this equation on B_1 and using that the projection of $B \cap D_1$ is trivial, we get

$$\underbrace{[B_1, \ldots, B_1]}_{n \text{ terms}} = X \leq Q_1.$$

So B_1/Q_1 is nilpotent. Hence also B/Q is nilpotent.

Suppose for the moment that B is abelian. Let $B_0 = \{b \in B \mid \sum b_i = 0\}$. So $[B, S] = [B, D] = B_0$ and $S' = B_0 \,\mathrm{Alt}(\Omega)$. As D is perfect, $D \leq S'$ and so $E = B \cap D \leq B_0$.

Put $L = \langle e_i - e_j \mid e \in E, 1 \leq i < j \leq n \rangle \leq K$ and $Y = \{b \in B \mid b_i \in L\}$. Note that $Y \cap B_0$ is generated by elements of the form $(0, \ldots, 0, e_i - e_j, 0, \ldots, 0, e_j - e_i, 0, \ldots, 0)$ where the non-zero entries are in arbitrary positions and $(e_1, \ldots, e_n) \in E$. Let $e = (e_1, \ldots, e_n) \in E$. Pick $s_1, s_2 \in R$ with $\pi(s_1) = (1, 2, 3)$ and $\pi(s_2) = (1, 4)(5, 6)$. Then

$$[e, \phi(s_1), \phi(s_2)] = [(e_3 - e_1, e_1 - e_2, e_2 - e_3, 0, \ldots, 0), \phi(s_2)] = (e_1 - e_3, 0, 0, e_3 - e_1, 0, \ldots, 0).$$

We conclude that $Y \cap B_0 \leq E$. Pick $e = (e_1, \ldots, e_n) \in B \cap D_i \leq E$. Since $B \cap D_i$ projects trivially on B_i, $e_i = 0$. Hence $e_j = e_j - e_i \in L$ for all $j \in \Omega$ and $e \in Y$. So $B \cap D_i \leq Y$ and the definition of E implies $E \leq Y$. Thus $Y \cap B_0 \leq E \leq Y \cap B_0$ and $E = Y \cap B_0$. As $E = B \cap D$ projects onto B_1, we get that $K = L$, $B = Y$ and $E = B_0$. So $D \leq S' \leq B_0 D = D$ and $S' = D$.

Thus the theorem holds if B is abelian. More importantly, note that all the above arguments are valid in S/B' and so $S' = DB'$. Hence $B_0 = EB'$ where $B_0 = S' \cap B$. For $j = 2, 3$, pick $d_j \in D$ with $1^{d_j} = j$. Then for $j = 2, 3$, $[B_1, d_j] \leq B_1 B_j$ and $[B_1, d_j]$ projects onto B_1. Thus

$$[[B_1, d_2], [B_1, d_3]] = [B_1, B_1].$$

Hence $B_0' = B'$ and $B_0 = EB_0'$. As B/Q is nilpotent and $Q \leq E \leq B_0$, $B_0'Q/Q \leq \Phi(B_0/Q)$. So $B_0 = EQ = E$ and $S' = DB' = DB_0' = DE' = D$. □

The preceding theorem is false for $|\Omega| = 6$ and thus our assumption that $|\Omega| \geq 7$ for all $(H, \Omega) \in \mathcal{A}$. Indeed, let $3.\mathrm{Alt}(6)$ be the 3-cover of $\mathrm{Alt}(6)$. Then $3.\mathrm{Alt}(6)$ has $\mathrm{Alt}(5)$ as a subgroup. Also, $3.\mathrm{Alt}(6)$ acts faithfully and block-natural on the cosets of $\mathrm{Alt}(5)$. But $3.\mathrm{Alt}(6)$ is not the derived group of a wreath-product.

5 Groups of 1-Type and ∞-Type

The main goal of this section is to prove theorems **??** and **??**. For this, we first prove a couple of technical lemmas.

Lemma 5.1 *Let H be a group, $\mathbb{K}$ a field and $0 \leq X \leq Y$ a chain of $\mathbb{K}H$-modules. Let P be its stabilizer in $GL_{\mathbb{K}}(Y)$. If H acts projectively non-trivially on Y/X, then*

$$X = [Y, [P, H]].$$

Proof: Let $Z = [Y, [P, H]]$ and suppose that $Z \neq X$. Then $Z \leq X$. Replacing $0 \leq X \leq Y$ by $0 \leq X/Z \leq Y/Z$, we may assume that $Z = 0$. So $[Y, [P, H]] = 0$ and $[P, H] = 1$. It is well-known and easily verified that $C_{GL_{\mathbb{K}(Y)}}(P) \leq Z(GL_{\mathbb{K}}(Y))P$. But this implies that H acts as scalars on Y/X, a contradiction. □

Lemma 5.2 *Let T be a finite group, $n \in \mathbb{Z}^+$ and $T_1, T_2, \ldots T_n$ non-trivial subgroups of T. Suppose that $T \leq \mathrm{PGL}_{\mathbb{K}}(V)$ and that $\mathrm{pdeg}_V(t) \geq (n+1)|T|^2$ for all $1 \neq t \in T$. Then there exists a T-invariant unipotent subgroup $Q \leq \mathrm{PGL}_{\mathbb{K}}(V)$ with $T \cap Q = 1$ and*

$$[[Q, T_1], [Q, T_2]], \ldots, [Q, T_n]] \neq 1.$$

Proof: For $1 \neq t \in T$, let U_t be a one-dimensional subspace of V such that $(U_t)^t \neq U_t$. Put $V_1 = \langle U_t^T \mid 1 \neq t \in T \rangle$. Then $\dim_{\mathbb{K}} V_1 \leq |T|^2$, T acts projectively faithfully on V_1 and $\mathrm{pdeg}_{V/V_1}(t) \geq n|T|^2$ for all $1 \neq t \in T$.

An easy induction argument now shows that there exists an ascending chain

$$0 = V_0 \leq V_1 \leq V_2 \leq V_n \leq V_{n+1} \leq V$$

of T-submodules so that T acts (projectively) faithfully on V_{i+1}/V_i and $\dim_{\mathbb{K}} V_{i+1}/V_i \leq |T|^2$ for all $0 \leq i \leq n$. Let $P \leq \mathrm{GL}_{\mathbb{K}}(V)$ be the stabilizer of this chain and Q the image in $\mathrm{PGL}_{\mathbb{K}}(V)$ of P. For $i = 1, \ldots, n$, let S_i be the pre-image in $\mathrm{GL}_{\mathbb{K}}(V)$ of T_i and $P_i = [P, S_i]$. Put $A_0 = C_P(V/V_1)$, $A_1 = [A_0, S_1] \leq P_1$ and inductively $A_i = [A_{i-1}, P_i]$ for $i = 2, \ldots, n$.

We will prove by induction that

$$(*) \quad [V_i, A_i] = 0 \text{ and } [V, A_i] = [V_{i+1}, A_i] = V_1$$

for all $1 \leq i \leq n$. Note first that $A_i \leq A_0$ for all i and so $[V_{i+1}, A_i] \leq [V, A_i] \leq [V, A_0] \leq V_1$. By **??** applied to $0 \leq V_1 \leq V_2$ and $H = S_1$, $[V_2, [A_0, S_1]] = V_1$. Thus (*) holds for $i = 1$.

Suppose that $(*)$ holds for $i - 1$. Then

$$[V_i, P_i, A_{i-1}] \leq [V_{i-1}, A_{i-1}] = 0$$

and

$$[V_i, A_{i-1}, P_i] = [V_1, P_i] = 0.$$

Thus by the Three Subgroup Lemma,

$$[V_i, [A_{i-1}, P_i]] = 0.$$

So the first statement in (*) holds.

By **??** applied to $0 \leq V_i/V_{i-1} \leq V_{i+1}/V_{i-1}$ and $H = S_i$,

$$[V_{i+1}, [P, S_i]] + V_{i-1} = V_i.$$

Taking the commutator with A_{i-1} on both sides (and using the induction assumption), we conclude

$$[V_{i+1}, P_i, A_{i-1}] = V_1.$$

Also

$$[V_{i+1}, A_{i-1}, P_i] \leq [V, A_0, P_i] \leq [V_1, P_i] = 0.$$

Thus by the Three Subgroup Lemma,

$$[V_{i+1}, [A_{i-1}, P_i]] = V_1.$$

Hence (*) holds for all $1 \leq i \leq n$. In particular, $[V, A_n] = V_1$ and $A_n \not\leq Z(\mathrm{GL}_{\mathbb{K}}(V))$. Note that

$$A_n \leq [[P, S_1], [P, S_2], \ldots, [P, S_n]]$$

and the lemma follows by considering the image of the last equation in $\mathrm{PGL}_{\mathbb{K}}(V)$. □

Theorem 5.3 *Let G be a locally finite, simple group of alternating type and F a finite subgroup of G. Then the following are equivalent :*

1. *F is regular.*
2. *Let L be a finite group and $\phi : F \to L$ an embedding. Then there exists a finite subgroup E of G, containing F, and an epimorphism $\xi : E \to L$ with $\phi = \xi \mid_F$.*

Proof: Suppose (1) holds. Let $\phi : F \to L$ be as in (2). By **??**(3), there exists $(H, \Omega) \in \mathcal{A}(F)$ so that F has at least $|L|/|F|$ regular orbits on Ω and $|\Omega| \geq |L| + 2$. In particular, there exists an F-invariant subset Λ of Ω of size $|L|$ so that all the orbits of F on Λ are regular. Let $\rho : L \to \mathrm{Sym}(\Lambda)$ be an embedding such that $\rho(L)$ acts regularly on Λ. Let $\psi : N_H(\Lambda) \to \mathrm{Sym}(\Lambda)$ be the homomorphism arising from the action of $N_H(\Lambda)$ on Λ. As both $\psi(F)$ and $\rho\phi(F)$ act semi-regularly on Λ, there exists an inner automorphism τ of $\mathrm{Sym}(\Lambda)$ with $\psi\mid_F = \tau\rho\phi$. Thus

$$\psi(F) = \tau\rho\phi(F) \leq \tau\rho(L) \leq \mathrm{Sym}(\Lambda).$$

As $|\Omega| \geq |L| + 2$, $|\Omega \setminus \Lambda| \geq 2$. Since $H^{\Omega} = \mathrm{Alt}(\Omega)$, $\psi(N_H(\Lambda)) = \mathrm{Sym}(\Lambda)$. Let $E = \psi^{-1}(\tau(\rho(L)))$. Since $\psi(F) \leq \tau\rho(L)$, $F \leq E$. Since $\rho : L \to \mathrm{Sym}(\Lambda)$ is one-to-one, there exists a partial inverse $\rho^* : \rho(L) \to L$. Put $\xi = \rho^*\tau^{-1}\psi|_E$: $E \to L$. As ψ, τ^{-1} and ρ^* are onto, $\xi(E) = L$ and

$$\xi \mid_F = \rho^*\tau^{-1}\psi \mid_F = \rho^*\tau^{-1}\tau\rho\phi = \phi.$$

So (1) implies (2).

Suppose that (2) holds. Let $s \geq f_{reg}(|F|)$ with s even, Ω a set with $|\Omega| = s|F|$ and $L = \mathrm{Alt}(\Omega)$. Let $\phi : F \to L$ be an embedding so that $\phi(F)$ is semi-regular on Ω. Let E and ξ be given by (2). Then $(E, \Omega) \in \mathcal{A}^*(F)$ and all the orbits for F on Ω are regular. Hence F is $\mathcal{A}$-regular and so by **??** F is regular. □

We remark that the preceding theorem remains true if in part 2. "$\phi : F \to L$ an embedding" is replaced by "$\phi : F \to L$ a homomorphism. " Indeed suppose that $\alpha : F \to L$ is a homomorphism. Define $\phi : F \to F \times L$, $f \to (f, \alpha(f))$ and $\pi : F \times L \to L$, $(f, l) \to l$. Then ϕ is one-to-one and $\alpha = \pi\phi$. So if $\xi : E \to F \times L$ is onto with $\xi \mid_F = \phi$, then $\pi\xi : E \to L$ is onto with $(\pi\xi) \mid_F = \alpha$.

Proof of Theorem ??:

Suppose first that G is of non-regular type. Let F be a finite, non-regular subgroup of G. Assume for a contradiction that G is not of 1-type. Then there exists a Kegel cover $\mathcal{K}$ for G, none of whose factors are alternating groups. By [**?**, Proposition 3.2(b)], we may assume that all the factors of $\mathcal{K}$ are of the form $PSL_{\mathbb{K}}(V)$ for some finite field $\mathbb{K}$ and a $\mathbb{K}$-vectorspace V. Let $(T, \Sigma) \in \mathcal{B}_F$ with $|\Sigma| \geq 10$. By **??**, we can choose $(L, M) \in \mathcal{K}(T)$ so that $\mathrm{pdeg}_{L/M}(t) \geq 9|T|^2$ for all $1 \neq t \in T$. For $i = 1, 2$ pick $\Sigma_i \subseteq \Sigma$ so that $\Sigma = \Sigma_1 \cup \Sigma_2$ and $|\Sigma \setminus \Sigma_i| \geq 5$. For $i = 1, 2$, let T_i^* be the minimal normal supplement to $C_T(\Sigma)$ in $C_T(\Sigma_i)$. By **??**, there exists a T-invariant unipotent subgroup Q/M of L/M so that $T \cap Q = 1$ and

$$[[Q/M, T_1^*], [Q/M, T_2^*]] \neq 1.$$

Note that Q/M is a p-group for some prime p. Put $H = TQ$. Then Σ is an H-set with Q acting trivially on Σ and $(H, \Sigma) \in \mathcal{B}_F$. Pick $A \in \mathcal{B}_F(H)$. By **??**(c), both T and H are faithful and Σ-block-diagonal on Ω_A. We use the notation introduced just before Lemma **??**. Let R be the minimal normal supplement to H_Σ in H. Note that $H_\Sigma = QT_\Sigma$. By **??**(a),

$$H_{\Sigma_i} = (H_{\hat{\Sigma}_i} \cap R)H_\Sigma = (H_{\hat{\Sigma}_i} \cap R)QT_\Sigma.$$

Hence $T_{\Sigma_i} = (((H_{\hat{\Sigma}_i} \cap R)Q) \cap T)T_\Sigma$ and the minimality of T_i^* implies $T_i^* \leq (H_{\hat{\Sigma}_i} \cap R)Q$. Since T_i^* is perfect and $(H_{\hat{\Sigma}_i} \cap R)Q/(H_{\hat{\Sigma}_i} \cap R)M$ is a p-group, $T_i^* \leq (H_{\hat{\Sigma}_i} \cap R)M$. Thus

$$[[Q, T_1^*], [Q, T_2^*]] \leq ([H_{\hat{\Sigma_1}}, H_{\hat{\Sigma_2}}] \cap R)M.$$

By **??**(c), $[[Q, T_1^*], [Q, T_2^*]] \leq M$, a contradiction.

So non-regular type implies 1-type.

Suppose next that G is locally regular. We will show that G is of ∞-type. So let $\mathcal{S}$ be a class of finite simple groups such that every finite group is embedded into a member of $\mathcal{S}$.

Let F be a finite subgroup of G. Pick $L \in \mathcal{S}$ such that F is embedded in L. By **??**, there exist a finite subgroup E of G and $M \trianglelefteq E$ such that $F \leq E$, $F \cap M = 1$ and $E/M \cong L$. Put $K_F = (E, M)$. Then $\{K_F \mid F \text{ is a finite subgroup of } G\}$ is a Kegel cover for G all of whose factors are isomorphic to members of $\mathcal{S}$.

So we proved that locally regular implies ∞-type.

Suppose next that G is of 1-type. Then clearly G is not of ∞-type and so also not locally regular. Thus G is non-regular.

Suppose finally that G is of ∞-type. Then G is clearly not of 1-type. As non-regular implies 1-type, we conclude that G is locally regular. □

Proposition 5.4 *Let G be a locally finite, simple group of alternating type and F a non-regular subgroup. Then there exists a finite $F \leq \tilde{F} \leq G$ such that for all finite $\tilde{F} \leq H \leq G$ and all maximal normal subgroups M of H with $M \cap \tilde{F} = 1$, there exists a finite set Ω such that*

(a) $H/M \cong \mathrm{Alt}(\Omega)$.

(b) *F has no regular orbit on Ω.*

(c) $(H, \Omega) \in \mathcal{B}_F$

Proof: Let $\mathcal{U}$ be the set of pairs (H, M) where H is a finite subgroup of G, M is a maximal subgroup of H and H/M is not isomorphic to an alternating group. By **??**(a), G is of 1-type and so $\mathcal{U}$ is not a Kegel cover for G. Hence there exists a finite subgroup F_1 with $\mathcal{U}(F_1) = \emptyset$. Let $s = f_{reg}(|F|)$. By **??**, there exists a finite $F \leq F_2 \leq G$ with $\mathcal{A}(F_2) \subseteq \mathcal{A}(F, s) = \mathcal{A}^*(F)$. Since $\mathcal{B}_F$ is a Kegel cover for G, there exists $(\tilde{F}, \Sigma) \in \mathcal{B}_F$ with $\langle F_1, F_2 \rangle \leq \tilde{F}$.

Let H and M be as in the proposition. Since $\mathcal{U}(F_1) = \emptyset$, $H/M \cong \mathrm{Alt}(\Omega)$ for some set Ω. Thus $(H, M) \in \mathcal{A}(\tilde{F})$. Since $F_2 \leq \tilde{F}$, $(H, \Omega) \in \mathcal{A}^*(F)$. In particular, since F is non-regular, F has no regular orbit on Ω. Finally, (c) follows from $(\tilde{F}, \Sigma) \in \mathcal{B}_F$ and **??**(b). □

Proof Of Theorem ?? :

Let F be a non-regular finite subgroup of G. Let $\tilde{F}$ be given by **??**. For $1 \neq d \in G$, we have that $G = \langle d^G \rangle = \langle d^{\langle d^G \rangle} \rangle$ and so we can choose a finite $\tilde{F} \leq F^* \leq G$ with

$$\tilde{F} \leq \langle d^{\langle d^{F^*} \rangle} \rangle$$

for all $1 \neq d \in \tilde{F}$.

Let $F^* \leq L \leq G$ be finite and put $H = \langle F^L \rangle$. Then $\tilde{F} \leq H$. Let $\{M_1, \ldots, M_n\}$ be the set of all maximal normal subgroups of H. Pick $i \in \{1, \ldots, n\}$. Since $\langle F^L \rangle \not\leq M_i$, $F \not\leq M_i^{l_i}$ for some $l_i \in L$. Suppose that $1 \neq d \in \tilde{F} \cap M_i^{l_i}$. Then

$$F \leq \tilde{F} \leq \langle d^{\langle d^{F^*} \rangle} \rangle \leq \langle d^H \rangle \leq M_i^{l_i},$$

a contradiction. So $\tilde{F} \cap M_i^{l_i} = 1$. Thus by **??**(a)(c), $H/M_i^{l_i} \cong \mathrm{Alt}(\Omega_i)$ for some set Ω_i and $(H, \Omega_i) \in \mathcal{B}_F$. Note that $M_i^{l_i} = C_H(\Omega_i)$. Let R_i be the minimal normal supplement to M_i in H. By **??**(d), there exists $A \in \mathcal{B}_F(H)$. By **??**(c), H is faithful and Ω_i-block-diagonal on Ω_A. By **??**, $R_i^{l_i} \cong R_i \cong (K_i \wr_{\Omega_i} \mathrm{Alt}(\Omega_i))'$ for some finite group K_i. As $R_1 \dots R_n$ lies in none of the M_i's, $H = R_1 R_2 \dots R_n$. Thus (a) holds.

Let $R = R_1$ and $T = R_2 \dots R_m$. To show (b), it suffices to show that $R \cap T = 1$. Note that $B_i = R_i \cap M_i$ and so $[R_i, F] \not\leq B_i$ just means $F \not\leq M_i$. Thus we can choose $l_i = 1$ for all $1 \leq i \leq m$. Suppose that $R \cap T \neq 1$. Pick an orbit Λ for H on Ω_A so that $R \cap T$ acts non-trivially on Λ. Then R_1 and at least one R_j with $2 \leq j \leq m$ act non-trivially on Λ, say $j = 2$. As Ω_A is Ω_i-block-diagonal, there exist H-invariant partitions Δ_i of Λ so that Δ_i is isomorphic to Ω_i as an H-set for $i = 1, 2$. Let $\omega \in \Omega_1$ be F-extreme and U_1 the corresponding element in Δ_1. For all $\lambda \in U_1$, $C_F(\lambda) \leq N_F(U_1) = C_F(\omega)$. Since ω is F-extreme, $C_F(\lambda) = C_F(\omega)$ for all $\lambda \in U_1$ and so $C_F(\omega)$ acts trivially on U_1. Let $U_2 \in \Delta_2$.

Suppose that $U_2 \cap U_1 = \emptyset$. As $M_1 \neq M_2$, $H = M_1 M_2$ and so M_1 acts transitively on Δ_2. So we conclude $U \cap U_1 = \emptyset$ for all $U \in \Delta_2$, a contradiction. Hence $U_2 \cap U_1 \neq \emptyset$ and so $C_F(\omega)$ fixes an element in U_2. Thus $C_F(\omega)$ normalizes U_2. As U_2 was arbitrary, $C_F(\omega)$ acts trivially on Δ_2 and so on Ω_2. Thus $1 \neq C_F(\omega) \leq M_2$, a contradiction to $F \cap M_2 = 1$. □

6 Non-Absolutely Simple Groups of 1-Type

In this section we present examples of non-absolutely simple, locally finite, simple groups of non-regular alternating type. The existence of such groups also follows from [**?**, 1.33] and Theorem **??**. Our class of examples is slightly larger than the one in [**?**] and also shows that one does not have any control over the quotient $L/\langle F^L \rangle$ in Theorem **??**. Since knowledge of most of details of [**?**, Section 6] is required, they are repeated here.

Lemma 6.1 *Let H be a perfect, finite group and Ω a faithful, finite H-set. Then there exist a perfect, finite group H^* containing H, a function X which associates to each subgroup A of H a subgroup $X(A)$ of H^*, and a faithful, finite H^*-set Ω^* such that*

(a) $H \leq \langle h^{H^*} \rangle$ *for all* $1 \neq h \in H$.

(b) $X(A) \cap H = A$ *for all* $A \leq H$.

(c) *If* $A \leq B \leq H$, *then* $A \trianglelefteq B$ *if and only if* $X(A) \trianglelefteq X(B)$.

(d) $X(H) \trianglelefteq \langle X(H)^{H^*} \rangle$ *but* $X(H)$ *is not a normal subgroup of* H^*.

(e) *Every non-trivial orbit for* H *on* Ω^* *is isomorphic to an orbit for* H *on* Ω.

(f) *There exists a finite* $X(H)$*-set* Λ *so that*

- (fa) $X(H)^\Lambda = \mathrm{Alt}(\Lambda)$
- (fb) H *acts faithfully on* Λ.

(fc) *Every non-trivial orbit for H on Λ is isomorphic to an orbit for H on Ω.*

Proof: Let I be a faithful, finite H-set so that every non-trivial orbit for H on I is isomorphic to an orbit for H on Ω. Let $S = \mathrm{Alt}(I)$ and $\alpha : H \to S$ be the monomorphism associated to the action of H on I.

Let T be any non trivial, finite, perfect group and J a faithful, finite T-set such that T acts transitively on J. We assume that $0 \in I$ and $\{0, 1\} \subseteq J$. Let $K = H \wr_I S$. For $i \in I$, let $\beta_i : H \to K$ be the canonical isomorphism between H and the i-th component of the base group of K and let β be the canonical monomorphism from S to K. Let $H^* = K \wr_J T$ and for $j \in J$, let $\gamma_j : K \to H^*$ be the canonical isomorphism between K and the j-th component of the base group of H^*. Define $\rho : H \to H^*$ by $\rho(h) = \gamma_0(\beta_0(h))\gamma_1(\beta(\alpha(h)))$. Then ρ is clearly a monomorphism. For $A \leq H$, let $X(A)$ be the set of elements in the base group of H^* such that the projection onto the 0-th component is contained in $\gamma_0(\prod_{i \in I} \beta_i(A))$. Identifying H with $\rho(H)$, we see immediately that (b) and (c) hold. Note that $\langle X(H)^{H^*} \rangle$ is the base group of H^* and so (d) holds. One easily checks that $\langle h^{H^*} \rangle$ is the base group of H^* for all $1 \neq h \in H$ and so (a) holds.

Note that $K = H \wr_I S$ acts faithfully on $\Omega \times I$ and $H^* = K \wr_J T$ acts faithfully on $\Omega^* := \Omega \times I \times J$. By definition of the embedding of H into H^*, we see that

- $\Omega \times \{0\} \times \{0\}$ is isomorphic to Ω as an H-set.
- H acts trivially on $\Omega \times \{i\} \times \{0\}$ for all $i \in I \setminus \{0\}$.
- $\{\omega\} \times I \times \{1\}$ is isomorphic to I as an H-set for all $\omega \in \Omega$.
- H acts trivially on $\Omega \times I \times \{j\}$ for all $j \in J \setminus \{0, 1\}$.

By assumption, every non-trivial orbit for H on I is isomorphic to an orbit for H on Ω. So (e) holds.

Put $\Lambda = \{\Omega \times \{i\} \times \{1\} \mid i \in I\}$. Then the base group of H^* normalizes Λ. Hence Λ is an $X(H)$-set. Since $\gamma_1(K) \leq X(H)$ and $S = \mathrm{Alt}(I)$, we get $X(H)^\Lambda = \mathrm{Alt}(I)$. Also, Λ is isomorphic to I as an H-set. So (f) holds. □

Proof of Theorem ??:

Let G_1 be a finite perfect group and Ω_1 a faithful, finite G_1-set so that G_1 has no regular orbits on Ω_1 . Inductively, for $i \geq 1$, let $G_{i+1} = G_i^*$, X_i a function from the subgroups of G_i to the subgroups of G_{i+1}, Ω_{i+1} a faithful, finite G_{i+1}-set and Λ_{i+1} a finite $X_i(G_i)$-set which fulfills **??**. Let $G = \bigcup_{i=1}^{\infty} G_i$. Then by **??**(a)(d), G is an infinite, locally finite simple group.

Put $M_{1,1} = 1$, $M_{1,2} = G_1$ and inductively, for $n \geq 1$, we put $M_{n+1,j} = X_n(M_{n,j})$ for $1 \leq j \leq 2n$, $M_{n+1,2n+1} = \langle X_n(G_n)^{G_{n+1}} \rangle$ and $M_{n+1,2n+2} = G_{n+1}$. Using induction on n and **??**(c)(d), we get that for all $n \geq 1$, $M_{n,i} \lhd M_{n,i+1}$ for $1 \leq i \leq 2n - 1$ and $M_{m,i} \cap G_n = M_{n,i}$ for all $m > n$ and $1 \leq i \leq 2n$. For $i \geq 1$, put $M_i = \bigcup_{n \geq \frac{i}{2}} M_{n,i}$. Then $G_n \leq M_{2n}$ for all

$n \geq 1$ and so $G = \bigcup_{i=1}^{\infty} M_i$. Also, $G_n \cap M_i = M_{n,i}$ for all $n \geq 1$ and $1 \leq i \leq 2n$. By **??**(d), $M_i \lhd M_{i+1}$ for all $i \geq 1$.

Hence G is not absolutely simple. Suppose G is finitary. Then by [**?**], G is an alternating group and so absolutely simple. Therefore G is not finitary. For $i \geq 1$, let $H_{i+1} = X_i(G_i)$. Then G_i acts faithfully on Λ_{i+1} and so $\mathcal{D} = \{(H_i, \Lambda_i) \mid i \geq 2\}$ is an alternating Kegel cover for G. By **??**(e)(fc) and induction on i, each non-trivial orbit for G_1 on Ω_i or Λ_i is isomorphic to a G_1-orbit on Ω_1 for all $i \geq 2$. Since G_1 has no regular orbits on Ω_1, we conclude that G_1 has no regular orbits on Λ_i for all $i \geq 2$. So by **??**, G_1 is non-regular and G is of non-regular alternating type. □

References

[Del] S. Delcroix, *Non-finitary Locally Finite Simple Groups* Universiteit Gent, (2000) Thesis.

[FM] N. Flowers and U. Meierfrankenfeld, *On the center of maximal subgroups in locally finite simple groups of alternating type* submitted to J. Group Theory

[Ha1] J.I. Hall, *Infinite alternating groups as finitary linear transformation groups* J. Algebra 119 (1988),337-359.

[Ha2] J.I. Hall, *Locally Finite Simple Finitary Groups*, in B. Hartley et al. (eds.) *Finite and Locally Finite Groups* Kluwer Academic Publishers, (1995), 147-188.

[Har] B. Hartley *Simple Locally Finite Groups*, in B. Hartley et al. (eds.) *Finite and Locally Finite Groups* Kluwer Academic Publishers, (1995), 1-44.

[Mei] U. Meierfrankenfeld, *Non-Finitary Locally Finite Simple Groups*, in B. Hartley et al. (eds.) *Finite and Locally Finite Groups* Kluwer Academic Publishers, (1995), 189-212.

[Sch] I. Schur, *Über die Darstellungen der symmetrischen und alternierenden Gruppen durch gebrochenen linearen Substitutionen*, Crelle J. 139 (1911), 155-250.