The Local $C(G,T)$ Theorem

D. Bundy, N. Hebbinghaus, B. Stellmacher

August 23, 2005

1 Introduction

In this article all groups considered are assumed to be finite. Moreover G always denotes a group and p always a prime.

We define $\mathcal{A}(G)$ to be the set of elementary abelian p-subgroups of G of maximal order and $\Omega(G)$ to be the subgroup generated by the elements of order p of G. Then

$$J(G) := \langle A \mid A \in \mathcal{A}(G)\rangle$$

is the **Thompson subgroup** of G (with respect to p), and

$$B(G) := \langle C_T(\Omega(Z(J(T)))) \mid T \in Syl_p(G)\rangle$$

is the **Baumann subgroup** of G (with respect to p).

Definition 1.1 *Let p divide the order of G, $T \in Syl_p(G)$ and $S \leq T$. Then*

$$\begin{aligned} C(G,S) &:= \langle N_G(C) \mid 1 \neq C \ char\ S\rangle, \\ C^*(G,T) &:= \langle C_G(\Omega(Z(T))), C(G,B(T))\rangle, \\ C^{**}(G,T) &:= \langle C_G(\Omega(Z(T))), N_G(J(T))\rangle. \end{aligned}$$

Notice that every characteristic subgroup of $B(T)$ is characteristic in T and $J(T)$ is characteristic in $B(T)$. In particular

$$C^{**}(G,T) \leq C^*(G,T) \leq C(G,T).$$

Definition 1.2 *A group G is of characteristic p if*

$$C_G(O_p(G)) \leq O_p(G) \ (\textit{or equivalently } F^*(G) = O_p(G)).$$

In this paper we will classify those groups G of characteristic p that are not equal to $C(G,T)$ with respect to some Sylow p-subgroup T; a result called the **Local $\boldsymbol{C(G,T)}$-Theorem**.

The investigation of groups of characteristic p in which $G \neq C(G,T)$ is a natural extension of work on failure of Thompson factorization as first studied by Glauberman [8] in response to the factorization theorems of Thompson [17]. Indeed Glauberman's Theorem is similar to that of our $C^{**}(G,T)$-Theorem for minimal parabolic subgroups (see Theorem 1.5) in the case when G is p-solvable but without the assumption that G is minimal parabolic.

The Local $C(G,T)$-Theorem in the case $p = 2$ was proven by Aschbacher [1] and there are some key features of Aschbacher's proof which we have reformulated for use in our proof. In particular, $B(T)$-blocks are a generalization of his *short* groups to the case of p any prime, together with the extra condition that they are normalized by $B(T)$. Aschbacher uses the word *block* for a short subnormal subgroup.

Some of the properties of $B(T)$-blocks resemble those of components and these are proven in Chapter 6. For example, distinct subnormal $B(T)$-blocks commute (6.11). Furthermore, our notations $\mathcal{O}_G(V)$ and $\mathcal{A}_G(V)$ are essentially the same as $\mathcal{P}(G,V)$ and $\mathcal{P}^*(G,V)$ of Aschbacher.

An alternative proof for $p = 2$ was also given by Gorenstein and Lyons [9]. Their proof avoids the use of some deep results needed in Aschbacher's proof. Instead it requires the K-group hypothesis (that any simple section of G is one of the known finite simple groups), which is sufficient for the purposes of the classification of the finite simple groups.

Our proof works for all primes p and does neither use the K-group assumption nor the deep results used in Aschbacher's proof. In fact, it is more or less self contained.

Our result can be considered as part of a project of Meierfrankenfeld et. al. [13] and we will use standard concepts from this project. In particular, the name *characteristic* p for groups with the property $C_G(O_p(G)) \leq O_p(G)$ and the L-Lemma originate there. Our abstract definition of a minimal parabolic group is also used extensively in this project, but was originally an idea of McBride.

The Baumann subgroup and the Baumann Argument (3.7) first appeared in (2.11.1) of [2], but we prefer to quote [15], where the result is explicitly stated in the form we require. This result is used to show that certain subgroups satisfy the hypothesis of a pushing up result [16] which was originally proven by Glauberman and Niles [14] and independently for the case $p = 2$ by Baumann [3].

Groups generated by conjugacy classes of transvections were classified by McLaughlin [10], [11] and some of our results in Section 4 follow easily from this classification, but we prefer to give an independent proof tailored to our particular situation.

The results of Section 2 are elementary and mainly well-known. We have given explicit proofs rather than searching for original references in order to keep things reasonably self contained.

To state the main result we need two further definitions.

Definition 1.3 *The symmetric group of degree m is denoted by S_m. Let X be a group and W be a finite simple $GF(p)X$-module. If $X \cong S_m$, $m \geq 3$, then W is a natural S_m-module (for X), if $p = 2$ and W is isomorphic to the unique non-trivial simple section of the $GF(2)S_m$-permutation module.*

If $X \cong SL_2(p^m)$, then W is a natural $SL_2(p^m)$-module (for X), if W is irreducible, $F := End_X(W) \cong GF(p^m)$, and W is a 2-dimensional FX-module.

Moreover, for A_m and $SL_2(p^m)'$ rather than S_m and $SL_2(p^m)$ the corresponding module is called a natural A_m-module and a natural $SL_2(p^m)'$-module, respectively.

It is easy to see that every finite simple $GF(2)S_m$-module with $|W/C_W(t)| = 2$ for a transposition $t \in S_m$ is a natural S_m-module.

Definition 1.4 *Let $T \in Syl_p(G)$. A subgroup $E \leq G$ is a* ***$B(T)$*-block** *of G if for $W := \Omega(Z(O_p(E)))$:*

(i) *$E = O^p(E) = [E, B(T)]$, $[O_p(E), E] = O_p(E)$, and $[E, \Omega(Z(T))] \neq 1$.*

(ii) *$E/O_p(E) \cong SL_2(p^n)'$ or $p = 2$ and $E/O_2(E) \cong A_{2m+1}$, and $W/C_W(E)$ is a natural $SL_2(p^n)'$- resp. A_{2m+1}-module for $E/O_p(E)$.*

(iii) *$O_p(E) = W$, or*

  (1) *$p = 3$, and $O_3(E)/W$ is a natural $SL_2(3^n)'$-module for $E/O_3(E)$,*

  (2) *$O_3(E)' = \Phi(O_3(E)) = Z(E) = C_W(E)$ and $|Z(E)| = 3^n$, and*

  (3) *no element of $B(T) \setminus C_{B(T)}(W)$ acts quadratically on $O_3(E)/Z(E)$.*

If $E/O_p(E) \cong SL_2(p^n)'$, then E is a **linear** *block, and in the other case E is a* **symmetric** *block. Moreover, if (1) – (3) in (iii) hold, then E is an* **exceptional** *block.*

We will prove the following theorem.

Theorem 1.5 (**Local $C^*(G,T)$-Theorem**) *Let G be of characteristic p with $T \in Syl_p(G)$ such that $G \neq C^*(G,T)$. Then there exist $B(T)$-blocks $G_1, \ldots, G_r$ of G such that the following hold:*

(a) *$\{G_1, \ldots, G_r\}^G = \{G_1, \ldots, G_r\}$.*

(b) *$[G_i, G_j] = 1$ for $i \neq j$.*

(c) *$G = C^*(G,T)G_0$, where $G_0 := \prod_{i=1}^r G_i$.*

(*d*) *Every $B(T)$-block of G that is not in $C^*(G,T)$ is contained in one of the $B(T)$-blocks $G_1,\dots,G_r$.*

(*e*) $C^*(G,T)\cap G_0=\prod_{i=1}^r(C^*(G,T)\cap G_i)$. *Moreover either*

(*i*) $G_i/O_p(G_i)\cong SL_2(p^m)$, $p^m>3$, *and* $C^*(G,T)\cap G_i=N_{G_i}(T\cap G_i)$, *or*

(*ii*) $p=2$, $G_i/O_2(G_i)\cong A_{2m+1}$, *and* $(C^*(G,T)\cap G_i)/O_2(G_i)\cong A_{2m}$,

(*iii*) $p=3$, $G_i/O_3(G_i)\cong SL_2(3)'$ *and* $(C^*(G,T)\cap G_i)/O_3(G_i)=Z(G_i/O_3(G_i))$.

Corollary 1.6 (**Local $C(G,T)$-Theorem**) *Let G be of characteristic p with $T\in Syl_p(G)$ such that $G\neq C(G,T)$. Then G has the same structure as given in Theorem* 1.5 *with the additional restriction that if G_i is a symmetric block, then $G_i/O_2(G_i)\cong A_{2^n+1}$.*

It is easy to see that under the assumption of Theorem 1.5 every proper subgroup L with $B(T)\le L$ and $L\not\le C^*(G,T)$ satisfies the hypothesis of 1.5 (see 2.3). Hence, those groups G, where $C^*(G,T)$ is the unique maximal subgroup containing $B(T)$, are the basis for an induction on the order of G. This leads to a class of groups that plays the same role for groups of local characteristic p as the class of minimal parabolic groups for groups of Lie type in characteristic p (see [13]).

Definition 1.7 *Let $T\in Syl_p(G)$. Then G is a* **minimal parabolic** *group (with respect to p), if T is not normal in G and there is a unique maximal subgroup of G containing T.*

The restricted structure of minimal parabolic groups allows us to prove a Local $C^{**}(G,T)$-Theorem that is of interest on its own:

Theorem 1.8 (**Local $C^{**}(G,T)$-Theorem for Minimal Parabolic Groups**) *Let G be a minimal parabolic group of characteristic p with $T\in Syl_p(G)$ such that $G\neq C^{**}(G,T)$, and let $V:=\Omega(Z(O_p(G)))$ and $\overline{G}:=G/C_G(V)$. Then there exist subgroups $E_1,\dots,E_r$ of G such that*

(*a*) $\overline{G}=\overline{J(G)T}$ *and* $\overline{J(G)}=\overline{E}_1\times\cdots\times\overline{E}_r$,

(*b*) $\overline{T}$ *acts transitively on* $\{\overline{E}_1,\dots,\overline{E}_r\}$,

(*c*) $V=C_V(\overline{E}_1\times\ldots\times\overline{E}_r)\prod_{i=1}^r[V,E_i]$, *with* $[V,E_i,E_j]=1$,

(*d*) $\overline{E}_i\cong SL_2(p^n)$ *or* $p=2$ *and* $\overline{E}_i\cong S_{2^n+1}$, *for some* $n\in\mathbb{N}$, *and*

(*e*) $[V,E_i]/C_{[V,E_i]}(E_i)$ *is a natural module for* E_i.

As a corollary of the Local $C^*(G,T)$-Theorem we get a pushing up result for minimal parabolic groups.

Corollary 1.9 (Pushing Up Theorem for Minimal Parabolic Groups) *Let G be a minimal parabolic group of characteristic p with $T \in Syl_p(G)$. Suppose that neither any non-trivial characteristic subgroup of $B(T)$ nor $\Omega(Z(T))$ is normal in G. Then G satisfies the conclusion of the Local $C(G,T)$-Theorem. Moreover $C^*(G,T) = (C^*(G,T) \cap \prod_{i=1}^{r} G_i)T$.*

2 Preliminary Results

Lemma 2.1 *Let $\mathcal{D}$ be a conjugacy class of subgroups of G and A and B be subgroups of G. Suppose that $G = \langle \mathcal{D} \rangle$ and*

$$\mathcal{D} = \{X \in \mathcal{D} \mid X \leq A\} \cup \{X \in \mathcal{D} \mid X \leq B\}.$$

Then $A = G$ or $B = G$.

Proof. Let

$$\mathcal{D}_0 := \{X \in \mathcal{D} \mid X \not\leq A\} \text{ and } D := \langle \mathcal{D}_0 \rangle.$$

We may assume that $A \neq G$, so $\mathcal{D}_0 \neq \emptyset$. Clearly $D \leq B$ and $\langle A, D \rangle \leq N_G(D)$. Moreover, every element of $\mathcal{D}$ is a subgroup of A or D, whence $G = \langle \mathcal{D} \rangle \leq N_G(D)$. Since $\mathcal{D}$ is a conjugacy class of G and $\mathcal{D}_0 \neq \emptyset$, this gives $G = D = B$. □

Lemma 2.2 *Let G be of characteristic p and $L \leq G$. Any of the following conditions implies that L is of characteristic p :*

- (a) $L \trianglelefteq\trianglelefteq G$.
- (b) $O_p(G) \leq L$.
- (c) $L \trianglelefteq\trianglelefteq \langle L, O_p(G) \rangle$.
- (d) $O_p(G)$ *normalizes* L.

Proof. (a): Since $L \trianglelefteq\trianglelefteq G$, $F^*(L) \leq F^*(G) = O_p(G)$.

(b): $O_p(G) \leq O_p(L)$, so $C_L(O_p(L)) \leq C_G(O_p(G)) \leq O_p(G) \leq O_p(L)$.

(c): By (b) $\langle L, O_p(G) \rangle$ has characteristic p. Thus (a) (with $\langle L, O_p(G) \rangle$ in place of G) shows that L has characteristic p.

(d): $L \trianglelefteq LO_p(G)$. So (d) follows from (c).

□

Lemma 2.3 *Let G be of characteristic p, $T \in Syl_p(G)$, and $Q \trianglelefteq T$ with $C_T(Q) \leq Q$. Suppose that L and P are subgroups of G such that $Q \leq L$ and $B(T) \leq T_0 \in Syl_p(P)$. Then the following hold:*

- (a) $C_G(Q) \leq Q$.
- (b) L *is of characteristic* p.
- (c) P *is of characteristic* p.
- (d) $C^*(P, T_0) \leq C^*(G, T)$.
- (e) *If P is minimal with respect to $T_0 \leq P$ and $P \not\leq C^*(G, T)$, then P is a minimal parabolic of characteristic p with $C^*(P, T_0) \neq P$.*

Proof. (a): Let $D := C_G(Q)$. Since $C_T(Q) \le Q$ and $O_p(G) \le T$, $C_{O_p(G)}(Q) \le Q \le C_G(D)$. So by the $P \times Q$-Lemma, $O^p(D)$ centralizes $O_p(G)$. Hence D is a p-group since G is of characteristic p. As T normalizes D, $D \le T$ and so $D \le C_T(Q) \le Q$.

(b): Let $L_0 := \langle Q^L \rangle$. Since $Q \trianglelefteq T$, $O_p(G)$ normalizes Q and so also L_0. Hence by 2.2(d) L_0 is of characteristic p. Let $C := C_L(O_p(L))$. Then $C \le C_L(O_p(L_0))$ and thus

$$[Q, C] \le L_0 \cap C \le C_{L_0}(O_p(L_0)) \le O_p(L_0) \le O_p(L) \text{ and } [Q, C, C] = 1.$$

Hence C normalizes $QO_p(L)$, so $[QO_p(L), C, C] = 1$ and $[Q, O^p(C)] = 1$. By (a) $O^p(C) = 1$, and C is a p-group. Thus $C \le O_p(L)$, and L is of characteristic p.

(c): Observe that $B(T) \trianglelefteq T$ and $C_T(B(T)) \le B(T)$. Hence (c) follows from (b).

(d): Let $T_0 \le \widetilde{T} \in Syl_p(G)$. Then $\widetilde{T} \le N_G(B(T)) \le C^*(G, T)$, so $C^*(G, T) = C^*(G, \widetilde{T})$. Thus we may assume that $T_0 \le T$. Then

$$\Omega(Z(T)) \le C_T(B(T)) \le B(T) \le T_0,$$

and so $\Omega(Z(T)) \le \Omega(Z(T_0))$. It follows that

$$C_P(\Omega(Z(T_0))) \le C_P(\Omega(Z(T))) \le C_G(\Omega(Z(T))) \le C^*(G, T).$$

Since $B(T) = B(T_0)$, we conclude that $C^*(P, T_0) \le C^*(G, T)$.

(e): From (c) and (d) we get that P is of characteristic p and

$$C^*(P, T_0) \le P \cap C^*(G, T) \ne P.$$

The minimal choice of P shows that $P \cap C^*(G, T)$ is the unique maximal subgroup of P containing T_0. As $N_P(T_0) \le N_P(B(T)) \le P \cap C^*(G, T)$, P is a minimal parabolic subgroup of G. □

Lemma 2.4 *Let $G = QN$, where N is a normal subgroup of G and Q is a non-abelian 2-subgroup with $Q \cap N = 1$. Suppose that there exists $1 \ne t \in Z(Q) \cap Q'$ such that*

$$(*) \qquad C_N(Q) = C_N(t).$$

Then $[N, Q]$ is solvable of odd order.

Proof. There exists $S \in Syl_2(N)$ such that $Q \le N_G(S)$. Let $g \in N$ such that $a := t^g \in tS$ and $[t, a] = 1$. Then

$$ta \in C_S(t) \overset{(*)}{=} C_S(Q),$$

so $[Q, a] = 1$, since $t \in Z(Q)$. Now $(*)$ implies

$$Q \le C_G(a) = Q^g \times C_N(Q^g).$$

Let Q_0 be the projection of Q in $C_N(Q^g)$. Then t centralizes Q_0, so by $(*)$ also $[Q, Q_0] = 1$. It follows that $Q' \leq Q^g \cap Q$ and $[Q', g] \leq Q'Q'^g \cap N \leq Q^g \cap N = 1$. But now $t = a$ since $t \in Q'$.

We have shown that t itself is the only conjugate of t in $\langle t \rangle S$ that commutes with t. It follows that t is not conjugate in G to any other element of $\langle t \rangle S$. Hence, Glauberman's Z^*-Theorem [6] together with $(*)$ implies that $[N, Q]$ is a group of odd order, and the Theorem of Feit-Thompson [5] yields the desired result. □

Lemma 2.5 *Let G be of characteristic p. Suppose that there exist subgroups $E \leq G$ and $N \trianglelefteq G$ with $[O^p(N), E] = 1$, $[O_p(G), E] \leq E$, and $E = O^p(E)$. Then $E \trianglelefteq N_G(EN)$.*

Proof. Let $E_0 := E[E, N_G(EN)] = \langle E^{N_G(EN)} \rangle$ and $R := E_0 \cap N$. Then

$$(*) \qquad E_0 = ER \text{ and } O^p(R) \leq Z(E_0).$$

Note that $O_p(G)$ normalizes E_0. Hence by 2.2 E_0 has characteristic p, so by $(*)$ $O^p(R) \leq O_p(E_0)$ and $O^p(R) = 1$. Thus R is a p-group. It follows that $R \leq O_p(N) \leq O_p(G)$. Then $O^p(E_0) = E_0$ and $[O_p(G), E] \leq E$ imply $R \leq E$ and $E \trianglelefteq N_G(EN)$. □

Lemma 2.6 *Let E be a group, $Q := O_3(E)$, $W := \Omega(Z(Q))$ and $Z := C_W(E)$. Suppose that the following hold:*

(i) $E/Q \cong SL_2(3^n)$.

(ii) Q/W and W/Z are natural $SL_2(3^n)$-modules for E/Q.

(iii) $Z = \Phi(Q) = Q' = Z(E)$ and $|Z| = 3^n$.

Then the image of $C_{Aut(Q)}(Z)$ in $Aut(Q/W)$ is isomorphic to $SL_2(3^n)$.

Proof. Let $W_0 := [W, E]$ and $q := 3^n$. Then $W = Z \times W_0$, and $\overline{Q} := Q/W_0$ is a special group of order q^3. Let $W \leq A \leq Q$ and $T \in Syl_3(E)$ such that $A/W = Z(T/W)$. Pick $a \in A \setminus W$. By (ii) and (iii) $[a, Q] = \langle [a, Q]^E \rangle = Q' = Z$ and thus $|Q/C_Q(a)| = q$. As also $\overline{C_Q(a)}$ is normalized by T and Q/W is a natural $SL_2(q)$-module, we get that $\overline{A} = \overline{C_Q(a)}$ and thus $A = C_Q(a)$; in particular A is abelian.

Let $\mathcal{D} := \{A^e \mid e \in E\}$. For $B \in \mathcal{D}$ we have:

$$(*) \qquad C_Q(b) = B \text{ for } b \in B \setminus W.$$

Moreover $|\mathcal{D}| = q + 1$, and the images of the elements of $\mathcal{D}$ form a partition of Q/W. This latter property together with $(*)$ shows that the elements in $\mathcal{D}$ are the only abelian subgroups of order q^4 in Q.

Pick $A, B \in \mathcal{D}$, $A \neq B$. Then $(*)$ implies

$$[a, b] \neq 1 \text{ for all } a \in A \setminus W \text{ and } b \in B \setminus W.$$

The action of E on $\overline{Q}$ shows that $C_E(\overline{A}/\overline{Z})$ acts regularly on $\mathcal{D} \setminus \{A\}$.

Now let $\alpha \in Y := C_{Aut(Q)}(Z)$. Assume that α centralizes $\overline{A}/\overline{Z}$. If α normalizes B, then for $b \in B$ and $a \in A$

$$[b, a] = [b, a]\alpha = [b\alpha, a],$$

so $b^{-1}(b\alpha) \in W$. Hence α centralizes $AB/W = Q/W$, and so

$$C_Y(\overline{A}/\overline{Z}) \cap N_Y(\overline{B}) = C_Y(Q/W).$$

With a similar argument $C_Y(\overline{a}\overline{Z}/\overline{Z}) \cap N_Y(\overline{B}) \leq C_Y(\overline{B}/\overline{Z})$, so

$$C_Y(\overline{a}\overline{Z}/\overline{Z}) \cap N_Y(\overline{B}) = C_Y(\overline{B}/\overline{Z}) \cap N_Y(\overline{A}) = C_Y(Q/W).$$

It follows that $|Y/C_Y(Q/W)| \leq q(q-1)(q+1)$, because there are $(q-1)(q+1)$ choices for $\overline{a}\overline{Z}$ and then q choices for B with $\overline{a} \notin \overline{B}$. As E induces $SL_2(q)$ on Q/W, we are done.

□

Definition 2.7 *Let V be a finite dimensional $GF(p)G$-module. Then $\mathcal{O}_G(V)$ is the set of subgroups A of G such that:*

(*i*) $[V, A] \neq 1$,

(*ii*) $|A/C_A(V)||C_V(A)| \geq |A^*/C_{A^*}(V)||C_V(A^*)|$ *for all subgroups A^* of A, and*

(*iii*) $A/C_A(V)$ *is an elementary abelian p-group.*

Moreover

$$\mathcal{O}_G^*(V) := \{A \in \mathcal{O}_G(V) \mid |A/C_A(V)||C_V(A)| > |V|\}.$$

Suppose that $\mathcal{O}_G(V) \neq \emptyset$. Then

$$m_G(V) := \max\{|A/C_A(V)||C_V(A)| \mid A \in \mathcal{O}_G(V)\},$$

and $\mathcal{A}_G(V)$ is the set of minimal (by inclusion) elements of the set

$$\{A \in \mathcal{O}_G(V) \mid |A/C_A(V)||C_V(A)| = m_G(V)\}.$$

Observe that property (*ii*) above with $A^* = 1$ gives $m_G(V) \geq |V|$.

Lemma 2.8 *Let V be a finite dimensional $GF(p)G$-module, $V_0 \leq C_V(O^p(G))$ be a $GF(p)G$-submodule, and $W \leq V$. Then the following hold for $A \in \mathcal{O}_G(V)$:*

(a) $|W/C_W(A)| \leq |A/C_A(W)|$.

(b) *Let* $A \in \mathcal{O}^*_G(V)$. *Then* $|W/C_W(A)| < |A/C_A(W)|$ *or* $C_A(W) \in \mathcal{O}^*_G(V)$.

(c) $A \in \mathcal{O}_{N_G(C_V(B))}(C_V(B))$ *for all* $B \leq A$ *with* $[C_V(B), A] \neq 1$.

(d) *Let* $O_p(G/C_G(V)) = 1$. *Then* $\mathcal{O}_G(V/V_0) \neq \emptyset$ *if* $\mathcal{O}_G(V) \neq \emptyset$, *and* $\mathcal{O}^*_G(V/V_0) \neq \emptyset$ *if* $\mathcal{O}^*_G(V) \neq \emptyset$.

(e) *Let* V *be an elementary abelian normal subgroup of* G. *Then* $\{A \in \mathcal{A}(G) \mid [A, V] \neq 1\} \subseteq \mathcal{O}_G(V)$.

Proof. (a), (b) and (c): Set $A_0 := C_A(W)$. By the definition of $\mathcal{O}_G(V)$

$$|A||C_V(A)| \geq |A_0||C_V(A_0)| \geq |A_0||WC_V(A)| = |A_0||W||C_V(A)||C_W(A)|^{-1},$$

and thus $|A/A_0| \geq |W/C_W(A)|$.

Moreover, if $A \in \mathcal{O}^*_G(V)$ and $|A/A_0| = |W/C_W(A)|$, then $|A/C_A(V)||C_V(A)| = |A_0/C_A(V)||C_V(A_0)| = m_G(V) > |V|$ and $A_0 \in \mathcal{O}^*_G(V)$.

Assume now that $W = C_V(B)$ for some $B \leq A$ and set $B^* := C_A(W)$, so $B \leq B^*$ and $C_V(B^*) = C_W(B^*)$. Then $C_W(A^*) = C_W(A^*B^*) = C_V(A^*B^*)$ for every $A^* \leq A$, so

$$|A||C_W(A)| = |A||C_V(A)| \geq |A^*B^*||C_V(A^*B^*)| \geq |A^*B^*||C_W(A^*)|$$

and

$$|A/B^*||C_W(A)| \geq |A^*B^*/B^*||C_W(A^*)| = |A^*/A^* \cap B^*||C_W(A^*)|.$$

Hence (c) follows.

(d): Let $\overline{V} := V/V_0$. Observe that $C_A(V) = C_A(\overline{V})$, since $O_p(G/C_G(V)) = 1$ and that $|\overline{V}/C_{\overline{V}}(A)| \leq |V/C_V(A)|$.

(e): Let $A \in \mathcal{A}(G)$. Then the maximality of $|A|$ gives for every $A^* \leq A$,

$$\begin{aligned} |A| = |AC_V(A)| &= |A||C_V(A)||V \cap A|^{-1} \geq |A^*C_V(A^*)| \\ &= |A^*||C_V(A^*)||V \cap A^*|^{-1} \geq |A^*||C_V(A^*)||V \cap A|^{-1}, \end{aligned}$$

and thus with $A^*C_A(V)$ in place of A^*

$$|A/C_A(V)||C_V(A)| \geq |A^*C_A(V)/C_A(V)||C_V(A^*)| = |A^*/C_{A^*}(V)||C_V(A^*)|.$$

Hence $A \in \mathcal{O}_G(V)$ if $[V, A] \neq 1$. $\square$

Notation 2.9 *In the following six lemmas we will give some elementary facts about* S_n *in its action on a natural* $GF(2)$*-module. For this purpose we fix the following notation.*

Let $G = S_n$, $n > 1$, and V^* be a $GF(2)G$-permutation module (written multiplicatively); so there exists a basis $\Omega := \{v_1, \ldots, v_n\}$ that is permuted by G. We set

$$W := \langle v_i v_j \mid 1 \leq i, j \leq n\rangle \text{ and } V_0 := \langle \prod_{i=1}^{n} v_i \rangle.$$

If n is odd, then $V := W$ is a natural $GF(2)G$-module, and if n is even, then $V := W/V_0$ is a natural $GF(2)G$-module. Furthermore we fix $T \in Syl_2(G)$, and Y is the subgroup generated by the transpositions contained in T.

Lemma 2.10 *Let $G = S_n$, $n \geq 4$. Then either*

(a) *n is even, and $N_G(Y)$ is transitive on the transpositions of G that are not in Y, or*

(b) *n is odd, and $N_G(Y)$ has two orbits on the transpositions not in Y. The elements of one orbit have a fixed point in common with Y and the elements of the other orbit do not.*

Proof. This is an elementary calculation in S_n. □

Lemma 2.11 *Let $G = S_n$, $n \geq 5$, and V be a natural $GF(2)G$-module. Then $\langle N_G(Y), C_G(C_V(T))\rangle \cong S_{n-1}$ if n is odd, and $G = \langle N_G(Y), C_G(C_V(T))\rangle$ if n is even.*

Proof. Set $M := \langle N_G(Y), C_G(C_V(T))\rangle$. Suppose first that n is odd. Then $V^* = V_0 \times V$ and

$$C_{V^*}(T) = C_V(T) \times V_0,$$

so $C_G(C_V(T)) = C_G(C_{V^*}(T))$.

There exists a unique $v \in \Omega$ such that $v \in C_{V^*}(Y)$. This element is centralized by $N_G(Y)$, and thus also by T. It follows that $\langle C_G(C_{V^*}(T)), N_G(Y)\rangle$ fixes v; in particular $M \neq G$. Since there are transpositions in $C_G(C_{V^*}(T))$ that are not in Y, 2.10 shows that M contains all transpositions that fix v. Hence $M \cong S_{n-1}$.

Suppose that n is even. It suffices to show that M contains a transposition that is not in Y. Since then by 2.10 M contains all the transpositions of G, so $M = G$.

Let $\Omega_1, \ldots, \Omega_r$ be the T-orbits of Ω, and let $\Lambda_1, \ldots, \Lambda_k$ be the proper subsets of Ω with $[\prod_{v\in\Lambda_i} v, T] \leq V_0$. Set

$$o_i := \prod_{v\in\Omega_i} v,\ i = 1, \ldots, r, \text{ and } \ell_i := \prod_{v\in\Lambda_i} v,\ i = 1 \ldots, k.$$

Then

$$C_W(T) = \langle o_1, \ldots, o_r\rangle \text{ and } C_V(T) = \langle \ell_1, \ldots, \ell_k\rangle / V_0.$$

Assume first that $C_G(C_V(T)) = C_G(C_W(T))$. Since $n \geq 5$, we may assume that $|\Omega_1| \geq 4$. Hence there exists a transposition $d \in N_G(\Omega_1) \setminus C_G(\Omega_1)$ with $d \notin Y$. Clearly $[o_i, d] = 1$ for $i = 1, \ldots, r$ and thus $d \in C_G(C_W(T)) = C_G(C_V(T)) \leq M$, so $M = G$.

Assume now that $C_G(C_V(T)) \neq C_G(C_W(T))$. Then there exists $i \in \{1, \ldots, k\}$ and $t \in T$ such that $[\ell_i, t] \neq 1$. It follows that $\Lambda_i \cup \Lambda_i^t = \Omega$, and $\{\Lambda_i, \Lambda_i^t\}$ is a T-invariant partition of Ω. In particular, every such t acts fixed-point-freely on Ω.

Observe that $\Lambda_i \cap \Omega_j \neq \emptyset$ for every $j \in \{1, \ldots, r\}$; in particular $C_T(\Omega_j) \leq N_G(\Lambda_i)$. If $r > 1$, then $C_T(\Omega_2)$ is transitive on Ω_1, so $\Omega_1 \subseteq \Lambda_i$ and consequently $T \leq N_G(\Lambda_i)$, which contradicts $t \notin N_G(\Lambda_i)$.

We have shown that T is transitive on Ω, so $[\ell_i, T] \neq 1$ for every $i \in \{1, \ldots, k\}$. Let $y \in T$ be a 4-cycle acting transitively on $\Omega_0 \subseteq \Omega$. As $n \geq 5$, y has a fixed point in Ω and thus $y \in N_G(\Lambda_i)$ (for every i). In particular either

$$\Omega_0 \subseteq \Lambda_i \text{ or } \Omega_0 \subseteq \Omega \setminus \Lambda_i.$$

In both cases for every $i \in \{1, \ldots, k\}$

$$S_4 \cong L := N_G(\Omega_0) \cap C_G(\Omega \setminus \Omega_0) \leq N_G(\Lambda_i) \leq C_G(\ell_i).$$

It follows that $L \leq C_G(C_V(T))$, but L contains transpositions that are not in Y. Again $M = G$. □

Lemma 2.12 *Let $G = S_n$, n odd, and V be a natural $GF(2)G$-module. Then the following hold:*

(*a*) $C_V(Y) = [V, Y]$.

(*b*) $C_G(C_V(Y)) = Y$.

(*c*) *Let t and t' be involutions in T. Then $t = t'$ or $C_V(t) \neq C_V(t')$.*

(*d*) *Let $d \in G$ with $d^3 = 1$ and $|[V, d]| = 4$. Then d is conjugate to* (123) *in G.*

(*e*) *If G is a minimal parabolic* (*with respect to* 2), *then $n = 2^m + 1$.*

Proof. Properties (a) – (c) are elementary consequences of the action of G on V^* and Ω.

(d): Let $v \in \Omega$ such that $[v, d] \neq 1$. Then $[V^*, d] \leq \langle v, v^d, v^{d^{-1}} \rangle$, so d fixes all but 3 elements in Ω. Hence d is conjugate to (123) in G.

(e): We may assume that $n \geq 5$, so by 2.11 n is odd. Let M be the unique maximal subgroup containing T. As n is odd, $M \cong S_{n-1}$ and M has a unique fixed point $v \in \Omega$.

Let $\Omega_1, \ldots, \Omega_r$ be the T-orbits on Ω with $\Omega_1 = \{v\}$. Then $T \leq N_G(\Omega \setminus \Omega_2) \leq M$, so $\Omega \setminus \Omega_2 = \{v\}$, and (e) follows. □

Lemma 2.13 *Let $G = S_n$, n odd, $T \in Syl_2(G)$, and U be a $GF(2)S_n$-module. Suppose that $U = [U, O^2(G)]C_U(T)$ and that $[U, O^2(G)]/C_{[U,O^2(G)]}(O^2(G))$ is a natural $GF(2)S_n$-module. Then*

$$U = C_U(O^2(G)) \times [U, O^2(G)],$$

in particular $[U, O^2(G)]$ is a natural $GF(2)S_n$-module.

Proof. Let $U_0 := C_U(O^2(G))$. It is well known that S_n is generated by $n-1$ transpositions $t_1, \ldots, t_{n-1}$ and it follows from the hypothesis that each of them acts as a transvection on U/U_0, so $|U/U_0| \leq 2^{n-1}$. As the natural $GF(2)S_n$-module has order 2^{n-1}, we conclude that $U = [U, O^2(G)]U_0$. Without loss of generality we may assume that $|U_0| = 2$.

It suffices to show that $|[U, t_i]| = 2$, since then $\langle [U, t_i] \mid i = 1, \ldots, n-1 \rangle = [U, G]$ is a $GF(2)S_n$-submodule of order at most 2^{n-1}, and as above $[U, G]$ has to be a natural $GF(2)S_n$-module.

Let c be an $(n-2)$-cycle in A_n. Then c is centralized by a transposition t. It is easy to calculate in the natural module that $|C_{U/U_0}(c)| = 4$, so

$$|C_U(c)| = 8 \text{ and } U = C_U(c) \times [U, c].$$

Then $[C_U(c)/C_U(c) \cap C_U(t)| = 2$ and $|[C_U(c), t]| = 2$. Moreover, t centralizes $[U/U_0, c]$ and thus also $[U, c]$. It follows that $|[U, t]| = 2$. □

Lemma 2.14 *Let $G = S_n$ and V be a natural $GF(2)S_n$-module for G, and let $F \leq G$ such that $F = O^2(F)$ and $[V, F]C_V(F)/C_V(F)$ is an irreducible $GF(2)F$-module. Then the following hold:*

(a) *F has a unique non-trivial orbit on Ω.*

(b) *Suppose that n is odd, $F \cong A_k$, k odd, and $[V, F]$ is a natural A_k-module for F. Then F is normalized by a conjugate of Y.*

(c) *Suppose that $F \cong SL_2(2^k)$ and $[V, F]/C_{[V,F]}(F)$ is a natural $SL_2(2^k)$-module for F. Then $k = 2$, and F has exactly $n-6$ fixed-points on Ω. In particular $[V, F]$ and $C_{[V,F]}(F)$ are normalized by a conjugate of Y.*

Proof. Observe that $C_V(F) = C_{V^*}(F)/C_{V^*}(G)$ since $F = O^2(F)$, so

$$[V^*, F]C_{V^*}(F)/C_{V^*}(F) \cong [V, F]C_V(F)/C_V(F) \cong [V, F]/C_{[V,F]}(F).$$

(a): For $v \in \Omega$, let $W_v := \langle v^F \rangle$. As Ω is a basis of V^*, we get for $v, \widetilde{v} \in \Omega$

$$W_v = W_{\widetilde{v}} \text{ and } v^F = \widetilde{v}^F \text{ or } W_v \cap W_{\widetilde{v}} = 1.$$

Now the irreducibility of $[V^*, F]C_{V^*}(F)/C_{V^*}(F)$ shows that $[W_v, F] = 1$ for all but one orbit v^F.

(b): According to (a) F has a unique non-trivial orbit $\Omega_0 \subseteq \Omega$. Set $m := |\Omega_0|$ and $W_0 := \langle \Omega_0 \rangle$. Then $|W_0| = 2^m$ and $|[W_0, F]| = 2^{m-1}$. As $[W_0, F]$ is also a natural A_k-module for F we also get that $|[W_0, F]| = 2^{k-1}$, so $k = m$. Moreover, since k and n are odd, $|\Omega \setminus \Omega_0|$ is even. Hence, there exists a conjugate of Y normalizing $\Omega \setminus \Omega_0$ and thus also F.

(c): As in the proof of (b) we define W_0 using the unique non-trivial orbit Ω_0 of F on Ω and set $m := |\Omega_0|$. Observe that $C_{W_0}(F) = \langle \prod_{w \in \Omega_0} w \rangle$ and that $[W_0, F]$ is the set of all products of an even number of elements of Ω_0. On the other hand $[W_0, F]C_{W_0}(F)/C_{W_0}(F)$ is a natural $SL_2(2^k)$-module for F, so F is transitive on the non-trivial elements of $[W_0, F]C_{W_0}(F)/C_{W_0}(F)$. It follows that every element of $[W_0, F] \setminus C_{W_0}(F)$ is either the product of $m-2$ or 2 elements of Ω_0. Since $|F| \geq 60$ we get $m \geq 5$ and $4 = m - 2$, so $m = 6$. In particular F is a subgroup of A_6 and thus $k = 2$.

We have that $[W_0, F] = [W_0, C_G(\Omega \setminus \Omega_0)]$ and $C_{[W_0,F]}(F) = C_{[W_0,F]}(C_G(\Omega \setminus \Omega_0))$. As there exists a conjugate of Y normalizing Ω_0 and $\Omega \setminus \Omega_0$, this conjugate also normalizes $[W_0, F]$ and $C_{[W_0,F]}(F)$. Now the additional statement in (c) follows.

□

Lemma 2.15 *Let $G = S_n$, $n \geq 3$ and n odd, and let V be a natural $GF(2)S_n$-module for G. Suppose that $A \in \mathcal{O}_G(V)$. Then the following hold:*

(*a*) *A is generated by commuting transpositions of G.*

(*b*) $[V, A, A] = 1$.

(*c*) $|V/C_V(A)| = |A|$.

Proof. We proceed by induction on n. The case $n = 3$ is trivial, so we assume that $n \geq 5$ and that the result holds for $n-2$. Since $V^* = V \times V_0$ we may as well calculate in V^* rather than V.

By the Timmesfeld Replacement Theorem [12] there exists $1 \neq A_0 \leq A$ such that $[V^*, A_0, A] = 1$ and $A_0 \in \mathcal{O}_G(V^*)$. Let $1 \neq a \in A_0$ and $v \in \Omega$ such that $v \neq v^a$ and let t be the transposition of G with $v^t = v^a$ and $V_t^* := C_{V^*}(t)$. Then $w := vv^a = vv^t \in C_{V^*}(A)$, so

$$A \leq C_G(w) = \langle t \rangle \times L, \; L \cong S_{n-2}.$$

Observe that $V_t^*/\langle w \rangle$ is the natural permutation module for L. Thus by induction and 2.8 $|A/C_A(V_t^*)| = |V_t^*/C_{V_t^*}(A)|$, and $A = \langle t_1, \ldots, t_r \rangle C_A(V_t^*)$, where $t_1, \ldots, t_r$ are commuting transpositions of G in L. Moreover, by 2.12 $C_A(V_t^*) \leq \langle t \rangle$, so (a) and (c) follow, and 2.12 (a) yields (b). □

Lemma 2.16 *Let V be a finite dimensional $GF(p)G$-module, $E \trianglelefteq\trianglelefteq G$, and $W := [V, E]$, and let $A \in \mathcal{O}_G(V)$ with $[E, A] \neq 1$. Suppose that*

(*i*) $E \cong SL_2(p^m)'$ *or* $p = 2$ *and* $E \cong A_{2m+1}$, *and*

(*ii*) $W/C_W(E)$ *is a natural* $SL_2(p^m)'$- *resp.* A_{2m+1}-*module for* E.

Then the following hold:

(*a*) $A \leq N_G(E)$.

(*b*) $\overline{EA} := EA/C_{EA}(W) \cong SL_2(p^m)$ *and* $\overline{A} \in Syl_p(\overline{EA})$, *or* $p = 2$, $\overline{EA} \cong S_{2m+1}$ *and* $\overline{A}$ *is generated by commuting transpositions.*

(*c*) $[W, A, A] = 1$.

(*d*) $|A/C_A(W)| = |W/C_W(A)|$.

(*e*) *For* $T \in Syl_p(EA)$ *there exists a unique maximal element* B *in* $\mathcal{O}_T(V)$, *and* $C_{EA}(C_V(B)) = B$.

Proof. We may assume that $G = \langle E, A \rangle$. Let $A_0 := C_A(E)$, $A = A_0 \times A_1$, and $V_0 := C_V(A_0)$. The $P \times Q$-Lemma shows that E acts faithfully on V_0. Moreover, $WC_V(A) \leq V_0$ since $W = [W, E]$ and $W/C_W(E)$ is an irreducible E-module. In addition, by 2.8 (c) $A_1 \in \mathcal{O}_G(V_0)$, so A_1, V_0 and E satisfy the hypothesis in place of A, V and E. Hence, we may assume $A_0 = 1$ and $V = V_0$.

(a): This follows from [4] if E is quasi-simple and from [12, 9.3.6] if E is solvable.

(b) – (e): Suppose first that $E \cong A_{2m+1}$. By (a) and 2.15 $\overline{EA} \cong S_{2m+1}$ and $W/C_W(E)$ is a natural S_{2m+1}-module. Now 2.13 yields $C_W(E) = 1$, and (b) – (d) follow. Moreover, again by 2.15, a maximal element $B \in \mathcal{O}_T(V)$ is generated by a set which corresponds to a maximal set of pairwise commuting transpositions in S_{2m+1}, so B is unique and 2.12 yields (e).

Suppose now that $E \cong SL_2(p^m)'$. As one can see in $Aut(SL_2(p^m))$, $|A| \leq p^m$ since A is abelian, so $|W/C_W(A)| \leq p^m$. On the other hand, A induces a group of semi-linear $GF(p^m)$-transformations on $W/C_W(E)$. It follows that $|W/C_W(A)| = |A| = p^m$ and $EA \cong SL_2(p^m)$. Now (b) – (e) are easy to verify. □

3 Minimal Parabolic Groups

Throughout this section we assume

Hypothesis 3.1 *Let P be a minimal parabolic group with respect to p, $T \in Syl_p(P)$, and let M be the unique maximal subgroup of P containing T.*

Lemma 3.2 (*L-Lemma*) *Let $A \leq T$ with $A \nleq O_p(P)$. Then there exists a subgroup L containing A such that the following hold:*

(*a*) *$AO_p(L)$ is contained in exactly one maximal subgroup M_0 of L, and $M_0 = L \cap M^g$ for some $g \in P$.*

(*b*) *$L = \langle A, A^x \rangle O_p(L)$ for all $x \in L \backslash M_0$.*

(*c*) *L is not contained in any P-conjugate of M.*

Proof. See [15].

□

Lemma 3.3 *Suppose $N \trianglelefteq P$. Then the following hold:*

(*a*) *If $N \leq M$, then $N \cap T \lhd P$.*

(*b*) *If $N \nleq M$, then $O^p(P) \leq N$.*

Proof. See [15, 1.3(b)]. □

Lemma 3.4 *Let N be a normal subgroup of P contained in M. Set $\overline{P} := P/N$. Then $\overline{P}$ is a minimal parabolic group and $O_p(\overline{P}) = \overline{O_p(P)}$.*

Proof. Observe that $\overline{T} \in Syl_p(\overline{P})$ and $\overline{M}$ is the unique maximal subgroup of $\overline{P}$ containing $\overline{T}$. Suppose that $\overline{T} \lhd \overline{P}$. Then TN is a normal subgroup of P. Since $TN \leq M$, Lemma 3.3 (a) gives $T = TN \cap T \lhd P$, which contradicts the assumption that P is minimal parabolic. Therefore $\overline{P}$ is a minimal parabolic group.

Let D be the inverse image of $O_p(\overline{P})$ in P. Then $D \trianglelefteq P$. Since $D \leq TN \leq M$, by Lemma 3.3 (a), we have that $D \cap T \lhd P$. Then using the Dedekind Identity, $D = (D \cap T)N \leq O_p(P)N$. Hence $O_p(\overline{P}) = \overline{D} \leq \overline{O_p(P)}$. The reverse inclusion always holds, so $O_p(\overline{P}) = \overline{O_p(P)}$. □

Lemma 3.5 *Let V be a faithful $GF(p)$-module for P. Suppose that there exists an elementary abelian subgroup $A \leq T$ such that:*

(i) $|V/C_V(A)| \leq |A|$ *and* $|A_0||C_V(A_0)| < |A||C_V(A)|$ *for every* $1 \neq A_0 < A$,

(ii) $[C_V(T), P] \neq 1$, *and*

(iii) $P = \langle A, A^x \rangle$ *for every* $x \in P \setminus M$.

Then $P \cong SL_2(q)$, $q := |A|$, $C_V(A) = [V, A]C_V(P)$, *and* $V/C_V(P)$ *is a natural* $SL_2(q)$*-module for* P.

Proof. We will use the following additional notation:

$$Z := C_V(T),\ W := \langle Z^P \rangle,\ \widetilde{V} := V/C_V(P),\ \overline{P} := P/C_P(W).$$

3.5.1 *A acts quadratically on V and* $[W, A] \neq 1$.

The first part follows from [12, 9.2.1] together with (i) and the second part follows from (ii) and (iii).

3.5.2 $O_p(\overline{P}) = C_{\overline{P}}(\widetilde{W}) = 1$ *and* $\overline{M}$ *is a maximal subgroup of* $\overline{P}$.

Note that $C_P(\widetilde{W})/C_P(W)$ is a p-group, so $C_P(\widetilde{W}) \leq O_p(\overline{P})$. Let C be the inverse image of $O_p(\overline{P})$. Then 3.3 implies that

$$C_P(W)T = P \text{ or } C \leq C_P(W)O_p(P) \leq M.$$

In the first case $P = C_P(Z)$, which contradicts (ii). In the second case $C = C_P(W)$, since $O_p(P) \leq C_P(W)$, so $O_p(\overline{P}) = 1$. Moreover, $\overline{M}$ is a maximal subgroup of $\overline{P}$, since $C \leq M$.

3.5.3 $C_{\widetilde{W}}(P) = 1$ *and* $\widetilde{C_W(A)} = C_{\widetilde{W}}(A)$.

Let $x \in P \setminus M$ and put $B := A^x$, so $P = \langle A, B \rangle$ by (iii). The quadratic action of A implies that

$$W = [W, A][W, B]Z \leq C_W(A)C_W(B) \leq W,$$

and we must have equality. Therefore

$$\widetilde{W} = \widetilde{C_W(A)}\widetilde{C_W(B)} \text{ and } \widetilde{C_W(A)} \cap \widetilde{C_W(B)} = \widetilde{C_W(P)} = 1.$$

As A and B are conjugate in P, we also get that

$$C_{\widetilde{W}}(A) \cap \widetilde{C_W(B)} = 1, \text{ and thus } \widetilde{C_W(A)} = C_{\widetilde{W}}(A).$$

Now $C_{\widetilde{W}}(P) = 1$ follows.

3.5.4 $|\widetilde{W}/C_{\widetilde{W}}(\overline{A})| \leq |\overline{A}|$.

Let $A_0 := C_A(\widetilde{W})$. By 3.5.2 $[A_0, W] = 1$. Hence (i) gives

$$|A_0||WC_V(A)| \leq |A_0||C_V(A_0)| \leq |A||C_V(A)|. \tag{1}$$

This shows that

$$|\widetilde{W}/C_{\widetilde{W}}(A)| \leq |W/C_W(A)| \leq |A/A_0|. \tag{2}$$

3.5.5 *There exists a field K with $|K| = |\overline{A}|$ such that $\widetilde{W}$ is a 2-dimensional vector space over K and $\overline{P} = SL(\widetilde{W}, K)$.*

According to 3.5.1 – 3.5.4 and (iii), $\overline{P}$ satisfies the hypothesis of [7, Theorem 2] and 3.5.5 follows from this theorem.

From 3.5.3 and 3.5.5 we get that

$$|W/C_W(A)| = |\overline{A}| \text{ and } C_W(A) = [W, A]C_W(P).$$

Hence (1) and (2) give

$$|A_0||WC_V(A)| = |A_0||C_V(A_0)| = |A||C_V(A)|,$$

so by (i), $A_0 = 1$, $|\overline{A}| = |A|$, and $|V/C_V(A)| = |A|$. From (iii) we get that $V = WC_V(P)$ and then $V = W$ since $C_V(P) \leq C_V(T) \leq W$. In particular $C_P(W) = C_P(V) = 1$, so $\overline{P} = P$. □

Theorem 3.6 *Let V be a faithful $GF(p)$-module for P. Suppose that $O_p(P) = 1$, $\mathcal{A}_P(V) \neq \emptyset$, and $C_P(C_V(T)) \leq M$. Then for every $A \in \mathcal{A}_P(V)$ there exists a subgroup $L_0 \leq P$ with $A \leq L_0$ such that the following hold:*

(*a*) $[V, A, A] = 1$.

(*b*) $L_0 \cong SL_2(q)$, $q := |A|$, $V/C_V(L_0)$ *is a natural* $SL_2(q)$*-module for* L_0*, and* $C_V(A) = [V, A]C_V(L_0)$*; in particular* $|V/C_V(A)| = |A|$.

(*c*) $C_V(A) = C_V(a)$ *for every* $a \in A^\sharp$.

(*d*) $|V/C_V(AB)| = |A||B|$ *for every* $B \in \mathcal{A}_P(V) \setminus \{A\}$ *with* $[A, B] = 1$.

Proof. Let $A \in \mathcal{A}_P(V)$. Then the maximality of $|A||C_V(A)|$ and minimality of A give

3.6.1 $|V/C_V(A)| \leq |A|$, *and* $|A_0||C_V(A_0)| < |A||C_V(A)|$ *for every* $1 \neq A_0 < A$.

We now apply the L-Lemma 3.2. Then there exists $A \leq L \leq P$ and $g \in P$ such that

3.6.2 *$L \cap M^g$ is the unique maximal subgroup of L containing $AO_p(L)$.*

3.6.3 *$L = \langle A, A^x \rangle O_p(L)$ for every $x \in L \setminus M^g$.*

3.6.4 *L is not contained in any P-conjugate of M.*

Among all $x \in L \setminus M^g$ we choose $B := A^x$ such that $L_0 := \langle A, B \rangle$ is minimal. We prove next:

3.6.5 *L_0 is minimal parabolic, and L_0 and V satisfy the hypothesis of* 3.5.

The first part of 3.6.5 follows from the fact that L is minimal parabolic by 3.6.3 and that $L = L_0 O_p(L)$. Hypothesis (i) of 3.5 follows from 3.6.1 and Hypothesis (iii) follows from the definition of L_0. Let $T_0 \in Syl_p(L_0)$ with $T_0 \leq T$ and suppose $[L_0, C_V(T_0)] = 1$. Then

$$L = O_p(L)L_0 \leq O_p(L)C_L(C_V(T_0)) \leq C_L(C_V(T)) \leq L \cap M,$$

which contradicts 3.6.4. Thus Hypothesis (ii) of 3.5 holds.

Now properties (a) – (c) follow from 3.5 and elementary properties of the natural $SL_2(q)$-module.

For the proof of (d), let $B \in \mathcal{A}_P(V)$ such that $[A, B] = 1$. If $A \cap B \neq 1$, then by (iii), $C_V(A) = C_V(B)$ and the maximality of $|A||C_V(A)|$ shows that $A = B$. If $A \cap B = 1$, then the maximality of $|A||C_V(A)| = |B||C_V(B)| = |V|$ gives

$$|AB| \leq |V/C_V(AB)| = |V/C_V(A) \cap C_V(B)| \leq |V/C_V(A)||V/C_V(B)| = |A||B| = |AB|.$$

□

Lemma 3.7 *Let P be of characteristic p and $W := \Omega(Z(O_p(P)))$. Suppose that neither $\Omega(Z(T))$ nor $B(T)$ is normal in P and that $P/C_P(W) \cong SL_2(p^n)$. Then $B(T) \in Syl_p(\langle B(T)^P \rangle)$ and $\Omega(Z(B(T)))W \trianglelefteq P$.*

Proof. See [15, 2.7]. □

Lemma 3.8 *Let G be of characteristic p, $C^*(G,T) \neq G$ for $T \in Syl_p(G)$, and $V \trianglelefteq G$ with*

$$\Omega(Z(T)) \leq V \leq \Omega(Z(O_p(G))).$$

Suppose that $G/C_G(V) \cong SL_2(p^n)$ or S_{2m+1} (with $p = 2$) and $V/C_V(G)$ is a natural $SL_2(p^n)$- resp. S_{2m+1}-module for $G/C_G(V)$. Then there exists a $B(T)$-block E of G such that $G = B(T)EC_G(V)$ and $[E, \Omega(Z(B(T)))] \leq V$.

Proof. Let $E := O^p(G)$. Assume first that $G/C_G(V) \cong SL_2(p^n)$. Clearly $C_G(V) \leq C^*(G,T)$ and with the Frattini argument $B(T) \not\leq C_G(V)$. Then $B(T)C_G(V) = TC_G(V)$, and $N_G(T)C_G(V)$ is the unique a maximal subgroup of G that contains $B(T)C_G(V)$; in particular $C^*(G,T) = N_G(B(T))C_G(V)$. By 2.3 (e) there exists a minimal parabolic subgroup P of characteristic p in G such that

$$B(T) \leq T_0 \in Syl_p(P),\ P \neq C^*(P,T_0),\ \text{and}\ P \not\leq C^*(G,T).$$

Thus $PC_G(V) \not\leq N_G(B(T))C_G(V)$ and $PC_G(V) = G$. So we may assume without loss that $P = G$ and by 3.7 that also $B(T) = T_0$ and $[O^p(P), \Omega(Z(B(T)))] \leq V$. In particular, no non-trivial characteristic subgroup of T_0 is normal in P. Now a standard pushing up result, see for example [16], shows that $O^p(P)$ is a $B(T)$-block and the result holds with $E := O^p(P)$.

Assume now that $p = 2$ and $\overline{G} := G/C_G(V) \cong S_{2m+1}$. Again by the Frattini argument $B(T) \not\leq C_G(V)$, so 2.8 (e) yields $A \in \mathcal{O}_G(V)$ with $[O^2(G), A] \neq 1$. Then by 2.16 $\overline{B(T)}$ is generated by a maximal set of pairwise commuting transpositions $\overline{t}_1, \ldots, \overline{t}_m$. Since $2m+1$ is odd, for every i there exists a 3-cycle $\overline{d}_i$ such that $[\overline{d}_i, \overline{t}_j] = 1$ for $i \neq j$ and

$$\langle \overline{d}_i, \overline{t}_i \rangle \cong S_3 \cong SL_2(2).$$

Let L_i be the inverse image of $\overline{B(T)}\langle \overline{d}_i, \overline{t}_i \rangle$ in G and $G_0 := \langle L_1, \ldots, L_m \rangle$. Then $G = G_0C_G(V)$; in particular $L_i \not\leq C^*(G,T)$ for $i = 1, \ldots, m$.

Now 2.3 shows that L_i satisfies the hypothesis with $L_i/C_{L_i}([V, L_i]) \cong SL_2(2)$. Hence, there exists a $B(T)$-block $E_i \leq L_i$ and $[\Omega(Z(B(T))), E_i] \leq V$. Let $E = \langle E_1, \ldots, E_m \rangle$. Then $[E, O_2(G)\Omega(Z(B(T)))] \leq V$ and thus $C_E(V) \leq O_2(G)$ since G is of characteristic 2. It follows that E is a $B(T)$-block with $E/O_2(E) \cong A_{2m+1}$ and $G = B(T)EC_G(V)$. □

4 Conjugacy Classes of Transvections

In this section we will work with the following hypotheses:

Hypothesis 4.1 *Let P be a group acting faithfully on an elementary abelian p-group V. Suppose that there exists a normal set*[1] *$\mathcal{D}$ of non-trivial elementary abelian p-subgroups of P such that the following hold for $A \in \mathcal{D}$:*

(*i*) $[V, A, A] = 1$.

(*ii*) $|V/C_V(A)| = |A|$ *and* $C_V(A) = C_V(a)$ *for every* $a \in A^\sharp$.

(*iii*) $|V/C_V(AB)| = |A||B|$ *for every* $B \in \mathcal{D}$ *with* $B \neq A$ *and* $[A, B] = 1$.

For $U \leq P$ we set

$$\mathcal{D} \cap U := \{A \mid A \in \mathcal{D}, A \leq U\} \text{ and } \mathcal{D}_P(U) := \cap_{g \in P}(\mathcal{D} \cap U^g).$$

Hypothesis 4.2 *Assume Hypothesis* 4.1 *and, in addition, that $T \in Syl_p(P)$ and $T \leq M \leq P$ with $\mathcal{D} \neq \mathcal{D}_P(M)$ such that*

(∗) $N_P(\mathcal{D} \cap T) \leq M$ *and* $C_P(C_V(T)) \leq M$.

Hypothesis 4.3 *Assume Hypothesis* 4.2 *and in addition that*

(∗∗) $|A||C_V(A)| \geq |X||C_V(X)|$ *for every $A \in \mathcal{D}$ and every elementary abelian p-subgroup $X \leq P$.*

Notation 4.4 *Assume Hypothesis* 4.2*. For $A \in \mathcal{D}$ we set*

$$\mathcal{M}(A) := \{M^g \mid g \in P,\ A \leq M^g\}.$$

By Λ we denote the set of all subgroups $L \leq P$ such that

(1) $L \cong SL_2(q)$ *and* $V/C_V(L)$ *is a natural* $SL_2(q)$*-module for* L,

(2) $\mathcal{D} \cap L$ *is the set of Sylow p-subgroups of* L,

(3) $\mathcal{M}(A) \neq \mathcal{M}(B)$ *for* $A \neq B \in \mathcal{D} \cap L$.

Moreover $\Lambda(A) := \{L \in \Lambda \mid A \leq L\}$.

Lemma 4.5 *Assume Hypothesis* 4.1*. Let $A, B \in \mathcal{D}$. Then $A = B$ or $A \cap B = 1$.*

---
[1] i.e., invariant under conjugation by G.

Proof. Let $x \in A \cap B$. Suppose that $x \neq 1$. By 4.1(ii)

$$C_V(A) = C_V(x) = C_V(B).$$

Now 4.1(i) gives $[V, A, B] \leq [C_V(A), B] = 1$ and similarly $[B, V, A] = 1$, so the Three Subgroups Lemma yields $[A, B, V] = 1$. Therefore $[A, B] = 1$, because P acts faithfully on V. Thus 4.1(iii) gives the result. □

Lemma 4.6 *Assume Hypothesis* 4.1. *Let $A, B \in \mathcal{D}$ such that $A \neq B$ and $[A, B] = 1$. Then $V = C_V(A)C_V(B)$ and AB acts quadratically on V.*

Proof. We have

$$|A||B||C_V(AB)| \overset{4.5}{=} |AB||C_V(AB)| \overset{4.1}{=} |A||C_V(A)|.$$

Hence

$$|B| = |C_V(A)/C_V(AB)| = |C_V(A)C_V(B)/C_V(B)| \leq |V/C_V(B)| \overset{4.1}{=} |B|,$$

and thus $V = C_V(A)C_V(B)$. In particular

$$[V, A] = [C_V(B), A] \leq C_V(B) \cap C_V(A)$$

and similarly $[V, B] \leq C_V(A) \cap C_V(B)$. □

Lemma 4.7 *Assume Hypothesis* 4.1. *Then $\langle \mathcal{D} \cap T \rangle$ is elementary abelian, and $\langle \mathcal{D} \cap T \rangle$ acts quadratically on V.*

Proof. If $\langle \mathcal{D} \cap T \rangle$ is abelian, then by 4.6 it also acts quadratically on V. Thus, it suffices to show that $\langle \mathcal{D} \cap T \rangle$ is abelian.

Suppose on the contrary that $\langle \mathcal{D} \cap T \rangle$ is not elementary abelian. Then there exist $A_1, A_2 \in \mathcal{D} \cap T$ with $[A_1, A_2] \neq 1$; in particular $A_1 \neq A_2$. Choose $\langle A_1, A_2 \rangle$ minimal with this property.

Since a p-group cannot be generated by conjugates of a proper subgroup, we have

4.7.1 $\langle A_1^{A_2} \rangle \neq \langle A_1, A_2 \rangle \neq \langle A_2^{A_1} \rangle$.

Then by the minimality of $\langle A_1, A_2 \rangle$:

4.7.2 $\langle A_1^{A_2} \rangle$ *and* $\langle A_2^{A_1} \rangle$ *are elementary abelian.*

If $A_1 \leq N_T(A_2)$ and $A_2 \leq N_T(A_1)$ then by 4.5

$$[A_1, A_2] \leq A_1 \cap A_2 = 1,$$

and $\langle A_1, A_2 \rangle$ is elementary abelian, which is a contradiction. Thus we may assume without loss that $A_2 \not\leq N_T(A_1)$.

Pick $a \in A_2 \setminus N_T(A_1)$. Then 4.7.2 and 4.6 show that

$$V = C_V(A_1)C_V(A_1^a) = C_V(A_1)C_V(A_1)^a = C_V(A_1)[V, a].$$

Since A_2 acts quadratically on V, we get

$$V = C_V(A_1)C_V(A_1^a) = C_V(A_1)C_V(A_2).$$

Observe that $C_V(A_1) \cap C_V(A_2) \leq C_V(A_1) \cap C_V(A_1^a)$. So 4.1 gives

4.7.3 $|A_2| = |V/C_V(A_2)| = \frac{|C_V(A_1)|}{|C_V(A_1) \cap C_V(A_2)|} \geq \frac{|C_V(A_1)|}{|C_V(A_1) \cap C_V(A_1^a)|} = |V/C_V(A_1^a)| = |A_1|.$

If also $A_1 \not\leq N_T(A_2)$, then a symmetric argument shows $|A_1| \leq |A_2|$, so $|A_1| = |A_2|$. If $A_1 \leq N_T(A_2)$, then $A_1A_1^a \cap A_2 \neq 1$ and by 4.1

$$C_V(A_1) \cap C_V(A_2) \leq C_V(A_1) \cap C_V(A_1^a) \leq C_V(A_1) \cap C_V(A_1A_1^a \cap A_2) = C_V(A_1) \cap C_V(A_2),$$

so $C_V(A_1) \cap C_V(A_2) = C_V(A_1) \cap C_V(A_1^a)$. This gives equality in 4.7.3 and again $|A_1| = |A_2|$. But then $A_1A_1^a = A_1A_2$, which contradicts 4.7.1. We have shown:

4.7.4 $|A_1| = |A_2|$ *and also* $A_1 \not\leq N_T(A_2)$.

Pick $b \in A_1 \setminus N_T(A_2)$. By 4.1 and 4.7.4

$$|V/C_V(A_1) \cap C_V(A_2)| \leq |A_1||A_2| = |A_1|^2 = |V/C_V(A_1) \cap C_V(A_1^a)|,$$

This gives $C_V(A_1) \cap C_V(A_2) = C_V(A_1) \cap C_V(A_1^a)$ and with a symmetric argument $C_V(A_1) \cap C_V(A_2) = C_V(A_2) \cap C_V(A_2^b)$.

On the other hand, by 4.7.2 and 4.6 both subgroups $A_1A_1^a$ and $A_2A_2^b$ act quadratically on V, so

$$[V, A_1] \leq C_V(A_1) \cap C_V(A_1^a) = C_V(A_1) \cap C_V(A_2)$$

and

$$[V, A_2] \leq C_V(A_2) \cap C_V(A_2^b) = C_V(A_1) \cap C_V(A_2).$$

It follows that $[V, A_1, A_2] = [V, A_2, A_1] = 1$, and the Three Subgroups Lemma yields $[A_1, A_2, V] = 1$. But then $[A_1, A_2] = 1$ since P is faithful on V, a contradiction. $\square$

Lemma 4.8 *Assume Hypothesis* 4.1. *Let* $A, B \in \mathcal{D}$ *such that* $[A, B] \neq 1$ *and set* $L := \langle A, B \rangle$. *Then for every* $C \in \mathcal{D} \cap L$ *with* $[C, A] = 1$ *either* $C \leq Z(L)$ *or* $C = A$. *In particular, for* $X, Y \in \mathcal{D}$ *either* X *and* Y *are conjugate in* $\langle X, Y \rangle$, *or* $[X, Y] = 1$.

Proof. Let L be a counterexample, so there exists $C \in \mathcal{D} \cap L$ such that $[C, A] = 1$ but $C \neq A$ and $[C, B] \neq 1$.

Assume first that C is conjugate to B. Then $|C| = |B|$, and 4.1 (iii) implies

$$|V/C_V(AC)| = |A||B|.$$

On the other hand by 4.1 (ii) $|V/C_V(L)| \leq |A||B|$, so we get that $C_V(L) = C_V(AC)$. Now 4.6 shows that $[V, A] \leq C_V(L)$. Hence $\langle A^L \rangle$ acts quadratically on V and $A \leq O_p(L)$. But then by 4.7 $[A, B] = 1$, a contradiction.

Assume now that C is not conjugate to B. Then there exists a Sylow p-subgroup of $L_0 := \langle C, B \rangle$ that contains B and a conjugate C^* of C; in particular by 4.7 $[C^*, B] = 1$. With the same argument as in the first case, this time applied to L_0, we get $C_V(L_0) = C_V(C^*B)$ and then $[V, B] \leq C_V(L_0)$, so as above $[C, B] = 1$, a contradiction.

We have shown that L has the desired properties. Let $x \in L$ such that $\langle B^x, A \rangle$ is a p-group. Then 4.7 implies $[B^x, A] = 1$ and thus $A = B^x$ since $B^x \not\leq Z(L)$. Now the second part of the assertion follows. □

Lemma 4.9 *Assume Hypothesis* 4.2. *Let $H \leq P$ such that $\mathcal{D} \cap T \subseteq \mathcal{D} \cap H$ and $\mathcal{D} \cap H \not\subseteq \mathcal{D} \cap M$. Then H satisfies Hypothesis* 4.2 *with respect to $\mathcal{D} \cap H$ and $M \cap H$.*

Proof. Let $T_0 \in Syl_p(H)$ such that $\mathcal{D} \cap T = \mathcal{D} \cap T_0$. Then $N_H(\mathcal{D} \cap T_0) \leq M \cap H$; in particular $T_0 \leq M$ and $T_0 \leq T^g$ for some $g \in M$. It follows that

$$C_V(T^g) \leq C_V(T_0) \text{ and } C_H(C_V(T_0)) \leq C_H(C_V(T^g)) \leq M \cap H.$$

□

Lemma 4.10 *Assume Hypothesis* 4.2. *Let $\mathcal{D}_0 \subseteq \mathcal{D}$ be a normal subset of P such that $\mathcal{D}_0 \not\subseteq \mathcal{D}_P(M)$. Then $\langle \mathcal{D}_0 \rangle$ satisfies Hypothesis* 4.2 *with respect to $\mathcal{D}_0$ and $M \cap \langle \mathcal{D}_0 \rangle$.*

Proof. Let $\mathcal{D}_1 := \mathcal{D} \setminus \mathcal{D}_0$, $P_0 := \langle \mathcal{D}_0 \rangle$, and $T_0 := P_0 \cap T$. Observe that by 4.8 $[P_0, \langle \mathcal{D}_1 \rangle] = 1$; in particular

$$\mathcal{D} \cap T = (\mathcal{D}_0 \cap T_0) \cup C_{\mathcal{D} \cap T}(P_0).$$

It follows that

$$N_{P_0}(\mathcal{D}_0 \cap T_0) \leq N_{P_0}(\mathcal{D} \cap T) \leq M \cap P_0.$$

As also

$$C_{P_0}(C_V(T_0)) \leq C_{P_0}(C_V(T)) \leq M \cap P_0,$$

the claim now follows from the fact that $\mathcal{D}_0 \not\subseteq \mathcal{D}_P(M)$. □

Lemma 4.11 *Assume Hypothesis* 4.2. *Let* $\mathcal{T}_0 \subseteq \mathcal{D} \cap T$ *be maximal* (*by inclusion*) *such that* $N := N_P(\mathcal{T}_0) \not\leq M$. *Then*

$$\mathcal{D} \cap N \neq \mathcal{D} \cap N \cap M \text{ and } \mathcal{D} \cap N \cap M \neq \mathcal{T}_0,$$

and $\langle A, B\rangle \in \Lambda$ *for every* $A \in (\mathcal{D} \cap M \cap N) \setminus \mathcal{T}_0$ *and* $B \in (\mathcal{D} \cap N) \setminus (\mathcal{D} \cap M)$.

Proof. Set $\mathcal{T} := \mathcal{D} \cap T$. Recall from 4.7 that the elements in $\mathcal{T}$ centralize each other, and from 4.2 that $N_P(T) \leq N_P(\mathcal{T}) \leq M$. The Frattini argument shows that the only P-conjugate of M containing $\mathcal{T}$ is M itself. Let $\mathcal{T}_1 \subseteq \mathcal{T}$. As $\mathcal{T} \subseteq N_P(\mathcal{T}_1)$, an elementary argument gives

4.11.1 *Either* $N_P(\mathcal{T}_1) \not\leq M$, *or* M *is the unique conjugate of* M *containing* $\mathcal{T}_1$.

In particular $\mathcal{D} \cap N \neq \mathcal{D} \cap N \cap M$, and $\mathcal{D} \cap N \cap M \neq \mathcal{T}_0$. Let

$$A \in (\mathcal{D} \cap N \cap M) \setminus \mathcal{T}_0,\ B \in (\mathcal{D} \cap N) \setminus (\mathcal{D} \cap M) \text{ and } L := \langle A, B\rangle$$

such that L is a minimal counterexample. We also set

$$\mathcal{D}^* := A^L \text{ and } H := L \cap M.$$

As $N_L(A) \leq N_P(\mathcal{T}_0 \cup \{A\})$, the maximality of $\mathcal{T}_0$ and 4.11.1 imply:

4.11.2 H *is the unique* L*-conjugate of* H *containing* A*; in addition*

$$N_L(A) \leq H,\ N_L(H) = H \text{ and } [A, B] \neq 1.$$

By 4.8 A and B are conjugate in L, so $q := |A| = |B|$. We now divide the proof into two cases.

4.11.3 *Case I* : *There exists* $X \in \mathcal{D}^*$ *such that* $L_0 := \langle A, X\rangle < L$ *and* $X \not\leq H$.

The minimality of L shows that $L_0 \in \Lambda$; in particular $L_0 \cong SL_2(q)$ and $V/C_V(L_0)$ is a natural $SL_2(q)$-module for L_0. By 4.1 (ii) $|V/C_V(L)| \leq |A||B| = q^2$ while $|V/C_V(L_0)| = q^2$. Since $C_V(L_0) \geq C_V(L)$ we get that $C_V(L) = C_V(L_0)$.

Let $A_0, \ldots, A_q$ be the Sylow p-subgroups of L_0 with $A_0 := A$. As $V/C_V(L_0)$ is a natural $SL_2(q)$-module, the groups $C_V(A_i)/C_V(L_0)$, $i = 0, \ldots, q$, form a partition of $V/C_V(L_0)$. Thus, there exists $i \in \{0, \ldots, q\}$ such that

$$C_V(L) = C_V(L_0) < C_V(B) \cap C_V(A_i).$$

Let $L_i := \langle A_i, B\rangle$. Then $C_V(L) < C_V(L_i)$, so $L_i < L$. The minimality of L shows that either $V/C_V(L_i)$ is a natural $SL_2(q)$-module, or $L_i \leq H^x$ where $x \in L$ with $B \leq H^x$.

The first case contradicts $|V/C_V(L_i)| < |V/C_V(L)| = q^2$. In the second case i is uniquely determined since any two different Sylow p-subgroups generate L_0 and $A \not\leq H^x$ by 4.11.2. It follows that $C_V(A_i) = C_V(B)$; in particular $[A_i, B] = 1$. Hence 4.1 (iii) yields $A_i = B$ and $L = L_0$. But then L is not a counterexample.

4.11.4 *Case II* : $X \leq H$ *for every* $X \in \mathcal{D}^*$ *with* $\langle A, X\rangle < L$.

Let $T_0 \in Syl_p(L)$ with $A \leq T_0$, and let $x \in L \setminus H$. By 4.11.2 $A^x \leq H$ implies $x \in H$. As we are in Case II, this shows that

$$L = \langle A, A^x\rangle \text{ for every } x \in L \setminus H.$$

By 4.11.2 $T_0 \leq T^h \leq M$, for some $h \in H$, so $C_V(T^h) \leq C_V(T_0)$, and thus by 4.2

$$C_L(C_V(T_0)) \leq C_L(C_V(T^h)) \leq H.$$

Now (A, L, H) satisfies the hypothesis of 3.5 in place of (A, P, M) and L is not a counterexample. □

Lemma 4.12 *Assume Hypothesis* 4.2. *For every* $A \in \mathcal{D} \setminus \mathcal{D}_P(M)$ *there exists* $g \in P$ *and* $L \in \Lambda(A)$ *such that* $A \leq M^g$ *and* $L \not\leq M^g$. *In particular* $\Lambda(A) \neq \emptyset$.

Proof. Let $\mathcal{D}_0$ be the set of all $A \in \mathcal{D}$ such that there exists a $g \in P$ and $L \in \Lambda(A)$ such that $A \leq M^g$ and $L \not\leq M^g$. We set

$$\mathcal{D}^* := \mathcal{D}_0 \cup \mathcal{D}_P(M) \text{ and } \mathcal{D}_* := \mathcal{D} \setminus \mathcal{D}^*.$$

We have to show that $\mathcal{D} = \mathcal{D}^*$.

Observe that $\mathcal{D}^*$ and $\mathcal{D}_*$ are normal sets in P, so no element of $\mathcal{D}_*$ is conjugate to an element of $\mathcal{D}^*$. Hence 4.8 shows that the elements of $\mathcal{D}_*$ centralize the elements of $\mathcal{D}^*$.

From now on we assume that $\mathcal{D}_* \neq \emptyset$ and derive a contradiction. Let $\mathcal{T}_1 := \mathcal{D}^* \cap T$. Then $\mathcal{D}_* \subseteq N_P(\mathcal{T}_1)$, so $N_P(\mathcal{T}_1) \not\leq M$, since $\mathcal{D}_*$ is a normal set. We now choose $\mathcal{T}_0 \subseteq \mathcal{D} \cap T$ maximal with respect to $\mathcal{T}_1 \subseteq \mathcal{T}_0$ and $N_P(\mathcal{T}_0) \not\leq M$. Observe that $\mathcal{D}^* \cap N_P(\mathcal{T}_0) = \mathcal{D}^* \cap T = \mathcal{T}_1$.

According to 4.11 there exist $A \in (\mathcal{D} \cap M \cap N_P(\mathcal{T}_0)) \setminus \mathcal{T}_0$ and $L \in \Lambda(A)$ such that $L \leq N_P(\mathcal{T}_0)$ and $L \not\leq M$; in particular $A \in \mathcal{D}^*$. It follows that $A \in \mathcal{D}^* \cap N_P(\mathcal{T}_0) = \mathcal{T}_1 \subseteq \mathcal{T}_0$, a contradiction.

□

Lemma 4.13 *Assume Hypothesis* 4.2. *Let* $L \in \Lambda$ *and* $B \in \mathcal{D}$ *such that* $[L, B] \neq 1$ *and* $B \not\leq L$. *Then there exists a unique* $A \in \mathcal{D} \cap L$ *such that the following hold for* $L^* := \langle L, B\rangle$, $q := |A|$ *and* $\overline{V} := V/C_V(L^*)$:

(*a*) $[A, B] = 1$,

(*b*) $C_V(L^*)[V, A] = C_V(L^*)[V, B] = C_V(AB)$,

(*c*) $|[\overline{V}, A]| = q$,

(*d*) $|\overline{V}| = q^3$, *and*

(*e*) $[\overline{V}, L] = [\overline{V}, L^*]$ *is a natural* $SL_2(q)$*-module for* L *invariant under* L^*.

Proof. Recall that $L \cong SL_2(q)$ and $V/C_V(L)$ is a natural $SL_2(q)$-module for L. Let $A_0, \ldots, A_q$ be the $q+1$ Sylow p-subgroups of L. We set

$$q_0 := |\overline{C_V(L)}|, L_i := \langle A_i, B\rangle, \text{and } V_i := C_V(L_i), \text{ for } i = 0, \ldots, q.$$

At least one of the groups L_i is non-abelian, so 4.8 implies that A_i is conjugate to B in L^*. In particular $|B| = q$ and $B \notin \mathcal{D}_P(M)$. From 4.1 we get that $|V/V_i| \leq q^2$ and $|\overline{V}| = q^2 q_0 \leq q^3$, so

$$|\overline{V}_i| \geq q_0 \text{ and } |\overline{C_V(A_i)}| = |\overline{C_V(B)}| = qq_0.$$

Suppose that (a) holds for some $A \in \mathcal{D} \cap L$. Then as L is generated by any two of its Sylow p-subgroups, A must be the unique element of $\mathcal{D} \cap L$ which commutes with B. Furthermore, by 4.6 we get $[V, A][V, B] \leq C_V(AB)$ and $|V/C_V(AB)| = q^2$. Since $V/C_V(L)$ is a natural $SL_2(q)$-module for L, this forces $q_0 = q = |[\overline{V}, A]| = |[\overline{V}, B]|$ and (b) - (e) hold.

It suffices to prove that (a) holds for some $A \in \mathcal{D} \cap L$, so we assume that $[A, B] \neq 1$ for all $A \in \mathcal{D} \cap L$ and aim for a contradiction.

Since $V/C_V(L)$ is a natural $SL_2(q)$ module for L and $|V/C_V(A_i)| = q$, the subgroups $C_V(A_i)/C_V(L)$, $0 \leq i \leq q$, form a partition of $V/C_V(L)$. Thus

$$(*) \qquad V = \bigcup_{i=0}^{q} C_V(A_i).$$

Hence for each $b \in B^\sharp$ there exists a $j \in \{0, \ldots, q\}$ with $[V, b] \cap C_V(A_j) \neq 1$. Note that B and so also L_j centralizes $[V, b] \cap C_V(A_j)$. As B and A_j are conjugate in L_j we get $[V, b] \cap C_V(A_j) \leq [V, b] \cap [V, A_j]$. Thus, we have:

4.13.1 *For every $b \in B^\sharp$, there exists $j \in \{0, \ldots, q\}$ such that $[V, b] \cap [V, A_j] \neq 1$.*

It follows from 4.6 that $1 \neq [V, b] \cap [V, \mathcal{D} \cap T^g] \leq C_V(\mathcal{D} \cap T^g)$, where $A_j \leq T^g$. Assume that there exists $M_j \in \mathcal{M}(A_j) \setminus \mathcal{M}(B)$. By 4.9 $H := C_P(C_V(\mathcal{D} \cap T^g) \cap [V, b])$ satisfies Hypothesis 4.2 with respect to $H \cap M_j$. But then by 4.12 there exists $\hat{L} \in \Lambda(B)$ with $\hat{L} \leq H$. By considering the action of $\hat{L}$ on the natural $SL_2(q)$-module $V/C_V(\hat{L})$ we get $[V, b] \cap C_V(\hat{L}) = 1$, which contradicts $[V, b] \cap C_V(\mathcal{D} \cap T^g) \leq C_V(H)$. We have shown that $\mathcal{M}(A_j) \subseteq \mathcal{M}(B)$, so $\mathcal{M}(B) = \mathcal{M}(A_j)$, since A_j and B are conjugate. Recall that $\mathcal{M}(A_j) \neq \mathcal{M}(D)$ for every $A_j \neq D \in \mathcal{D} \cap L$. Hence

4.13.2 $C_V(A_i) \cap [V, b] = 1$ *for every $i \neq j$ and $b \in B^\sharp$.*

On the other hand, by 4.1 $|[V, b]| = q$. As the subgroups $C_V(X)/C_V(L)$, $X \in \mathcal{D} \cap L$, form a partition of $V/C_V(L)$, $(*)$ implies that $[V, b] \leq C_V(A_j)$ for every $b \in B^\sharp$. Using the Three Subgroups Lemma and the faithful action of P on V this gives $[A_j, B] = 1$, which is a contradiction. □

Theorem 4.14 *Assume Hypothesis 4.2 and $|\mathcal{D} \cap T| = 1$. Then $\langle \mathcal{D} \rangle \cong SL_2(q)$, $q = |A|$, and $V/C_V(\langle \mathcal{D} \rangle)$ is a natural $SL_2(q)$-module.*

Proof. By 4.12 there exists $L \in \Lambda$ and by 4.13 $L = \langle \mathcal{D} \rangle$.

□

Lemma 4.15 *Assume Hypothesis* 4.3. *Let* $A, B \in \mathcal{D} \cap T$ *and* $L \in \Lambda(A)$ *with* $L \not\leq M$ *and* $A \neq B$. *Then* $[L, B] = 1$.

Proof. Assume that $[L, B] \neq 1$ and recall that $[A, B] = 1$ by 4.7. We apply 4.13 and use the notation given there. Then

$$(*) \qquad C_V(L^*)[V, A] = C_V(AB), \ |\overline{V}| = q^3, \text{ and } |V/C_V(AB)| = q^2.$$

Let $W := [V, L]C_V(L^*)$. By 4.13 $\overline{W}$ is a natural $SL_2(q)$-module for L and L^*-invariant. For every $1 \neq x \in AB$ and $A \neq D \in \mathcal{D} \cap L$ we have $[L, D^x] \neq 1$, since $[A, D^x] \neq 1$. Hence 4.13 also applies to $\hat{L} := \langle L, D^x \rangle$, if $D^x \not\leq L$. In particular we get $C_V(L^*) = C_V(\hat{L})$ and $[\overline{V}, D^x] = [\overline{V}, Y]$ for some $Y \in \mathcal{D} \cap L$ with $A \neq Y$. This shows that AB acts on the set

$$\Omega_0 := \{[\overline{V}, D] \mid D \in \mathcal{D} \cap L \text{ and } D \neq A\}.$$

As $|\Omega_0| = q$ and $|AB| = q^2$, we get that $|N_{AB}([\overline{V}, D])| = q$ for $D \in \mathcal{D} \cap L$ with $A \neq D$. On the other hand, $[\overline{V}, AB] = [\overline{V}, A]$, so $C := C_{AB}(\overline{W})$ has order q. Since $[\overline{V}, L^*] = \overline{W}$, we conclude that $C \leq O_p(L^*)$.

Let $AB \leq T_0 \in Syl_p(L^*)$. From 4.13 we get that $C_V(AB)$ is T_0-invariant. Observe that $C_{T_0}(C_V(AB)) \cap C_{T_0}(V/C_V(AB))$ is elementary abelian.

Hence 4.3 and 4.6 show that

$$AB = C_{T_0}(C_V(AB)) \cap C_{T_0}(V/C_V(AB)),$$

so AB is normal in T_0. In particular $[V, AB] = [V, A][V, B]$ is T_0-invariant. This gives

$$[V, \mathcal{D} \cap T, T_0] \leq [C_V(AB), T_0] = [V, A, T_0] \leq [V, AB] \leq [V, \mathcal{D} \cap T],$$

so T_0 normalizes $[V, \mathcal{D} \cap T]$. In particular, $\langle (\mathcal{D} \cap T)^{T_0} \rangle$ acts quadratically on V and so is p-group. Hence, T_0 normalizes $\mathcal{D} \cap T$ and $T_0 \leq M$. Then there exists $x \in M$ with $C_V(T^x) \leq C_V(T_0)$, and $[C_V(T_0), L] \neq 1$ since $L \not\leq M$. This shows that $W \leq \langle C_V(T_0)^{L^*} \rangle$, so $O_p(L^*)$ centralizes W and acts quadratically on V. In particular $O_p(L^*)$ is elementary abelian. Hence 4.3 implies $C = O_p(L^*)$ and thus $[L, C] = 1$. Now $C_V(L)$ is AB-invariant and so $C_V(L) \leq C_V(AB)$. But then $|V/C_V(AB)| = q$ which contradicts 4.1. □

Lemma 4.16 *Assume Hypothesis* 4.3. *Let* $A, B \in \mathcal{D}$ *with* $[A, B] = 1$ *and* $A \notin \mathcal{D}_P(M)$. *Then* $\mathcal{D} \cap AB = \{A, B\}$.

Proof. We apply 4.12. Then, possibly after replacing A by a conjugate, we may assume that $A \leq T$ and that there exists $L \in \Lambda(A)$ with $L \not\leq M$. Hence, by 4.15 $|\mathcal{D} \cap AC| = 2$ for every $C \in \mathcal{D} \cap T$ with $C \neq A$. On the other hand, $A, B \in \mathcal{D} \cap T^g$ for some $g \in P$, and by 4.7 $\mathcal{D} \cap T^g$ and $\mathcal{D} \cap T$ are both in $C_P(A)$. Hence conjugation in $C_P(A)$ gives the claim for $|\mathcal{D} \cap AB|$. □

Lemma 4.17 *Assume Hypothesis* 4.3. *Let* $B \in \mathcal{D}$ *and* $L \in \Lambda$ *with* $|X| \geq 3$ *for every* $X \in \mathcal{D} \cap L$. *Then either* $B \leq L$ *or* $[L, B] = 1$.

Proof. We may assume that $[L, B] \neq 1$ and $B \not\leq L$. As before we set

$$L^* := \langle L, B \rangle \text{ and } \overline{V} := V/C_V(L^*).$$

By 4.13 there exists a unique $A \in \mathcal{D} \cap L$ such that

$$(*) \qquad [A, B] = 1 \text{ and } C_V(AB) = C_V(L^*)[V, A].$$

We now use the fact that $q := |A| \geq 3$. Let K be a complement for A in $N_L(A)$. Then $|K| = q - 1 \geq 2$ and by $(*)$ $C_V(AB)$ is K-invariant. Hence $A\langle B^K \rangle$ acts quadratically on V, and thus is abelian. On the other hand, by 4.1

$$|C_V(A)||A| = |V| = |C_V(AB)||AB| \leq |C_V(AB)||A\langle B^K \rangle|,$$

so 4.3 implies that $AB = A\langle B^K \rangle$. In particular AB is K-invariant and by 4.16 K normalizes B and $C_V(B)$.

Observe that K acts fixed-point-freely on the natural $SL_2(q)$-module $V/C_V(L)$. Thus

$$\overline{V} = [\overline{V}, K] \times \overline{C_V(K)} \text{ and } C_{\overline{V}}(K) = \overline{C_V(K)} = \overline{C_V(L)}.$$

It follows that $\overline{C_V(K)} \cap \overline{C_V(B)} = 1$ and $\overline{C_V(B)} \leq [\overline{V}, K]$. As $\overline{C_V(B)} \cap \overline{C_V(L)} \leq \overline{C_V(L^*)} = 1$, the action of K on $V/C_V(L)$ yields either

$$\overline{C_V(B)} = [\overline{V}, A] \text{ or } \overline{C_V(B)} = [\overline{V}, K].$$

In the first case $C_V(B) = C_V(AB)$, which contradicts 4.1.

Thus we have $\overline{C_V(B)} = [\overline{V}, K]$. By 4.13 $[\overline{V}, K]$ is L-invariant. It follows that $\langle B^L \rangle$ acts quadratically on V, so $\langle B^L \rangle$ is abelian. Now 4.1 (iii) shows that B is normal in L^*, so $[L, B] = 1$, which contradicts our assumption. □

Theorem 4.18 *Assume Hypothesis* 4.3. *Then there exist subgroups* $E_1, \ldots, E_r$ *of* P *such that the following hold for* $W_i := [V, E_i]$ *and* $i, j \in \{1, \ldots, r\}$:

(*a*) $\mathcal{D} = \mathcal{D}_P(M) \cup (\mathcal{D} \cap E_1) \cup \cdots \cup (\mathcal{D} \cap E_r)$ *and* $\langle \mathcal{D} \rangle = \langle \mathcal{D}_P(M) \rangle \times E_1 \times \cdots \times E_r$.

(b) $[W_i, E_j] = [W_i, \langle \mathcal{D}_P(M)\rangle] = 1$ *for* $i \neq j$ *and* $V = W_i C_V(E_i)$.

(c) $E_i \cong SL_2(q_i)$ *where* $q_i = |A|$ *for* $A \in \mathcal{D} \cap E_i$, *or* $E_i \cong S_m$, *m odd,* $E_i \cap M \cong S_{m-1}$, *and* $|A| = 2$ *for* $A \in \mathcal{D} \cap E_i$.

(d) $E_i \cong SL_2(q_i)$ *and* $W_i/C_{W_i}(E_i)$ *is a natural* $SL_2(q_i)$*-module for* E_i*,or* $E_i \cong S_m$ *and* W_i *is a natural* S_m*-module for* E_i*. Moreover, in the second case* $\mathcal{D} \cap E_i$ *acts as the conjugacy class of transpositions on* W_i.

Proof. We will prove 4.18 by induction on $|\mathcal{D}| + |P|$. Let P be a minimal counterexample. Then by 4.9:

4.18.1 $P = \langle \mathcal{D} \rangle$.

According to 4.8 and 4.10 there exists a partition of $\mathcal{D}$ satisfying

4.18.2 $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_r$ *such that for* $E_i := \langle D_i \rangle$:

(1) $\mathcal{D}_0 = \mathcal{D}_P(M)$ *and* $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ *for* $i \neq j$.

(2) $[E_i, E_j] = 1$ *for* $i \neq j$, *and* $\mathcal{D}_i$ *is a conjugacy class of* E_i *for* $i \geq 1$.

(3) *For* $i \geq 1$, $\mathcal{D}_i$ *and* E_i *satisfy Hypothesis 4.3 with respect to* $M \cap E_i$.

Assume that $\mathcal{D} \neq \mathcal{D}_i$ for $i \geq 1$. Then induction and 4.18.2 (3) show that (a) – (d) hold for E_i and $\mathcal{D}_i$; in particular $W_i/C_{W_i}(E)$ is an irreducible E_i-module. Hence $[W_i, E_j] = 1$ for $i \neq j$, and (b) – (d) hold for P. Since $W_i C_V(E_i) = V$, we also get that $E_1 \cdots E_r$ is the direct product of the subgroups E_j and also (a) holds. But then P is not a counterexample. We have shown:

4.18.3 $\mathcal{D} = \mathcal{D}_1$ *and* $P = E_1$.

Assume next that $|A| \geq 3$ for $A \in \mathcal{D}$. Then by 4.12 and 4.17 $P \cong SL_2(q)$ where $q = |A|$, and again (a) – (d) follow. Thus we have:

4.18.4 $|A| = 2 = |V/C_V(A)|$ *for* $A \in \mathcal{D}$.

Then 4.7 and an elementary argument using dihedral groups yields

4.18.5 *Let* $A \in \mathcal{D}$ *and* $D \in \mathcal{D} \setminus C_{\mathcal{D}}(A)$. *Then* $L := \langle A, D \rangle \cong SL_2(2)$, *and* $V/C_V(L)$ *is a natural* $SL_2(2)$*-module for* L.

Let $A \in \mathcal{D} \cap T$. According to 4.7 and 4.9 either $C_{\mathcal{D}}(A) \subseteq \mathcal{D} \cap M$ or $C_P(A)$ satisfies Hypothesis 4.3 with respect to $C_{\mathcal{D}}(A)$ and $C_M(A)$. In the first case by 4.18.5 there exists $L \in \Lambda(A)$ with $L \not\leq M$. Hence by 4.15 $[B, L] = 1$ for every $B \in C_{\mathcal{D}}(A) \setminus \{A\}$, so

$$C_{\mathcal{D}}(L) = C_{\mathcal{D}}(D) \setminus \{D\} \text{ for every } D \in \mathcal{D} \cap L.$$

Now 4.13 implies that $P = L$ and P is not a counterexample. We have shown that

4.18.6 *$C_{\mathcal{D}}(A) \not\subseteq \mathcal{D} \cap M$; in particular $C_{\mathcal{D}}(A) \neq \{A\}$ and $C_P(A)$ satisfies Hypothesis 4.3 with respect to $C_{\mathcal{D}}(A)$ and $C_M(A)$.*

Let $\mathcal{D}_A := C_{\mathcal{D}}(A) \setminus \{A\}$. Assume first that $\mathcal{D}_A$ is not a conjugacy class of $\langle \mathcal{D}_A \rangle$. Choose $\mathcal{D}^* \subseteq \mathcal{D}_A$ such that $\mathcal{D}^*$ is a conjugacy class of $\langle \mathcal{D}^* \rangle$ and $|\mathcal{D}^*|$ is maximal with that property. By our assumption there exists $B \in \mathcal{D}_A \cap T$ with $B \notin \mathcal{D}^*$, and by 4.8 $[\langle \mathcal{D}^* \rangle, B] = 1$ for every such B. Hence the maximality of $\mathcal{D}^*$ shows that $\langle \mathcal{D}^* \rangle$ is normal in $\langle \mathcal{D}_A \rangle$ and $\langle \mathcal{D}_B \rangle$.

Let $D \in \mathcal{D}$ with $D \not\leq M$. Then $\mathcal{M}(D) \neq \mathcal{M}(B)$ and by 4.18.5 either $D \in \mathcal{D}_B$ or $\langle D, B \rangle \in \Lambda$. In the former case D normalizes $\mathcal{D}^*$ and in the latter case 4.15 implies that $D \in \mathcal{D}_A$, so again D normalizes $\mathcal{D}^*$. It follows that

$$\mathcal{D} = (\mathcal{D} \cap M) \cup (\mathcal{D} \cap N_P(\mathcal{D}^*)).$$

But then 2.1 shows that $P = M$ or $P = N_P(\mathcal{D}^*)$. The first case contradicts 4.2 and the second case contradicts $\mathcal{D} \neq \mathcal{D}^*$ and the fact that $\mathcal{D}$ is a conjugacy class by 4.18.2 and 4.18.3. We have shown that $\mathcal{D}_A$ is a conjugacy class, so 4.18.4, 4.18.6 and induction give

4.18.7 *$\langle \mathcal{D}_A \rangle \cong S_n$, with n odd, $M \cap \langle \mathcal{D}_A \rangle \cong S_{n-1}$, $W := [V, \langle \mathcal{D}_A \rangle]$ is a natural S_n-module for $\langle \mathcal{D}_A \rangle$, and $\mathcal{D}_A$ acts as the conjugacy class of transpositions on W.*

Using the usual generators and relations for S_n we get from 4.18.7

4.18.8 *There exist $T_1, \ldots, T_{n-1} \in \mathcal{D}_A$ such that $T_i \in \mathcal{D} \cap M$ for $1 \leq i \leq n-2$, and*

$$[T_i, T_j] = 1 \iff |i - j| \neq 1 \text{ and } \langle T_i, T_j \rangle \cong SL_2(2) \iff |i - j| = 1.$$

By the same elementary observation as above $\mathcal{D} \not\subseteq M \cup C_P(A)$. Hence by 4.18.5 there exists $D \in \mathcal{D}$ such that $D \notin M$ and $\langle A, D \rangle \in \Lambda(A)$. Now 4.15 gives

$$\mathcal{D} \cap M \cap \langle \mathcal{D}_A \rangle \subseteq C_{\mathcal{D}}(D);$$

in particular $[D, T_i] = 1$ for $1 \leq i \leq n-2$.

Set $T_{n+1} := A$ and $T_n := D$. Then $T_1, \ldots, T_{n+1}$ generate a subgroup isomorphic to S_{n+2} provided we can show that $[D, T_{n-1}] \neq 1$. Assume that $[D, T_{n-1}] = 1$. Then

$\mathcal{D}_A = \mathcal{D}_D$, and as above 4.13, applied to $\langle A, D\rangle$, gives $P = \langle A, D\rangle$, and P is not a counterexample.

We have shown that $T_1, \ldots, T_{n+1}$ generate a subgroup U isomorphic to S_{n+2} in P. In particular $C_{\mathcal{D}}(X) \subseteq \mathcal{D} \cap U$ for every $X \in \mathcal{D} \cap \langle A, D\rangle$. Now 4.13 implies that $P = U$, and P is not a counterexample.

$\square$

5 The Proof of the Local $C^{**}(G,T)$-Theorem for Minimal Parabolic Groups

In this section we work with the following two hypotheses.

Hypothesis 5.1 *Let p be a prime, P a minimal parabolic group acting faithfully on an elementary abelian p-group V, and let $T \in Syl_p(P)$ and $M \leq P$ be the unique maximal subgroup of P containing T. Suppose also that*:

(i) $O_p(P) = 1$,

(ii) $\mathcal{O}_P(V) \neq \emptyset$,[2] *and*

(iii) $C_P(C_V(T)) \leq M$ *(so* $[C_V(T), P] \neq 1$*).*

Hypothesis 5.2 *Let P be a minimal parabolic group of characteristic p with $T \in Syl_p(P)$ and $C^{**}(P,T) \neq P$, and let M be the unique maximal subgroup of P containing T.*

Lemma 5.3 *Assume Hypothesis* 5.1*. Then Hypothesis* 4.3 *holds for $\mathcal{A}_P(V)$; in particular $|A||C_V(A)| = |V|$ for every $A \in \mathcal{A}_P(V)$. Moreover, $N_P(A)$ acts irreducibly on $V/C_V(A)$ for every $A \in \mathcal{A}_P(V)$.*

Proof. From 3.6 we get that P satisfies Hypothesis 4.1 with respect to $\mathcal{A}_P(V)$ and that $N_P(A)$ acts irreducibly on $V/C_V(A)$ for every $A \in \mathcal{A}_P(V)$. In addition, since P is minimal parabolic and $O_p(P) = 1$, we also get Hypothesis 4.2. Now Hypothesis 4.3 follows from the definition of $\mathcal{A}_P(V)$. □

Lemma 5.4 *Assume Hypothesis* 5.2 *and let*

$$V := \Omega(Z(O_p(P))) \text{ and } \overline{P} := P/C_P(V).$$

Then $\overline{P}$ and V satisfy Hypothesis 5.1*, and*

$$|A/C_A(V)||C_V(A)| = |V| \text{ for every } A \in \mathcal{A}(T) \text{with } A \not\leq C_P(V).$$

[2]Here $\mathcal{O}_P(V)$ is the set introduced in 2.7.

Proof. Since $C_P(O_p(P)) \le O_p(P) \le T$, we have $\Omega(Z(T)) = C_V(T)$. Hence

$$C_P(V) \le C_P(C_V(T)) \le C^{**}(P,T) \le M.$$

By Lemma 3.4 it follows that $O_p(\overline{P}) = 1$. It remains to show that $\mathcal{O}_{\overline{P}}(V) \neq \emptyset$.

We first show that $J(T) \not\le C_P(V)$. Suppose on the contrary that $J(T) \le C_P(V)$. Then $J(T) \le C_T(V) \in Syl_p(C_P(V))$ and, as $J(T) = J(C_T(V))$ *char* $C_T(V)$, the Frattini Argument gives

$$P = C_P(V)N_P(C_T(V)) \le C_P(\Omega(Z(T)))N_P(J(T)) \le C^{**}(P,T),$$

whichis a contradiction.

Therefore $J(T) \not\le C_P(V)$ and there exists $A \in \mathcal{A}(T)$ with $A \not\le C_P(V)$. Let $A_0 \le A$. Then

$$|A| \ge |A_0 C_V(A_0)| = \frac{|A_0||C_V(A_0)|}{|A_0 \cap V|} \ge \frac{|A_0||C_V(A_0)|}{|C_V(A)|}.$$

Thus $A \in \mathcal{O}_P(V)$ and it follows immediately that $\overline{A} \in \mathcal{O}_{\overline{P}}(V)$. Now 5.3 gives the additional statement. □

Theorem 5.5 *Assume Hypothesis* 5.1 *holds. Let* $\mathcal{D} := \mathcal{A}_P(V)$. *Then there exist subgroups* $E_1, \ldots, E_r$ *of* P *so that, for each* $1 \le i \le r$:

(*a*) $P = (E_1 \times \ldots \times E_r)T$,

(*b*) T *acts transitively on* $\{E_1, \ldots, E_r\}$,

(*c*) $\mathcal{D} = (\mathcal{D} \cap E_1) \cup \cdots \cup (\mathcal{D} \cap E_r)$,

(*d*) $V = C_V(E_1 \times \ldots \times E_r)\prod_{i=1}^{r}[V, E_i]$, *with* $[V, E_i, E_j] = 1$,

(*e*) $E_i \cong SL_2(p^n)$ *or* $p = 2$ *and* $E_i \cong S_{2^n+1}$, *for some* $n \in \mathbb{N}$, *and*

(*f*) $[V, E_i]/C_{[V,E_i]}(E_i)$ *is a natural module for* E_i.

Proof. By 5.3, $\mathcal{D}$ satisfies Hypothesis 4.3, so we are allowed to apply 4.18 with the notation given there. Since P is minimal parabolic we get from 3.3 that $O^p(P) \le E_1 \times \ldots \times E_r$ and as $O_p(P) = 1$, $\mathcal{D}_P(M) = \emptyset$. Therefore (a) – (d) and (f) hold.

For the proof of (e) it suffices to show that $m = 2^n + 1$ if $E_i \cong S_m$. Observe that $N_T(E_i)E_i = C_T(E_i)E_i$, so $N_T(E_i)E_i$ is a minimal parabolic group. Now (e) follows from 2.12 (e). □

The proof of the Local $C^{**}(G,T)$-Theorem for minimal parabolic groups: Let $\overline{P} := P/C_P(V)$. By 5.4 $\overline{P}$ satisfies the hypothesis of 5.5. Thus the only thing that remains to be proven is

$$\overline{J(P)} = \overline{E}_1 \times \cdots \times \overline{E}_r =: \overline{E}.$$

Let $A \in \mathcal{A}(T)$. Suppose that $\overline{A} \not\leq \overline{E}$ and that $|\overline{A}|$ is minimal with this property.

By 5.3 and 5.4 there exists $\overline{B} \leq \overline{A}$ with $\overline{B} \in \mathcal{A}_{\overline{P}}(V)$ and

$$|V| = |\overline{A}||C_V(\overline{A})| = |\overline{B}||C_V(\overline{B})|.$$

Moreover, $N_{\overline{P}}(\overline{B})$ acts irreducibly on $V/C_V(\overline{B})$. The latter fact shows that there exists a unique $k \in \{1, \ldots, r\}$ such that $\overline{B} \leq \overline{E}_k$.

Assume that $\overline{E}_k \cong SL_2(q)$. Then $\overline{B} \in Syl_p(\overline{E}_k)$ and the structure of $Aut(SL_2(q))$ gives

$$\overline{A} = \overline{B} \times \overline{A}_0,\ A_0 := C_A(\overline{E}_k).$$

This shows that also $A_0 C_V(A_0) \in \mathcal{A}(T)$, and the minimal choice of $\overline{A}$ gives $\overline{A}_0 \leq \overline{E}$. But then also $\overline{A} \leq \overline{E}$, which contradicts the choice of A.

Assume next that $\overline{E}_k \cong S_{2^n+1}$. Then $|\overline{B}| = 2$ and by 2.16 (b)

$$\overline{A} = \overline{B} \times \overline{A}_0 \text{ with } A_0 \leq A, \text{ and } C_V(\overline{A}_0) \not\leq C_V(\overline{B}).$$

Similarly, as in the previous case, this shows that $A_0 C_V(A_0) \in \mathcal{A}(T)$ and then that $\overline{A}_0 \leq \overline{E}$.

Lemma 5.6 *Let p be a prime and P be a minimal parabolic group acting faithfully on an elementary abelian p-group V. Suppose that $O_p(P) = 1$ and $\mathcal{O}_P(V) \neq \emptyset$. Then $[C_V(T), P] \neq 1$ for every $T \in Syl_p(P)$.*

Proof. Let $V_0 := C_V(O^p(P))$ and $\widetilde{V} := V/V_0$. By 3.3 P also acts faithfully on $\widetilde{V}$. We also have $[C_{\widetilde{V}}(T), P] \neq 1$, for otherwise $O^p(P)$ would centralize the inverse image of $C_{\widetilde{V}}(T)$, contradicting the definition of V_0. Moreover, 2.8 shows that $\mathcal{O}_P(\widetilde{V}) \neq \emptyset$. Hence $(P, \widetilde{V})$ satisfies the hypothesis of 5.5, so we get (a) – (f) with $\widetilde{V}$ in place of V.

Let $A \in \mathcal{O}_T(V)$. Then there exists $i \in \{1, \ldots, r\}$ such that $[E_i, A] \neq 1$. Hence 2.16 shows that $A \leq E_i C_P([V, E_i])/C_P([V, E_i])$ and $[V, E_i, A] \leq C_V(T \cap E_i)$; in particular $C_V(T \cap E_i) \not\leq C_V(O^p(P))$.

If $E_i \cong SL_2(p^n)$, with $p^n > 2$, then let K be a complement for $T \cap (E_1 \cdots E_r)$ in $N_{E_1 \cdots E_r}(T \cap (E_1 \cdots E_r))$. Then $T = (T \cap (E_1 \cdots E_r)) N_T(K)$ and

$$C_V(O^p(P)) \prod_{i=1}^{r} C_{[V,E_i]}(T \cap E_i) = C_V(O^p(P)) \times [C_V(T \cap (E_1 \cdots E_r)), K].$$

Since $N_T(K)$ normalizes $[C_V(T \cap (E_1 \cdots E_r)), K]$, it follows that $C_V(T) \not\leq C_V(O^p(P))$.

If $E_i \cong S_{2m+1}$, then 2.13 shows that $V = C_V(O^2(P)) \times [V, O^2(P)]$ and again $C_V(T) \not\leq C_V(O^p(P))$. □

6 $B(T)$-Blocks

In this section we assume

Hypothesis 6.1 *Let G be of characteristic p and $T \in Syl_p(G)$.*

Notation 6.2 *Let $\mathcal{B}(T)$ be the set of $B(T)$-blocks of G. We set*

$$\mathcal{B}(G) := \bigcup_{g \in G} \mathcal{B}(T^g).$$

Moreover, $\mathcal{B}^*(G)$ is the set of maximal elements of $\mathcal{B}(G)$ with respect to inclusion and

$$\mathcal{B}^*(T) := \mathcal{B}^*(G) \cap \mathcal{B}(T).$$

For $E \in \mathcal{B}(G)$ we set $W_E := [\Omega(Z(O_p(E))), E]$.

Lemma 6.3 *Let $E \in \mathcal{B}(T)$. Suppose that Q is a p-subgroup of G normalized by $B(T)E$. Then $Q \leq N_G(E)$.*

Proof. As $B(B(T)Q) = B(T)$, Q normalizes $B(T)$. Moreover, from $E = [E, B(T)]$ we get that $EB(T) = \langle B(T)^E \rangle$. Hence Q normalizes $EB(T)$ and thus also $E = O^p(EB(T))$.

□

Lemma 6.4 *Let $E \in \mathcal{B}(T)$. Then the following hold:*

(*a*) *$E = O^p(EO_p(G))$ and $W_E \leq \Omega(Z(O_p(G)))$.*

(*b*) *Assume that E is not exceptional. Then*

$$O_p(E) \leq \Omega(Z(O_p(G))) \text{ and } [O_p(G), E] = W_E.$$

(*c*) *Assume that E is exceptional. Then $Z(E)W_E = \Omega(Z(O_3(E))) \leq \Omega(Z(O_3(G)))$ and either*

$$O_3(E) \leq O_3(G) \text{ or } [O_3(G), E] = W_E.$$

(*d*) *$[W_E, J(T)] \neq 1$.*

Proof. (a): From 6.3 with $Q := O_p(G)$ we get $O_p(G) \leq N_G(E)$. The first part now follows from the fact that $E = O^p(E)$. Since $W_E Z(E)/Z(E)$ is an irreducible E-module, $[W_E, O_p(G)] \leq Z(E)$. Hence the Three Subgroups Lemma gives

$$[W_E, O_p(G)] = [W_E, E, O_p(G)] = 1,$$

so $W_E \leq \Omega(Z(O_p(G)))$, since G is of characteristic p.

(b): Note that $W_E = O_p(E)$ and $W_E = [W_E, E]$, so the result follows from (a).

(c): Since $[O_3(E), O_3(G)] \leq \Omega(Z(O_3(E)))$ the Three Subgroups Lemma gives

$$[O_3(E), O_3(E), O_3(G)] = 1.$$

It follows that $Z(E) \leq \Omega(Z(O_3(G)))$ and by (a)

$$Z(E)W_E = \Omega(Z(O_3(E))) \leq \Omega(Z(O_3(G))).$$

The other statement in (c) is a direct consequence of the structure of $O_3(E)$ and the fact that $E = O^3(E)$.

(d): From the definition of a $B(T)$-block we get $E = [E, B(T)]$ and $[W_E, E] \neq 1$. Hence $W_E \not\leq Z(B(T))$ and (d) follows. □

Lemma 6.5 *Let $E \in \mathcal{B}(T)$ be an exceptional $B(T)$-block. Then*

$$O^3(N_G(E) \cap C_G(W_E)) \leq C_G(E).$$

Proof. We fix the following notation:

$$R := O^3(N_G(E) \cap C_G(W_E)), \qquad M := N_G(E) \cap C_G(Z(E)),$$

$$M_2 := C_M(O_3(E)/Z(E)W_E), \qquad \overline{N_G(E)} := N_G(E)/C_G(E).$$

We first show:

$$(*) \qquad \overline{M}/O_3(\overline{M}) \cong SL_2(3^n).$$

We put $E^* := E$ if E is non-solvable. If $E/O_3(E) \cong Q_8$, then there exists $a \in B(T)$ such that $E\langle a\rangle/O_3(E) \cong SL_2(3)$ and we put $E^* := E\langle a\rangle$. Then 2.6 applies to E^* and we get $\overline{M} = \overline{E^*}\,\overline{M}_2$.

Note that $\overline{E} \cap \overline{M}_2 \leq O_3(\overline{E}) \leq O_3(\overline{M})$. Moreover, $C_{\overline{M}_2}(W_E)$ centralizes an E-chief series of E, so $C_{\overline{M}_2}(W_E) \leq O_3(\overline{M})$. Hence Schur's Lemma implies that $\overline{M}_2/O_3(\overline{M})$ is a cyclic group whose order divides $3^n - 1$. In particular, M_2 normalizes $C_{W_E}(B(T))$ and so $[\overline{B(T)}, \overline{M}_2] \leq O_3(\overline{M})$. This shows that $B(T)C_G(E)$ is normalized by M_2.

If $\overline{M}_2 = O_3(\overline{M})$, then $\overline{M} = \overline{E}^* O_3(\overline{M})$ and $(*)$ follows. So assume that $\overline{M}_2 \neq O_3(\overline{M})$. Then there exists a non-trivial $3'$-subgroup $\overline{Q} \leq \overline{M}_2$ and this subgroup normalizes $\overline{B(T)}$. Hence

$$\overline{B(T)} = \overline{A}(\overline{B(T)} \cap O_3(\overline{M})), \text{ with } \overline{A} := C_{\overline{B(T)}}(\overline{Q}).$$

But then A leaves invariant the decomposition

$$O_3(\overline{E}) = C_{O_3(\overline{E})}(\overline{Q}) \times \overline{W}_E,$$

and acts quadratically in each factor. This contradicts the definition of an exceptional $B(T)$-component and finishes the proof of $(*)$.

According to $(*)$, $\overline{R} \cap \overline{M} \leq O_3(\overline{M})$. Thus we may assume that $\overline{R} \not\leq \overline{M}$, for otherwise the result follows. Consider $R_0 := C_R(O_3(E)/W_E Z(E))$. Then $[O_3(E), R_0] \leq Z(E) W_E$ and the Three Subgroups Lemma yields

$$[O_3(E), O_3(E), R_0] = [Z(E), R_0] = 1,$$

so $\overline{R}_0 \leq O_3(\overline{M})$. Again Schur's Lemma shows that $\overline{R}/\overline{R} \cap O_3(\overline{M})$ is a cyclic $3'$-group. Let $\overline{Q}$ be a non-trivial $3'$-subgroup of $\overline{R}$.

As $B(T)$ normalizes R, we get

$$[\overline{R}, \overline{B(T)}] \leq \overline{R} \cap \overline{M} = O_3(\overline{M}).$$

It follows that $\overline{R}$ normalizes $\overline{B(T)}$. In particular

$$\overline{B(T)} = \overline{A}(\overline{B(T)} \cap O_3(\overline{M})) \text{ with } \overline{A} := C_{\overline{B(T)}}(\overline{Q}).$$

As in the proof of $(*)$, this contradicts the definition of an exceptional $B(T)$-block. □

Theorem 6.6 *Let $E \in \mathcal{B}(T)$. Then $E \trianglelefteq E C_G(W_E)$.*

Proof. We fix the following notation:

$$W := W_E,\ C := C_G(W),\ C_0 := C_G(O_3(E)/Z(E)W),\ R := [C, E],\ \overline{G} := G/O_3(G).$$

Let G be a minimal counterexample. Then $G = CEB(T)$ and $W \trianglelefteq G$. We will prove the result in a sequence of steps.

6.6.1 *E is exceptional and $O_3(E) \leq O_3(G)$; in particular $E/O_3(E) \cong SL_2(q)'$, $q = 3^n$.*

Assume that E is not exceptional or $p = 3$ and $O_3(E) \not\leq O_3(G)$. Then by 6.4 $[O_p(G), E] \leq W$. Hence $[E, C_G(W)]$ centralizes W and $O_p(G)/W$, so

$$[E, C_G(W)] \leq O_p(G).$$

Now 6.4 (a) implies that E is normal in $EC_G(W)$ and G is not a counterexample.

We now fix in addition an involution $t \in E$ with $[t, E] \leq O_3(E)$ and $O_3(G) \leq Y \leq C$ such that $\overline{Y} = C_{\overline{C}}(\overline{t})$. Note that $Y = C_Y(t)O_3(G)$.

6.6.2 *Let $N \leq C$ be an $EB(T)$-invariant subgroup. Then either $C = N(C \cap EB(T))$ and $O^3(C) \leq N$, or $N \leq N_G(E)$.*

If $NEB(T) < G$, then by induction $N \leq N_G(E)$, and if $NEB(T) = G$, then $C = N(C \cap EB(T))$. Since $C \cap EB(T) \leq O_3(EB(T))$, the latter case gives $O^3(C) \leq N$.

6.6.3 *$O^3(\overline{C}) = F^*(\overline{C})$, and $O^3(C) \not\leq N_G(E)$.*

Let F be the inverse image of $F^*(\overline{C})$ in G. Assume first that $F \leq N_G(E)$. Then by 6.5 $O^3(F) \leq C_G(E)$, so $[\overline{F}, \overline{E}] = 1$. It follows that $\overline{R} \leq C_{\overline{C}}(\overline{F}) \leq \overline{F}$. Hence $R \leq F$ and $O^3(R) \leq C_G(E)$. Now 2.5 (with $N := R$) implies that E is normal in G, a contradiction.

We have shown that $F \not\leq N_G(E)$, and thus by 6.6.2 $O^3(\overline{C}) = F^*(\overline{C})$.

6.6.4 *Either $C = Y$, or $O^3(\overline{C})$ is an r-group, r a prime different from 2 and 3.*

Note that Y is $EB(T)$-invariant. Hence by 6.6.2 either $C = Y(C \cap EB(T))$ or $Y \leq N_G(E)$. As $[t, EB(T)] \leq O_3(E) \leq O_3(G)$, the first case gives $C = Y$.

Assume that $Y \leq N_G(E)$. Then

$$[\overline{Y}, \overline{E}] \leq \overline{Y} \cap \overline{E} \leq \overline{C} \cap \overline{E} = 1,$$

since W is a faithful $\overline{E}$-module. It follows that $\overline{Y} = C_{\overline{C}}(\overline{S})$, where $\overline{S}$ is a Sylow 2-subgroup of $\overline{E}$. As $\overline{S}$ is a quaternion group we conclude from 2.4 that $\overline{U} := [\overline{C}, \overline{t}]$ is solvable of odd order. In particular $\overline{C} = \overline{YU}$, so the inverse image U is not in $N_G(E)$. As U is $EB(T)$-invariant, 6.6.2 yields $C = U(C \cap EB(T))$, and thus $O^3(C) \leq U$. Now 6.6.3 shows that $\overline{U} = F(\overline{C})$. Let r be a prime dividing $|\overline{U}|$, so $r \notin \{2, 3\}$. Then, again using 6.6.2, $\overline{U} = O_r(\overline{C})$.

6.6.5 *$C \neq Y$, so $O^3(\overline{C})$ is an r-group, r a prime different from 2 and 3.*

Assume that $C = Y$. Then $C = C_C(t)O_3(G)$ and both $O_3(G)$ and $C_C(t)$ normalize $[O_3(G), t] = O_3(E)$. From $G = CEB(T)$ we conclude that $O_3(E) \trianglelefteq G$. By 2.6, EC_0 is normal in G, so $R \leq EC_0$.

Note that R centralizes $O_3(G)/O_3(E)$, $Z(E)$ and W, so $R \cap C_0 \leq O_3(G)$. It follows that either $[E, R] \leq O_3(G)$ or $t \in RC_0$.

In the first case by 6.4 (a) $R \leq N_G(E)$, and thus by 6.5 $O^3(R) \leq C_G(E)$. Now 2.5 shows that G is not a counterexample.

In the second case there exists an involution $a \in R$ such that $t \in aC_0$ and $[a, E] \leq R \cap C_0 \leq O_3(G)$. Now again 6.4 (a) and 6.5 give $a \in C_G(E)$, and a centralizes $O_3(G)/O_3(E)$ and $O_3(E)$, which contradicts the fact that G is of characteristic 3.

We derive a final contradiction. Let $Q := [O_3(E), G]$, $D := \Phi(Q)$, and $\widetilde{Q} := Q/WD$. Note that $O_3(G)$ centralizes $\widetilde{Q}$, so $\overline{G}$ acts on $\widetilde{Q}$. The action of t on Q shows that

$$\widetilde{Q} = [\widetilde{Q}, E] \times C_{\widetilde{Q}}(E) \text{ and } [\widetilde{Q}, E] = \widetilde{O_3(E)}.$$

If $[\widetilde{Q}, E] = 1$, then $O_3(E) \leq WD$, and thus $Q = O_3(E) = W$, which is impossible. Hence $[\widetilde{Q}, E]$ is a natural $SL_2(3^n)'$-module for E.

Let $A := T \cap E$ and C_1 be the inverse image of $O^3(C)$ in G. Then $\overline{A}$ acts quadratically on $\widetilde{Q}$ and $C_{\widetilde{Q}}(A) = C_{\widetilde{Q}}(\overline{a})$ for every $\overline{a} \in \overline{A}^\sharp$. Recall from 6.6.5 that $\overline{C}_1$ is a $3'$-group.

Assume first that $q > 3$. Then $\overline{C}_1 = \langle C_{\overline{C}_1}(\overline{a}) \mid \overline{a} \in \overline{A}^\sharp \rangle$ and each $C_{\overline{C}_1}(\overline{a})$ normalizes $C_{\widetilde{Q}}(A) = C_{\widetilde{Q}}(\overline{a})$. Hence $[\overline{C}_1, A] = [\overline{C}_1, A, A]$ centralizes $C_{\widetilde{Q}}(A)$ and $\widetilde{Q}/C_{\widetilde{Q}}(A)$. As $[C, A]$ also centralizes W and $O_3(G)/Q$, we conclude that $O^3([C_1, A])$ centralizes $O_3(G)$, and thus $[C_1, A] \leq O_3(G)$. But then also $[C_1, E] \leq O_3(G)$, which using 6.4(a) implies that C_1 normalizes E. This contradicts 6.6.3.

Assume now that $q = 3$, so $E/O_3(E) \cong Q_8$. For $x \in C_1$ set $L := \langle E, E^x \rangle$. Then either $[E, x] \leq O_3(G)$, and thus $x \in N_G(E)$, or $C_1 \cap L \not\leq O_3(G)$. According to 6.6.3 we may assume that $C_1 \cap L \not\leq O_3(G)$.

Observe that L acts on $\widetilde{Q}_0 := [\widetilde{Q}, t][\widetilde{Q}, t^x]$ and $|\widetilde{Q}_0| \leq 3^4$. Let L_0 be the kernel of this action. If $L \cap C_1 \not\leq L_0$, then by the order of $GL_4(3)$ and 6.6.5, $L \cap C_1/L_0 \cap C_1$ is a cyclic group of order 5 or 13 which is normalized by $E/O_p(E) \cong Q_8$, but it is easily checked that this is impossible in $GL_4(3)$. Therefore $L \cap C_1 \leq L_0$. Hence $O^3(L \cap C_1)$ centralizes the L-series $D \leq DW \leq Q_0 \leq O_3(G)$, and thus $L \cap C_1 \leq O_3(L)$. But this contradicts $L \cap C_1 \not\leq O_3(G)$ and 6.6.5. □

Lemma 6.7 *Let $E \in \mathcal{B}(T)$ and $F \in \mathcal{B}(G)$ such that $[E, F] \leq E$. Then either $F = E$, or $[F, E] = 1$, or $p = 2$ and the following hold:*

(*a*) *$F \leq E$ and $O_2(F) \leq O_2(E)$.*

(*b*) $FO_3(E)/O_2(E) \cong A_{2r+1}$ *and* $E/O_2(E) \cong A_{2m+1}$, *for some* $r \leq m$.

(*c*) *There exists* $g \in E$ *such that* $E, F \in \mathcal{B}(T^g)$.

Proof. If $[W_E, F] = 1$, then 6.5 implies $[E, F] = 1$ and if $[W_F, E] = 1$, then by 6.6 $E \leq N_G(F)$ and again 6.5 implies $[E, F] = 1$. Thus we may assume that $[W_E, F] \neq 1$ and $[W_F, E] \neq 1$. As W_E is normalized by F, we get that $W_F \leq W_E$.

We fix the following notation:

$$R := C_G(W_E)E,\ \overline{FR} := FR/C_G(W_E),\ \widetilde{R} := R/O_p(R).$$

Then $\overline{F}$ induces automorphisms in $\overline{E} \cong SL_2(p^n)'$ or A_{2m+1}.

6.7.1 *The case* $F \leq R$.

Let $F_0 := EF \cap C_G(W_E)$. Then

$$\widetilde{E}\widetilde{F} = \widetilde{E} \times \widetilde{F}_0.$$

By 6.6 $C_G(W_E) \leq C_G(W_F) \leq N_G(F)$, so $[F, F_0] \leq O_p(F)$ and $F' \leq EO_p(F)$. It follows that $\widetilde{F} \leq \widetilde{E}$, or one of the following two cases holds:

(i) $p = 2$ and $F/O_2(F) \cong C_3$, or

(ii) $p = 3$, $F/O_3(F) \cong Q_8$ and $FE/O_3(FE) \cong SL_2(3^m)' \times C_2 \times C_2$.

In case (i) neither E nor F are exceptional. Hence 6.4 (b), applied to E and F, gives $[O_2(G), EF] = W_E$. Then $[O_2(G), O^2(F_0), O^2(F_0)] = 1$. As G is of characteristic 2, this shows that F_0 is a 2-group and $\widetilde{F} \leq \widetilde{E}$.

In case (ii) let t be an involution in F. Then $t \in O^3(F') \leq E$ and $[t, E] \leq O_3(E)$. It follows that $O_3(E) = [O_3(E), t] = O_3(F)$ and $[O_3(G), EF] \leq O_3(E)$. If E is exceptional, then 6.5 implies $[E, O^3(F_0)] = 1$, so $[O_3(G), O^3(F_0), O^3(F_0)] = 1$. If E is not exceptional, then $O_3(E) = W_E$ and again $[O_3(G), O^3(F_0), O^3(F_0)] = 1$. Thus, we have the same property as in case (i). As there we get that $\widetilde{F} \leq \widetilde{E}$.

Thus, in all cases we have established that $\widetilde{F} \leq \widetilde{E}$. Now 6.4 (a) implies $E = O^p(EO_p(R))$, and thus $F = O^p(F) \leq E$.

Suppose that E is a linear block. Then the p'-elements of $\overline{E}$ act fixed-point-freely on $W_E/C_{W_E}(E)$. It follows that $W_E = W_F C_{W_E}(E)$, and thus $E = F$.

Suppose E is a symmetric block. We first treat the case where F is a linear block, so $\overline{F} \cong SL_2(2^k)'$. Suppose $k > 1$. Then by 2.14 (c) $k = 2$ and there exists $g \in E$ such that $J(T)^g$ normalizes W_F and $C_{W_F}(F)$. Put $\check{W}_F = W_F/C_{W_F}(E)$. By 2.16 (b) there exist elements in $J(T)^g$ acting as transvections on $\check{W}_F$. On the other hand, $F \in \mathcal{B}(T^h)$

for some $h \in G$. So $J(T^h)$ normalizes F and $\check{W}_F$. It follows that $J(T^h)$ acts $GF(4)$-semilinearly on $\check{W}_F$ and so no element of $J(T^h)$ acts as a transvection on $\check{W}_F$. But $J(T^h)$ and $J(T^g)$ are conjugate in $N_G(\check{W}_F)$, a contradiction. This contradiction gives $k = 1$, so F is also a symmetric block.

We have shown that F is always a symmetric block; in particular (a) and (b) hold. By 2.16 (b), (e) $\overline{B(T)}$ is generated by a maximal set of commuting transpositions on W_E. Hence 2.14 (b) implies (c).

6.7.2 *The case $F \not\leq R$.*

Since both $C_{Aut(W_E)}(\overline{E})$ and $Out(\overline{E})$ are solvable, $\overline{FE}/\overline{E}$ is solvable. Thus $\overline{F} \not\leq \overline{E}$ implies $F \neq F'$, so $p = 2, 3$ and $F/O_p(F) \cong SL_2(p)'$. Moreover, if E is a symmetric block, then $|\overline{FE}/\overline{E}| \leq 2$, while $|F/\overline{O_2(F)}| = 3$, a contradiction. Hence $\overline{E} \cong SL_2(p^k)$ with $k > 1$, $O_p(\overline{F}) \leq \overline{E}$ and by 3.7 $\overline{B(T)} \in Syl_p(\overline{E})$. In particular $W_F C_{W_E}(E) < W_E$.

Assume that $[W_F, B(T)^h] = 1$ for some $h \in E$. As $F \in \mathcal{B}(G)$, there exists $g \in G$ such that $F \in \mathcal{B}(T^g)$; so $[W_F, B(T^g)] \neq 1$ while $[W_F, B(T)^h] = 1$. But this is impossible since $B(T)^h$ and $B(T^g)$ are conjugate in $N_G(W_F)$. We have shown that $[W_F, B(T)^h] \neq 1$ for every $h \in E$.

If $O_p(\overline{F}) \neq 1$, then $\overline{F}$ normalizes a Sylow p-subgroup of $\overline{E}$ and thus a conjugate $\overline{B(T)}^{\overline{x}}$, $x \in E$. If $O_p(F) \leq C_G(W_E)$, then $F/C_F(W_E)$ is a p'-group and $C_{W_E}(F) \not\leq C_{W_E}(E)$. Hence also in this case $\overline{F}$ normalizes a Sylow p-subgroup of $\overline{E}$ and thus a conjugate $\overline{B(T)}^{\overline{x}}$, $x \in E$.

As we have seen above $W_F \not\leq C_{W_E}(B(T)^h)$ for every $h \in E$. Since W_F is an irreducible F-module, we get from the module structure of W_E

$$(*) \qquad [C_{W_E}(B(T)^x), F] = 1 \text{ and } W_E = W_F \times C_{W_E}(B(T)^x).$$

In particular $\overline{E} \cong SL_2(p^2)$ and $O_p(\overline{F}) = 1$. As $|Syl_p(\overline{E})| = 5$ resp. 10 and $|\overline{F}| = 3$ resp. 8, there exists a second conjugate $\overline{B(T)}^{\overline{y}}$, $y \in E$, normalized by $\overline{F}$. But then also $[C_{W_E}(B(T)^y), F] = 1$, which contradicts $(*)$ since $W_E = C_{W_E}(B(T)^y) C_{W_E}(B(T)^x)$. $\square$

Lemma 6.8 *Let $E \in \mathcal{B}(T)$ be a symmetric block with $E \not\leq C^*(G,T)$. Then there exists $F \in \mathcal{B}(T)$ such that $F \leq E$, $F \not\leq C^*(G,T)$ and $F/O_2(F) \cong A_3 \cong SL_2(2)'$.*

Proof. Note that $A \in \mathcal{A}(T)$ satisfies $(*)$ of 2.16. Hence by 6.4 (d) and 2.16 (b) there exist $A \leq B(T)$ and $E^* = EA$ such that $\widetilde{E^*} := E^*/O_2(E^*) \cong S_{2n+1}$ and $\widetilde{A}$ is generated by a maximal set of commuting transpositions. We can choose $\widetilde{d} \in \widetilde{E}$ of order 3 to be inverted by one of these transpositions and commute with the others such that $d \notin C^*(G,T)$. Then $F := \langle d \rangle [W_E, d]$ has the required properties. $\square$

Lemma 6.9 *Let $\mathcal{B}(T)_{max}$ be the set of maximal elements of $\mathcal{B}(T)$. Then*

$$\mathcal{B}(T)_{max} = \mathcal{B}^*(T).$$

Proof. Let $F \in \mathcal{B}(T)_{max}$ and $F \leq E \in \mathcal{B}^*(G)$. By 6.7 (c) there exists $g \in G$ such that $F, E \leq \mathcal{B}(T^g)$. Then there exists $h \in N_G(F)$ such that $B(T^{gh}) = B(T)$. Hence $F \leq E^h \in \mathcal{B}(T)$, so $F = E^h$, since $F \in \mathcal{B}(T)_{max}$. It follows that $E = F$ and $F \in \mathcal{B}^*(G)$. □

Lemma 6.10 *Let $E \in \mathcal{B}^*(T)$. Then E is the unique element of $\mathcal{B}^*(G)$ in $EC_G(W_E)$ that is not contained in $C_G(W_E)$.*

Proof. Let $F \in \mathcal{B}^*(G)$ and $F \leq EC_G(W_E)$. Then by 6.6 $[E, F] \leq E$, and thus by 6.7 either $[E, F] = 1$ or $F \leq E$. In the latter case the maximality of F implies $F = E$. □

Lemma 6.11 *Let $E, F \in \mathcal{B}(G)$. Suppose that E and F are subnormal in G. Then $E = F$ or $[E, F] = 1$.*

Proof. Let $\overline{G} := G/O_p(G)$. The subnormality of E implies that either $\overline{E}$ is a component of $\overline{G}$ or $\overline{E} \leq F(\overline{G})$.

If $[\overline{E}, \overline{F}] \leq \overline{E} \cap \overline{F}$, then by 6.4 (a) $[E, F] \leq E \cap F$, and 6.7 gives $E = F$ or $[E, F] = 1$. Thus we may assume

$$(*) \qquad [\overline{E}, \overline{F}] \not\leq \overline{E} \cap \overline{F}.$$

In particular, $(*)$ shows that E and F are both solvable, so $\overline{E} \cong \overline{F} \cong C_3$ or Q_8.

Let $L := \langle E, F \rangle$ and $W := [\Omega(Z(O_p(G))), L]$. Then $C_L(W) \leq O_2(L)$, since $C_L(W)$ centralizes $O_2(G)/W$ and W. As L is also subnormal in G, we get $\overline{C_L(W)} = 1$.

Assume first that $\overline{E} \cong C_3$. Then by 6.4 $|W| \leq 2^4$ and $[E, F] \leq C_L(W)$ since $GL_4(2)$ has abelian Sylow 3-subgroups. Thus $\overline{C_L(W)} = 1$ gives $[\overline{E}, \overline{F}] = 1$, which contradicts $(*)$.

Assume that $\overline{E} \cong Q_8$. If $Z(\overline{E})$ is normal in $\overline{L}$, then also $[W, Z(\overline{E})] = W_E$ is L-invariant. As $GL_2(3) \setminus SL_2(3)$ does not contain elements of order 4, F normalizes $EC_G(W_E)$, and thus by 6.10 also E. But this contradicts $(*)$.

Suppose that $Z(\overline{E})$ is not normal in $\overline{L}$. There exists $y \in L$ such that $E^y \neq E$ but $[E, E^y] \leq E \cap E^y$. Hence as already seen, $[E, E^y] = 1$ and $\overline{E} \times \overline{E}^y \cong Q_8 \times Q_8$. On the other hand, similarly to the above, $\overline{L}$ is a subgroup of $SL_4(3)$. Since a Sylow 2-subgroup of $SL_4(3)$ has order 2^8, we get that $\overline{F} \cap (\overline{E} \times \overline{E}^y) \neq 1$. Hence $Z(\overline{F}) \leq Z(\overline{E}) \times Z(\overline{E}^y)$ and thus $Z(\overline{F}) = Z(\overline{E})$ or $Z(\overline{E}^y)$. In both cases $Z(\overline{E})$ is normal in $\overline{L}$, a contradiction. □

Theorem 6.12 *Let $E \in \mathcal{B}(G)$. Suppose that E is subnormal in G. Then the following hold:*

(a) $E \trianglelefteq B(G)$.

(b) $E \in \mathcal{B}(T^x)$ *for every* $x \in G$.

(c) *For every* $F \in \mathcal{B}(G)$ *either* $F \leq E$ *or* $[F, E] = 1$.

Proof. Observe that $E \in \mathcal{B}^*(G)$, since E is subnormal in G. Let

$$V := \langle \Omega(Z(T))^G \rangle, \ \overline{G} := G/C_G(V).$$

(a): We may assume that E is a $B(T)$-block. By 2.16 $J(T^x) \leq N_G(\overline{E})$ for all $x \in G$, so by 6.10 $J(T^x)$ also normalizes E. It follows that $W_E \cap Z(B(T^x)) \not\leq Z(E)$, so $[W_E, E^y] \neq 1$ for all $y \in B(T^x)$. Now 6.11 implies that $B(T^x) \leq N_G(E)$. Hence

$$E \trianglelefteq \langle B(T^x) \mid x \in G \rangle = B(G).$$

(b): For every $x \in G$, $B(T)$ and $B(T^x)$ are conjugate in $B(G)$. Thus, (a) implies (b).

(c): Let $F \in \mathcal{B}(G)$. By (a) F normalizes E. Now 6.7 shows that $F \leq E$ or $[E, F] = 1$. □

7 The Proof of the Local C^*(G,T)-Theorem

In this section we investigate a minimal counterexample to the Local $C^*(G,T)$-Theorem. We assume in this section:

Hypothesis 7.1 *Let G be a group of characteristic p with $T \in Syl_p(G)$ such that G is a minimal counterexample to the Local $C^*(G,T)$-Theorem.*

Notation 7.2 *We use the notation introduced in* 6.2. *In addition we define*

$$\begin{aligned} \mathcal{B}_*(T) &:= \{E \in \mathcal{B}^*(T) \mid E \not\leq C^*(G,T)\}, \quad \mathcal{B}_*(G) := \cup_{g\in G}\mathcal{B}_*(T^g), \\ V &:= \langle \Omega(Z(T))^G \rangle, \quad Z := \Omega(Z(B(T))), \quad \overline{G} := G/C_G(V). \end{aligned}$$

Observe that $O_p(\overline{G}) = 1$ (see for example [13, 2.0.1]).

Moreover, $\mathcal{L}(T)$ is the set of proper subgroups $L < G$ satisfying:

$$B(T) \leq L \text{ and } L \not\leq C^*(G,T).$$

Set $\mathcal{L}(G) := \cup_{g\in G}\mathcal{L}(T)$.

Lemma 7.3 *Every $L \in \mathcal{L}(G)$ satisfies the hypothesis and conclusion of the Local $C^*(L,S)$-Theorem for $S \in Syl_p(L)$.*

Proof. This follows from 2.3 and the minimality of G as a counterexample. □

Lemma 7.4 *Let $E \in \mathcal{B}_*(G)$. Then E is not subnormal in G.*

Proof. Let Ω be the set of all elements in $\mathcal{B}_*(G)$ that are subnormal in G and assume that $\Omega \neq \emptyset$. We will show that G is not a counterexample to the Local $C^*(G,T)$-Theorem. Set

$$G_0 := \prod_{E\in\Omega} E, \; R := C_G([V, G_0]).$$

Clearly no element of Ω is contained in R; in particular RT is a proper subgroup of G. Now 7.3 implies that $R \leq C^*(G,T)$, since R is normal in G.

By 6.12 G satisfies (a), (b), and (d) of the Local $C^*(G,T)$-Theorem; in particular $G_0 \trianglelefteq G$. It remains to show (c) and (e) to get the desired contradiction.

Let $E, \widetilde{E} \in \Omega$ with $E \neq \widetilde{E}$. Then by 6.3 $[V,E] \leq E$ and by 6.12 (c) $\widetilde{E} \leq C_G([V,E])$. The Dedekind identity then yields

$$EC_G([V,E]) \cap \widetilde{E}C_G([V,\widetilde{E}]) = E\widetilde{E}C_G([V,E][V,\widetilde{E}]).$$

Now an elementary induction argument shows that

$$\bigcap_{E\in\Omega}(B(T)EC_G([V,E])) = B(T)\bigcap_{E\in\Omega}(EC_G([V,E])) = B(T)G_0R.$$

Let $x \in G$. By 6.12 and 2.16

$$B(T)^x \le B(T)EC_G([V,E]) \text{ for every } E \in \Omega.$$

It follows that $B(T)G_0R = B(T)^xG_0R$, and $B(T)G_0R$ is normal in G. So the Frattini argument gives

$$G = G_0RN_G(B(T)) = G_0C^*(G,T).$$

Thus also (c) of the Local $C^*(G,T)$-Theorem holds.

Using 6.12 and 2.16 we get that $B(T)E/O_p(B(T)E) \cong SL_2(p^m)$ or S_{2m+1} for $E \in \Omega$. In the first case $B(T) \in Syl_p(B(T)E)$ and $N_{B(T)E}(B(T))$ is a maximal subgroup of $B(T)E$. In the second case $N_{B(T)E}(B(T)) = N_{B(T)E}(Y)$, where $YO_2(E)/O_2(E)$ is a subgroup of S_{2m+1} generated by a maximal set of commuting transpositions. Furthermore, we get from 2.13 that $W := [V,E]\Omega Z(T) = C_W(E) \times [V,E]$ and then from 2.11 that

$$\langle N_{B(T)E}(B(T)), C_{B(T)E}(\Omega(Z(T\cap B(T)E)))\rangle/O_2(B(T)E) \cong A_{2m};$$

in particular $\langle N_{B(T)E}(B(T)), C_{B(T)E}(\Omega(Z(T\cap B(T)E)))\rangle$ is a maximal subgroup of $B(T)E$. We conclude that in both cases $C^*(G,T) \cap B(T)E$ is a maximal subgroup of $B(T)E$ since $B(T)E \not\le C^*(G,T)$. Now also (e) of the Local $C^*(G,T)$-Theorem holds. But then G is not a counterexample. □

Lemma 7.5 *G is not a minimal parabolic group.*

Proof. Assume that G is minimal parabolic. Then G satisfies the hypothesis of the $C^{**}(G,T)$-Theorem for minimal parabolic groups because $C^{**}(G,T) \le C^*(G,T)$. Hence, we can apply this theorem to G, as it was already proven in Chapter 5.

Let $U := \Omega(Z(O_p(G)))$. Then there exists a subnormal subgroup E_1 of G with

$$E_1 \not\le C^*(G,T) \text{ and } C_G(U) \le E_1$$

such that

$$E_1/C_{E_1}(U) \cong SL_2(p^n) \text{ or } S_{2^m+1} \text{ (and } p=2),$$

and $[U,E_1]/C_{[U,E_1]}(E_1)$ is the corresponding natural module. Moreover, every other conjugate of E_1 in G centralizes $[U,E_1]$, and $U = C_U(E_1)[U,E_1]$. As $C_{[U,E_1]}(J(T)) \not\le C_{[U,E_1]}(E_1)$, this gives $B(T) \le N_G(E_1)$.

Let $H := B(T)E_1$ and $W := \Omega(Z(O_p(H)))$. Note that $[O_p(H),E_1] \le O_p(E_1) \le O_p(G)$ and that $[U,E_1] = [U,E_1,E_1]$ since $U = C_U(E_1)[U,E_1]$. As $[U,E_1]/C_{[U,E_1]}(E_1)$ is irreducible, the Three Subgroups Lemma yields that $[U,E_1] \le W$.

By 2.3 (c) H is of characteristic p. The action of E_1 on $[U, E_1]$ also shows that $H = E_1 C_H([U, E_1])$, so H satisfies the hypothesis of 3.8. Thus there exists a $B(T)$-block E with $H = B(T)EC_H(W)$; in particular $E \not\leq C^*(G,T)$. As $W_E \leq W$, we get from 6.6 that E is normal in H. Since $E = O^p(E)$ and $O^p(H) = O^p(E_1) \trianglelefteq\trianglelefteq G$ we conclude that E is subnormal in G. But this contradicts 7.4. □

Lemma 7.6 *There exists $F \in \mathcal{B}(T)$ such that $F \not\leq C^*(G,T)$. Moreover, for every $F \in \mathcal{B}(T)$ with $F \not\leq C^*(G,T)$ there exists $E \in \mathcal{B}_*(T)$ such that $F \leq E$. In particular $\mathcal{B}_*(G) \neq \emptyset$.*

Proof. By 7.5 G is not a minimal parabolic group. Hence there exists a proper subgroup $L \leq G$ with $T \leq L$ and $L \not\leq C^*(G,T)$. Then $L \in \mathcal{L}(T)$, and by 7.3 there exists $F \in \mathcal{B}(T)$ such that $F \not\leq C^*(G,T)$.

Let $F \leq E \in \mathcal{B}(T)$, where E is a maximal element of $\mathcal{B}(T)$. By 6.9 $E \in \mathcal{B}^*(G)$, and as $F \not\leq C^*(G,T)$, also $E \not\leq C^*(G,T)$. Hence $E \in \mathcal{B}_*(G)$. □

Lemma 7.7 *Let $E \in \mathcal{B}_*(T)$. Then $EB(T)$ is contained in a unique maximal element L of $\mathcal{L}(G)$, and $E \trianglelefteq\trianglelefteq L$.*

Proof. Let $\mathcal{U}$ be the set of all $L \in \mathcal{L}(G)$ containing $EB(T)$. For every $L \in \mathcal{U}$ define

$$\Sigma_L := \{E^g \mid g \in G,\ E^g \trianglelefteq\trianglelefteq L\}.$$

Let $L \in \mathcal{U}$ and $E^g \in \Sigma_L$. Since $E^g = O^p(E^g)$, the subnormality of E^g in L gives $O_p(L) \leq N_L(E^g)$ and thus $[\Omega(Z(O_p(L))), E^g] = W_{E^g}$. Using 2.8 (e) and 2.16 $J(T) \leq N_L(E)$. Since E^g is a $B(T^g)$-block, $J(T)$ is conjugate to $J(T^g)$ in $N_L(E)$ and so $B(T) \leq N_L(E)$. Therefore:

7.7.1 *Every element of Σ_L is a subnormal $B(T)$-block of L.*

Now let N be the subgroup generated by all subnormal $B(T)$-blocks of L. By 6.12 either E is one of these $B(T)$-blocks or $[N, E] = 1$.

Assume the second case, so $E \leq C_L(N)$. Let $B(T) \leq S \in Syl_p(L)$. As $C_L(N)$ does not contain any subnormal $B(T)$-block of L, we get from 7.3 that

$$E \leq C_L(N)S \leq C^*(L,S).$$

In particular $E \leq C^*(G,T^g)$ for $S \leq T^g$. But then $g \in N_G(B(T)) \leq C^*(G,T)$ and so $C^*(G,T) = C^*(G,T^g)$. This contradicts $E \in \mathcal{B}_*(T)$. We have shown that E is subnormal in L. Hence

7.7.2 $E \in \Sigma_L$ *for every $L \in \mathcal{U}$.*

Now let $\widetilde{L} \in \mathcal{U}$ and $K \in \Sigma_{\widetilde{L}}$. Suppose that $K \leq L$. From 7.7.1, applied to $\widetilde{L}$, we get that K is a $B(T)$-block. On the other hand, $K = E^g$ for some $g \in G$, so K is also a $B(T^g)$-block, and $B(T)$ and $B(T^g)$ are conjugate in $N_G(K)$. This shows that $K \not\leq C^*(G,T)$. Hence as above, K does not centralize all the subnormal $B(T)$-blocks of L, and 6.12 shows that K has to be one of these blocks. We have shown

7.7.3 *Let $K \in \Sigma_{\widetilde{L}}$ and $K \leq L$. Then $K \in \Sigma_L$.*

Now [12, 6.7.3] shows that $B(T)E$ is contained in a unique maximal element of $\mathcal{L}(G)$. □

Lemma 7.8 *Suppose that $[V,Z]=1$. Then $O_p(E) \leq O_p(G)$ for every $E \in \mathcal{B}(T)$ with $E \not\leq C^*(G,T)$.*

Proof. Observe that $C_T(Z) = B(T)$, so $[V,Z]=1$ implies $V \leq B(T)$. Pick $E \in \mathcal{B}(T)$ with $E \not\leq C^*(G,T)$; in particular $[V,E] \neq 1$ and $W_E \leq V$. If E is not exceptional, then $O_p(E) = W_E \leq V$, and we are done. Thus we may assume that E is exceptional. If $\underline{O_3(E)} \leq C_G(V)$, then 6.6 shows that $O_3(E) \leq O_3(G)$. Hence, we may also assume that $\overline{O_3(E)} \neq 1$, so W_E is the only non-central E-chief factor in $C_{EB(T)}(V)$. Set

$$V^* := \langle V\alpha \mid \alpha \in Aut(B(T))\rangle.$$

As no element of $B(T) \setminus C_{B(T)}(W_E)$ acts quadratically on $O_3(E)/Z(E)$, V^* centralizes W_E and

$$W_E \leq C_{B(T)}(V^*) \leq C_{EB(T)}(V).$$

It follows that $[C_{B(T)}(V^*), E] = W_E \leq C_{B(T)}(V^*)$. But $C_{B(T)}(V^*)$ is a non-trivial characteristic subgroup of $B(T)$, and thus $E \leq C^*(G,T)$, a contradiction. □

Lemma 7.9 *Let $E \in \mathcal{B}(T)$ with $W_E \leq V$ and $T_E := C_T(\overline{E})$. Then $[W_E, T_E] = 1$ and $[V, T_E, E] = 1$.*

Proof. Note that T_E normalizes W_E. Hence $[W_E, T_E] \leq Z(E)$ since $W_E/Z(E)$ is an irreducible E-module. As $[W_E, E] = W_E$, a first application of the Three Subgroups Lemma gives $[W_E, T_E] = 1$. But then $[V, E, T_E] = [W_E, T_E] = 1$ and $[E, T_E, V] = 1$, and another application of the same lemma also yields $[V, T_E, E] = 1$. □

Notation 7.10 *We use Definition 2.7. Recall that $G \neq C_G(V)N_G(J(T))$, so $J(T) \not\leq C_G(V)$. Hence by 2.8 $\mathcal{O}_T(\widetilde{V}) \neq \emptyset$, where $\widetilde{V} := V/V_0$ and $V_0 := C_V(O^p(G))$. We set*

$$Q_T(V) := \langle A \mid A \in \mathcal{O}_T^*(V)\rangle.$$

Moreover, set $T_0 := Z$ if $\overline{Z} \neq 1$ and $T_0 := Q_T(V)$ if $\overline{Z} = 1$.

Lemma 7.11 *Let $E \in \mathcal{B}(T)$ such that $E \not\leq C^*(G,T)$ and $\langle E, T_0\rangle \leq L \in \mathcal{L}(T)$. Then $T_0 \leq C_T(\overline{E})$.*

Proof. By 7.3 there exists $E \leq F \trianglelefteq\trianglelefteq L$ with $F \in \mathcal{B}(T)$. ¿From 3.8 we get that $[F, Z] \leq V$, so $[\overline{F}, \overline{Z}] = 1$. Thus $[\overline{F}, \overline{T}_0] = 1$ if $T_0 = Z$. Assume that $\overline{Z} = 1$. Then 7.8 gives $O_p(\overline{F}) = 1$. On the other hand, 2.16 implies $[F, A] \leq O_p(F)$ for $A \in \mathcal{O}^*_T(V)$. Hence also in this case $[\overline{F}, \overline{T}_0] = 1$. This shows that $[\overline{E}, \overline{T}_0] = 1$.

□

Lemma 7.12 *There exists $E \in \mathcal{B}_*(T)$ such that $T_0 \leq C_T(\overline{E})$.*

Proof. Let $T \leq P \leq G$ such that $|P|$ is minimal with $P \not\leq C^*(G, T)$. Then by 2.3 P is minimal parabolic, and thus by 7.5 $P \in \mathcal{L}(T)$. Hence 7.3 gives a $B(T)$-block $F \leq P$ with $F \not\leq C^*(G, T)$, in particular $[\overline{F}, \overline{T}_0] = 1$ by 7.11. According to 6.9 there exists $E \in \mathcal{B}_*(T)$ with $F \leq E$. By 6.7 we may assume that F and E are both symmetric and $F < E$. In particular $p = 2$ and $O_2(F) \leq O_2(E) \leq V$, so $\overline{E} \cong A_m$ and $\overline{F} \cong A_{m'}$, $3 \leq m' < m$, m' and m odd.

Pick $t \in B(T)$ such that $R := [W_E, t]$ has order 2 and $[\overline{F}, \overline{t}] \neq 1$, and set $E_t := O^2(C_E(\overline{t}))$. Then $\overline{E}_t \cong A_{m-2}$ and also E_t is a $B(T)$-block not in $C^*(G, T)$. Moreover $R \leq W_F$, and thus by 7.11 $\langle B(T)E_t, T_0\rangle \leq C_G(R)$. Observe that $\langle F, E_t\rangle = E$ and that $C_G(R) \in \mathcal{L}(T)$. So applying 7.11 we see that T_0 centralizes $\overline{E}_t$ and so also $\overline{E}$. □

Lemma 7.13 $\overline{T}_0 = 1$.

Proof. By way of contradiction we assume that $\overline{T}_0 \neq 1$. Recall that $O_p(\overline{G}) = 1$; so $N_{\overline{G}}(\overline{T}_0)$ is a proper subgroup of $\overline{G}$. We further set

$$T_E := C_T(\overline{E}),\ Q := \langle A \mid A \in \mathcal{O}^*_T(\widetilde{V})\rangle.$$

According to 7.12 there exists $E \in \mathcal{B}_*(T)$ with $T_0 \leq T_E$; in particular $\overline{EB(T)} \leq N_{\overline{G}}(\overline{T}_0)$. By 7.7 $EB(T)$ is contained in a unique maximal subgroup H of G and $E \trianglelefteq\trianglelefteq H$.

7.13.1 *$\overline{H}$ is the unique maximal subgroup of $\overline{G}$ containing $\overline{EB(T)}$; in particular $\overline{T} \leq N_{\overline{G}}(\overline{T}_0) \leq \overline{H}$.*

By 6.6 $C_G(V) \leq N_G(E)$ and by 7.4 $N_G(E) \leq H$; so $\overline{H}$ is a maximal subgroup of $\overline{G}$, and the uniqueness property of H implies that of $\overline{H}$. As $\overline{EB(T)} \leq N_{\overline{G}}(\overline{T}_0)$ and $\overline{T}_0$ is normal in $\overline{T}$, we also get the additional assertion.

7.13.2 *$[E, Q] \leq O_p(E)$, and $N_G(Q) \leq H$ if $Q_T(V) \neq 1$.*

Note that $C_H(\widetilde{V}) = C_H(V)$ because $O_p(\overline{G}) = 1$. Then 2.16, applied to $\overline{H}$ and $\widetilde{V}$, shows that $[E, Q] \leq O_p(E)$. On the other hand, Q is normal in $N_G(QO_p(E))$, so $EB(T) \leq N_G(Q)$. The uniqueness of H gives either $N_G(Q) \leq H$ or $\overline{Q} = 1$. In the second case 2.8 implies that also $Q_T(V) = 1$.

7.13.3 *$N_G(\overline{T}_1) \leq \overline{H}$ for every $\overline{B(T)}$-invariant subgroup $1 \neq \overline{T}_1 \leq \overline{T}_E$.*

As $O_p(\overline{G}) = 1$, $N_{\overline{G}}(\overline{T}_1)$ is a proper subgroup of $\overline{G}$ containing $\overline{EB(T)}$. Hence 7.13.1 implies $N_G(\overline{T}_1) \leq \overline{H}$.

According to 7.5 G is not minimal parabolic. Thus there exists a proper subgroup $P \leq G$ with $T \leq P$ and $P \not\leq H$. We choose P such that $|P|$ is minimal with that property. Then P is minimal parabolic since $N_G(T) \leq H$. Observe that $G = \langle P, E \rangle$ by the uniqueness of H. Set $A := Z \cap O_p(P)$ and $S := \langle A^P \rangle$.

7.13.4 *Either $\overline{A} = 1$ or $[W_E, S] \neq 1$.*

Recall that E has a unique non-central chief factor in V. Assume that $[W_E, S] = 1$. Then $C_V(S)$ is P- and $B(T)E$-invariant, so $C_V(S)$ is G-invariant. But now the definition of V shows that $V = C_V(S)$ and $\overline{S} = 1$.

7.13.5 $\overline{S} = 1$.

Assume that $\overline{S} \neq 1$. Then $T_0 = Z$ and according to 7.13.4 there exists $y \in P$ such that $[W_E, Y] \neq 1$ for $Y := A^y$. If Y normalizes W_E, then by 7.3 Y also normalizes $\overline{E}$, and $[W_E, E] = W_E$ implies that $[W_E, Y, E] \neq 1$. The action of $B(T)$ on W_E shows that $[W_E, Y] \cap Z \not\leq Z(G)$. If Y does not normalize W_E, then by 7.3 Y also does not normalize $(W_E \cap Z)C_V(G)$ and $[W_E \cap Z, Y] \not\leq Z(G)$.

Hence in both cases $R := [V, Y] \cap Z \not\leq Z(G)$, so $C_{\overline{G}}(R)$ is a proper subgroup of $\overline{G}$. On the other hand, by 7.11

$$R \leq [V, Y] \leq [V, T_E^y],$$

and so by 7.9 $[E^y, R] = 1$. Thus also $[B(T)^y E^y, R] = 1$ since $R \leq Y \leq Z^y$. The uniqueness of H^y implies

$$B(T) \leq C_G(R) \leq H^y.$$

In particular $B(T)$ and $B(T)^y$, and thus also Z and Z^y are conjugate in H^y. It follows from 7.13.1 that

$$\overline{EB(T)} \leq N_{\overline{G}}(\overline{Z}) \leq \overline{H}^y.$$

The uniqueness of H yields $H = H^y$ and $y \in H$. Now 6.12 shows that E is also an $B(T)^y$-block and by 7.9 and 7.11 $[W_E, Y] = 1$, contradicting the choice of Y.

Let $W_0 := C_V(O^p(P))$ and choose $1 \neq W \leq V$ minimal such that $W = [W, O^p(P)]$. Then $U := W/W \cap W_0$ is an irreducible P-module. Observe that by 3.3

$$C_T(W) \leq C_T(U) = O_p(P).$$

7.13.6 *$\overline{Z} \neq 1$, so $T_0 = Z$.*

Assume that $\overline{Z} = 1$. Then $T_0 = Q_T(V)$, so $\mathcal{O}_T^*(V) \neq \emptyset$ and thus by 2.8(c) also $\mathcal{O}_T^*(\widetilde{V}) \neq \emptyset$. Moreover, by 7.13.2 and 7.8 $Q \leq T_E$ and by 7.9 $[V, Q, E] = 1$. This shows that $G = \langle E, P\rangle \leq N_G(\mathcal{O}^*_{O_p(P)}(\widetilde{V}))$ and so

$$\mathcal{O}^*_{O_p(P)}(\widetilde{V}) = \emptyset.$$

Let $Q_0 := Q \cap O_p(P)$, and $W_1 := [W, Q_0]$. Then $[W_1, O^p(P)] = 1$ and by 7.9 $[W_1, E] = 1$, so $W_1 \leq V_0$ and $\widetilde{W_1} = 1$. Furthermore, let $A \in \mathcal{O}_T^*(\widetilde{V})$ and $A_0 := A \cap O_p(P)$. Then 2.8 (b) implies that $|A/A_0| > |\widetilde{W}/C_{\widetilde{W}}(A)|$, and thus also

$$|A/C_A(U)| > |U/C_U(A)|,$$

since $C_T(U) = O_p(P)$ and $C_A(U) = A_0$. On the other hand, by 5.6 and 3.6 applied to $P/C_P(U)$ we get $|A/C_A(U)| = |U/C_U(A)|$, a contradiction.

7.13.7 $[W, O_p(P)] = 1$.

By 7.13.5 and 7.13.6 $[\overline{O_p(P)}, \overline{Z}] = 1$ and $\overline{Z} \not\leq \overline{O_p(P)}$, so $[O_p(P), O^p(P)] \leq C_P(W)$ using 3.3. The Three Subgroups Lemma gives $[W, O_p(P)] = 1$, since $W = [W, O^p(P)]$.

We now derive a final contradiction using 7.13.6 and 7.13.7. From 3.3 we get that $C_T(U) = C_T(W) = O_p(P)$; in particular $O_p(P/C_P(W)) = 1$. Hence again 5.6 and 3.6 imply

$$(*) \qquad |A/C_A(W)||C_W(A)| = |W| \text{ for } A \in \mathcal{O}_P(W).$$

If $[W, J(T)] = 1$, then $Z \leq J(T) \leq O_p(P)$, which contradicts 7.13.5. Thus we have $[W, J(T)] \neq 1$. But now an elementary argument using $(*)$ gives

$$(A \cap O_p(P))W \in \mathcal{A}(T) \text{ for every } A \in \mathcal{A}(T),$$

so $W \leq J(T)$ and $Z \leq C_T(W) = O_p(P)$, again a contradiction to 7.13.5. □

Lemma 7.14 *Let $E \in \mathcal{B}_*(T)$. Then there exists $A \leq B(T)$ such that the following hold:*

(a) $[E, A] = E$ *and* $[V, A, A] = 1$.

(b) $|V/C_V(A)| = |\overline{A}|$ *and* $C_V(A) = C_V(a)$ *for every* $a \in A \setminus C_A(V)$.

Proof. If E is a symmetric block we let $F \le E$ be the $B(T)$-block given by 6.8 with $F/O_p(F) \cong SL_2(2)'$ and otherwise set $F := E$. Thus in all cases F is a linear block not in $C^*(G,T)$. Hence 7.8 and 7.13 give $O_p(F) \le O_p(G)$. The action of F on W_F shows that

$$(1) \qquad |B(T)/C_{B(T)}(W_F)| = q.$$

Observe that by 7.13 $[V,Z] = 1$, so $V \le B(T)$. Set

$$W^* := \langle W_F\alpha \mid \alpha \in Aut(B(T))\rangle \text{ and } V^* := \langle V\alpha \mid \alpha \in Aut(B(T))\rangle.$$

Assume first that F is not exceptional. Then

6.3 implies $[O_p(G), F] \le W_F \le W^*$. As $F \not\le C^*(G,T)$, this shows that $W^* \not\le O_p(G)$. Hence there exists $\alpha \in Aut(B(T))$ such that $A := W_F\alpha \not\le O_p(G)$; in particular $A \not\le O_p(F)$ and $[E,A] = E$. The action of A on W_F and (1) give

$$(2) \qquad q = |W_F/C_{W_F}(A)| = |V/C_V(A)| \text{ and } [V,A,A] = 1.$$

As $W_F/C_{W_F}(F)$ is a 2-dimensional $SL_2(q)$-module, we also get

$$(3) \qquad C_V(A) = C_V(a) \text{ for every } a \in A \setminus C_A(V).$$

Hence, A satisfies (a) and (b).

Assume now that F is exceptional; so $F = E$. Then no element in $B(T)\setminus C_{B(T)}(W_E)$ acts quadratically on $O_3(E)/Z(E)$. It follows that W^* is elementary abelian and $[V^*, W_E] = 1$. By 3.7 $O_3(E) \le B(T)$, so there exists $\alpha \in Aut(B(T))$ with $A := O_3(E)\alpha \not\le C_{B(T)}(E/O_3(E))$ for otherwise $\langle O_3(E)\alpha \mid \alpha \in Aut(B(T))\rangle$ is a characteristic subgroup of $B(T)$ normalized by E, contradicting $E \not\le C^*(G,T)$. In particular $[E,A] = E$.

Observe that $V^* \le C_{B(T)}(W_E\alpha) \cap C_{B(T)}(W_E)$, so

$$[V^*, O_3(E)] \le \Omega(Z(O_3(E))) \text{ and } [E, C_V(E)\alpha^{-1}] \le O_3(E).$$

This shows that

$$[O_3(E), O_3(E)C_V(E)\alpha^{-1}] = Z(E)[O_3(E), C_V(E)\alpha^{-1}] \le [O_3(E), C_V(E)\alpha^{-1}]Z(B(T)).$$

As $[O_3(E), O_3(E)C_V(E)\alpha^{-1}]$ is an E-submodule of $\Omega(Z(O_3(E)))$, we get that either

$$W_E \le [O_3(E), C_V(E)\alpha^{-1}]Z(B(T)) \text{ or } [O_3(E), C_V(E)\alpha^{-1}] \le Z(E).$$

In the first case $W_E\alpha \le [A, C_V(E)]Z(B(T)) \le C_V(E)Z(B(T))$ and thus $O_3(E) \le C_{B(T)}(W_E\alpha)$. But then

$$[A, O_3(E)] \le Z(A) \text{ and } [O_3(E), A, A] = 1,$$

which contradicts the definition of an exceptional $B(T)$-block.

So we are in the second case, in particular

$$[O_3(E), C_V(E)\alpha^{-1}, E] = 1 \text{ and } [E, O_3(E), C_V(E)\alpha^{-1}] = [O_3(E), C_V(E)\alpha^{-1}].$$

Observe that either $[C_V(E)\alpha^{-1}, E] = O_3(E)$ or $[C_V(E)\alpha^{-1}, E] \le Z(O_3(E))$. Hence the Three Subgroups Lemma gives

$[O_3(E), C_V(E)\alpha^{-1}] = Z(E)$ or $[O_3(E), C_V(E)\alpha^{-1}] = 1$, respectively.

Assume that $[O_3(E), C_V(E)\alpha^{-1}] = Z(E)$. Then $Z(E)\alpha = [A, C_V(E)] \le C_V(E)$ and thus

$$[O_3(E), A, A] = [O_3(E) \cap A, A] \le Z(E)\alpha \le C_V(E) \cap O_p(E) = Z(E).$$

Now A acts quadratically on $O_3(E)/Z(E)$, which contradicts the definition of an exceptional $B(T)$-block.

Thus, we have $[O_3(E), C_V(E)\alpha^{-1}] = 1$ and so $[A, C_V(E)] = 1$. Now as above (2) and (3) hold for A, so A satisfies (a) and (b). □

Theorem 7.15 *No group satisfies Hypothesis* 7.1.

Proof. Let $\mathcal{Y}$ be the set of all subgroups $A \le B(T)$ for which there exists $E \in \mathcal{B}_*(T)$ such that A and E satisfy (a) and (b) of 7.14, and let

$$\mathcal{D} := \cup_{g \in G} \mathcal{Y}^g \text{ and } \overline{\mathcal{D}} := \{\overline{A} \mid A \in \mathcal{D}\}.$$

We will show that $\overline{\mathcal{D}}$ satisfies Hypothesis 4.3.

It is evident from 7.14 that $\overline{\mathcal{D}}$ satisfies (i) and (ii) of 4.1. Moreover, 7.13 shows that property $(**)$ of 4.3 holds. Next we prove (iii) of 4.1.

Let $A, B \in \mathcal{D}$ such that $[\overline{A}, \overline{B}] = 1$. If $C_V(A) = C_V(B)$, then 4.3 $(**)$ yields $\overline{A} = \overline{B}$. Assume that $C_V(A) \ne C_V(B)$. Then by 4.1 (ii) $\overline{A} \cap \overline{B} = 1$ and so $|\overline{AB}| = |\overline{A}||\overline{B}|$. On the other hand, by 4.1 (ii) $|V/C_V(AB)| \le |\overline{A}||\overline{B}|$, so again $(**)$ gives $|\overline{AB}||C_V(AB)| = |V|$. This proves (iii) of 4.1.

Finally, we show property $(*)$ of 4.2 with $M := \overline{C^*(G,T)}$. Let $L := N_G(\mathcal{D} \cap T)$ and recall that $C_G(V) \le C^*(G,T)$. Hence

$$N_{\overline{G}}(\overline{\mathcal{D}} \cap \overline{T}) \le M \iff L \le C^*(G,T),$$

so we may assume by way of contradiction that $L \not\le C^*(G,T)$. Then $L \in \mathcal{L}(T)$, and by 7.3 there exists a $B(T)$-block E in L which is not in $C^*(G,T)$. According to 7.6

$E \leq F \in \mathcal{B}_*(T)$. But then by 7.14 there exists $A \in \mathcal{D} \cap B(T)$ such that $[F, A] = F$. This contradicts $A \in \mathcal{D} \cap T$ and $F \leq L$.

We have shown that $\overline{\mathcal{D}}$ satisfies Hypothesis 4.3. By 4.18 there exists a subnormal subgroup E^* in G such that $C_G(V) \leq E^* \not\leq C^*(G,T)$ and $\overline{E}^*$ satisfies (c) and (d) of 4.18. Moreover, by the definition of $\mathcal{D}$ and 4.18 (a) there exists $E \in \mathcal{B}_*(G)$ and $A \in \mathcal{D}$ such that $EA \leq E^*$; in particular $B(T)$ normalizes E^*. Now 3.8, applied to $B(T)E^*$, together with 6.6 shows that E is normal in E^*, and thus subnormal in G. This contradicts 7.4. □

The Proof of Corollary 1.9: Let M be the unique maximal subgroup of G containing T. As every characteristic subgroup X of $B(T)$ is also characteristic in T, we get $T \leq N_G(X)$. Hence $N_G(X) \leq M$ if X is non-trivial. Similarly $C_G(\Omega(Z(T))) \leq M$. It follows that $C^*(G,T) \leq M$; in particular $C^*(G,T) \neq G$. Hence G satisfies the hypothesis of the Local $C^*(G,T)$-Theorem and the Local $C^{**}(G,T)$-Theorem for Minimal Parabolic Groups. In particular, for every subnormal symmetric $B(T)$-block E not in $C^*(G,T)$, $E/O_2(E) \cong A_{2^n+1}$. Thus, G satisfies the conclusion of the Local $C(G,T)$-Theorem. Moreover, (e) of the Local $C^*(G,T)$-Theorem together with the fact that G is a minimal parabolic gives the additional statement in the conclusion of 1.9.

Acknowledgement. We would like to thank U. Meierfrankenfeld and Ch. Parker for reading an earlier draft of the manuscript and for their helpful suggestions.

References

[1] M. Aschbacher, A Factorization Theorem for 2-constrained Groups, Proc. LMS(3) 43 (1981), 450–477.

[2] B. Baumann, Endliche nichtauflösbare Gruppen mit einer nilpotenten maximalen Untergruppe, J. Alg. 38 (1976), 119–135.

[3] B. Baumann, Über endliche Gruppen mit einer zu $L_2(2^n)$ isomorphen Faktorgruppe. Proc. AMS 74 (1979), 215–222.

[4] A. Chermak, Quadratic action and the $\mathcal{P}(G,V)$-theorem in arbitrary characteristic, J. Group Theory 2 (1999), no. 1, 1–13.

[5] W. Feit, J. Thompson. Solvability of groups of odd order, Pacific J. Math. 13 (1963), 775-1029.

[6] G. Glauberman, Central elements in core-free groups, J. Alg. 4 (1966), 403–420.

[7] G. Glauberman, Weakly Closed Elements of Sylow Subgroups, Math. Z. 107 (1968) 1–20.

[8] G. Glauberman, Failure of factorization in p-solvable groups. Quart. J. Math. Oxford Ser. (2) 24 (1973), 71–77.

[9] D. Gorenstein, R. Lyons, On Aschbacher's local $C(G;T)$ theorem. Israel J. Math. 82 (1993), no. 1-3, 227–279.

[10] J. McLaughlin, Some groups generated by transvections. Arch. Math. 18 (1967), 364–368.

[11] J. McLaughlin, Some subgroups of $SL_n(F_2)$. Illinois J. Math. 13 (1969), 108–115.

[12] H. Kurzweil, B. Stellmacher, The Theory of Finite Groups, Springer Universitext, New York, 2004, xii+387 pp.

[13] U. Meierfrankenfeld, B. Stellmacher, G. Stroth, Finite groups of local characteristic p: an overview, Groups, combinatorics and geometry, Durham 2001, 155–192, World Sci. Publishing, River Edge, NJ, 2003.

[14] R. Niles, Pushing-up in finite groups. J. Algebra 57 (1979), 26–63.

[15] Ch. Parker, G. Parmeggiani, B. Stellmacher, The $P!$-theorem, J. Alg. 263 (2003), no. 1, 17–58.

[16] B. Stellmacher, Pushing Up, Arch. Math. 46 (1986), 8–17.

[17] J. Thompson, Factorizations of p-solvable groups. Pacific J. Math. 16 (1966), 371–372.