Groups of local characteristic p

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G is a finite $\mathcal{K}_p$-group, and p a fixed prime.

G has **characteristic** p if $C_G(O_p(G)) \leq O_p(G)$.

p-**local subgroup**: Normalizer of a non-trivial p-subgroup.

G has **local characteristic** p if all p-local subgroups of G have characteristic p.

Object of the talk: Describe the current status of the project to understand and classify the finite groups of local characteristic p with $O_p(G) = 1$.

Disclaimer: For p odd we do not expect to be able to achieve a complete classification. Some groups with a relatively small p-local structure will remain unclassified. In particular, we currently have no idea how to treat the case where G has a strongly p-embedded subgroup.

Motivation

1. We are trying to understand why the p-local subgroups of the finite simple groups look the way they do.

2. We hope that the classification of the groups of local characteristic 2 will serve as the first step in a third generation proof for the classification of the finite simple groups.

Future plans

1. Understand and classify all groups of parabolic characteristic p.

(Here a parabolic subgroup of G is a subgroup which contains a Sylow p-subgroup. And G is of parabolic characteritic p if all p-local, parabolic subgroups of G have characteristic p.)

2. Classify all finite simple groups which are not of parabolic characteristic 2.

Characteristics of the simple groups

Groups of Lie–Type

Let G be a finite simple group of Lie type defined over a field of characteristic r.

If $p = r$, then G is of local characteristic p.

If $p \neq r$ and a Sylow p-subgroup of G is not cyclic, then G is usually not of parabolic characteristic p.

Some exceptions:

$U_3(3) \cong G_2(2)'$, $Sp_4(2)' \cong L_2(9)$, $P\Omega_5(3) \cong \Omega_6^-(2)$, $L_3(4)$ and $U_4(3)$ all have local characteristics 2 and 3.

$L_4(3)$ has parabolic characteristics 2 and 3.

Alternating groups

The alternating groups usually have no local characteristic. But Alt($p^n + \epsilon$),$\epsilon \leq 2$ has parabolic characteristic p.

Characteristics of the sporadics

Group	local char.	parabolic char.
M_{11}	3	3
M_{12}		2, 3
J_1		
M_{22}	2	2
J_2		2
M_{23}	2	2
HS		2
J_3	2	2
M_{24}	2	2
McL	3	3
He		2
Ru		2, 5
Suz		2
ON	7	7
Co_3		3, 5
Co_2	2	3, 5
Fi_{22}	2	2
HN		2, 3, 5
Ly	5	5
Th	2, 5	2, 3, 5
Fi_{23}		3
Co_1		2, 3, 5
J_4	2, 11	2, 11
Fi'_{24}		2, 3, 7
B		2, 3, 5
M		2, 3, 5, 7, 13

Here we only listed cases with non-cyclic Sylow p-subgroup.

Notation

G is a group of local characteristic p with $O_p(G) = 1$.

$\mathcal{L} = \mathcal{L}_G = \{L \leq G \mid C_G(O_p(L)) \leq O_p(L)\}$

Note that $\mathcal{L}$ contains all the p-local subgroups of G.

$\mathcal{M}$ is the set of maximal members of $\mathcal{L}$ (by inclusion), i.e., the set of maximal p-local subgroups of G.

If $\mathcal{T}$ is a set of subgroups of G and $A \leq G$, then

$\mathcal{T}(A) = \{T \in \mathcal{T} \mid A \leq T\}$ and

$\mathcal{T}_A = \{T \in \mathcal{T} \mid T \leq A\}$.

S is a Sylow p-subgroup of G.

$Z = \Omega_1 \mathrm{Z}(S)$.

p-**core** of G with respect to S: $\langle \mathcal{M}(S) \rangle$.

The Pushing Up Theorem

Let H be a finite group and $T \in Syl_p(H)$. The group

$$P_H(T) := O^{p'}(C_H(\Omega_1 Z(T)))$$

is called the **point-stabilizer** of H with respect to T.

Theorem Let T be a p-group and let Σ be a set of groups such that for all $L \in \Sigma$

i) L is of characteristic p.

ii) $T \leq L$ and $T = O_p(N_L(T))$.

iii) $N_L(T)$ contains a point stabilizer of L.

Suppose that no non-trivial subgroup of T is normal in all $L \in \Sigma$. Then there exist $L \in \Sigma$ and $H \leq L$ with $B(T) \leq H$ such $O^p(H)$ has one of the following structures

$q^n SL_n(q)'$;

$q^{2n} Sp_{2n}(q)', p$ odd;

$q^{1+2n} Sp_{2n}(q)', p = 2$;

$2^6 G_2(2)', p = 2$;

$q^{1+6+8} Sp_6(q), p = 2$;

$2^{1+4+6} L_4(2), p = 2$; or

$q^{1+2+2} SL_2(q)', p = 3$.

(where q is a power of p)

Strongly p-embedded subgroups

We say that H is a strongly p-embedded subgroup of G if $H \neq G$ and $H \cap H^g$ is a p'-group for all $g \in G \setminus H$.

An elementary argument shows that G has a strongly p-embedded subgroup if and only if $\langle N_G(T) \mid 1 \neq T \leq S \rangle$ is a proper subgroup of G.

Bender classified all groups with a strongly 2-embedded subgroup.

For $p \neq 2$ no such theorem exists (independent from the CFSG).

The Open "Strongly p-embedded"-Problem

Determine all groups (of local characteristic p) with a strongly p-embedded subgroup and non-cyclic Sylow p-groups.

Proper p-core

Suppose now that G has no strongly p-embedded subgroup but the p-core $H := \langle \mathcal{M}(S) \rangle = \langle N_G(T) \mid 1 \neq T \trianglelefteq S \rangle$ is a proper subgroup of G.

Choose $L \in \mathcal{L}$ such that, in consecutive order, $L \not\leq H$, $|L \cap H|_p$ maximal, and L is minimal. An application of the Pushing Up Theorem gives us that $O^p(L) \sim q^{\epsilon} q^2 SL_2(q)'$, $\epsilon \in \{0, 1\}$.

For $p = 2$, Andreas Hirn is currently trying to obtain a contradiction in this situation.

The case $G = \langle \mathcal{M}(S) \rangle$

From now on we assume that G is equal to its p-core.

The basic idea here is to determine the structure of sufficiently many members L of $\mathcal{L}(S)$ to be able to identify a geometry on which G acts.

Let H and $\widehat{H}$ be finite groups and T and $\widehat{T}$ Sylow p-subgroups of H and $\widehat{H}$, respectively. We say that H has residual parabolic type $\widehat{H}$ if there exists a subset Λ of $\mathcal{L}_H(T)$ with $H = \langle \Lambda \rangle$ and an inclusion preserving bijection $\mathcal{L}_{\widehat{H}}(\widehat{T}) \to \Lambda, \widehat{L} \mapsto L$ such that for all $\widehat{L} \in \mathcal{L}_{\widehat{H}}(T)$, $L/O_p(L) \cong \widehat{L}/O_p(\widehat{L})$.

Often the residual parabolic type of a group is enough to identify it. So one of our main tasks is to derive information about $L/O_p(L)$ for at least some members of $\mathcal{L}(S)$. Our favorite method for this is to study the action of L on p-reduced normal subgroups, i.e. elementary abelian normal p-subgroups Y of L with

$$O_p(L/C_L(Y)) = 1.$$

Y_L is the largest p-reduced subgroup of L.

Modules

Let H be a finite group, V a p-reduced $\mathbb{F}_pH$-module and A an elementary abelian p-subgroup of V with $[V, A] \neq 1$.

If $|V/C_V(A)| \leq |A/C_A(V)|$, then A is an **offender** on V, and V is a **FF-module** for G.

If (i) A is an offender on $C_V(a)$, for all $a \in A \setminus C_A(V)$ (ii) $[V, A, A, A] = 1$ and (iii) $|V/C_V(A)| \leq |A/C_A(V)|^2$, then A is **near offender** on V, and V is a **near FF-module** for G.

If $[V, A, A] = 1$, then A is **quadratic** on V, and V is a **quadratic module** for G.

Note that FF- and near FF-modules are special cases of 2F-modules ($|V/C_V(A)| \leq |A/C_A(V)|^2$). So a list of FF-modules and near 2F-modules for quasi-simple groups can be easily obtained once the work of Guralnick and Malle on 2F-modules is complete.

Unfortunately the action of L on Y_L does not yield any information about $C_L(Y_L)$. An elementary argument shows that $Z := \Omega_1 \mathrm{Z}(S) \leq Y_L$ and so $C_L(Y_L) \leq C_G(Z)$.

So to make up for this misfortune we also study the group $N_G(Z)$. For this we pick

$$\widetilde{C} \in \mathcal{M} \text{ with } N_G(Z) \leq \widetilde{C}.$$

For a group H, define $F_p^*(H)$ by

$$F_p^*(H)/O_p(H) = F^*(H/O_p(H)).$$

To work with a group which is a little bit more manageable than $\widetilde{C}$ we define

$$E := O^p(F_p^*(C_{\widetilde{C}}(Y_{\widetilde{C}}))).$$

We now distinguish two cases:

E-**uniqueness** ($E!$): $\mathcal{M}(E) = \{\widetilde{C}\}$

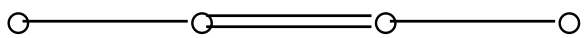
and

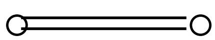
non E-**uniqueness** ($\neg E!$): $|\mathcal{M}(E)| \geq 2$.

$\neg E!$, an example

Here is an example for the $\neg E!$ case which illustrates why we look at overgroups of E despite the fact that these overgroups might not contain a Sylow p-subgroup.

Let $p = 2$ and $G = F_4(q).2$, where the 2 induces a graph automorphism. We would like to identify G via the F_4-building



But due to the graph automorphisms, not all of the parabolics of $F_4(q)$ are contained in parabolics of G. Now $E \leq F_4(q)$, namely E is the -parabolic. So E is contained in two different maximal parabolics M_1 and M_4 of $F_4(q)$.

Let $\Sigma = \{M_1, M_4\}$ and $R = O_2(M_1 \cap M_4)$. Then it is not too difficult to see that R and Σ fulfill the assumption of the Pushing Up Theorem.

$\neg E!$, a second example

Consider $G = E_8(q) \wr \mathrm{Sym}(p^k)$. Here E helps us to detect that G is not of local characteristic p.

Let H be the normalizer of a root subgroup in $E_8(q)$, i.e. the E_7-parabolic. Then $\tilde{C}$ is $H \wr \mathrm{Sym}(p^k)$, and E is a direct product of p^k copies of H. Hence, E is contained in the p-local subgroup L which is a direct product of $p^k - 1$ copies of H and $E_8(q)$.

$\neg E!$

The general idea of the $\neg E!$ case is to find a subgroup R of G and $\Sigma \subseteq \mathcal{L}(RE)$ such that we can apply the Pushing Up Theorem to R and Σ.

For this we make the following choices:

X is a point-stabilizer of some subnormal subgroup of $\widetilde{C}$, such that X is maximal with respect to $\mathcal{M}(EX) \neq \{\widetilde{C}\}$.

Next choose L such that in consecutive order:

$L \in \mathcal{L}(EX)$ with $L \not\leq \widetilde{C}$.

$|\widetilde{C} \cap L|_p$ is maximal.

$S_{\widetilde{C}}(L)$ is maximal, here $S_{\widetilde{C}}(L)$ is the largest subnormal subgroup of $\widetilde{C}$ contained in L.

$\widetilde{C} \cap L$ is maximal.

L is minimal.

Define $R = O_p(L \cap \widetilde{C})$. The following two situations need to be treated differently:

(PU-L): $\quad N_{\widetilde{C}}(R) \neq L \cap \widetilde{C}$.

(¬ PU-L) : $\quad N_{\widetilde{C}}(R) = L \cap \widetilde{C}$.

In the (PU-L)-Case put $H = N_{\widetilde{C}}(R)$ and $\Sigma = L^H$. A short and elementary argument shows that we can apply the Pushing Up Theorem.

The (¬ PU-L)-Case is more difficult. Here we choose an $\widetilde{C} \cap L$ invariant subnormal subgroup N of $\widetilde{C}$ minimal with respect to $N \not\leq L$. Put $H = N(\widetilde{C} \cap L)$ and $\Sigma = (H, L)$. If $Y_H \leq O_p(L)$ a rather lengthy amalgam type argument shows that the Pushing Up Theorem* can be applied. This leaves us with

*Actually one needs a stronger (not yet finished) version of the Pushing Up Theorem than stated above

The Open "$\neg E!, b = 1$"-Problem

In the $\neg E!$ and ($\neg$ PU-L) Case, determine the structure of H and L if $Y_H \not\leq O_p(L)$.

$E!$

We usually apply $E!$ through an intermediate property we call Q-uniqueness. Let $Q = O_p(\widetilde{C})$.

(Q!) $\quad C_G(x) \leq \widetilde{C}$ for all $1 \neq x \in C_G(Q)$.

An application of Thompson's $P \times Q$-Lemma shows that $[x, E] = 1$ for all $x \in \Omega_1 \mathrm{Z}(Q)$. Hence $E \leq C_G(x)$ and so $E!$ implies $C_G(x) \leq \widetilde{C}$. Thus

$E!$ implies $Q!$

Elementary consequences of $Q!$

For $L \in \mathcal{L}$ define $L^\circ = \langle Q^g \mid g \in G, Q^g \leq L \rangle$.

Lemma Suppose $Q!$.

(a) $\widetilde{C}^\circ = Q$, in particular, any p-subgroup of G contains at most one conjugate of Q.

(b) If $L \in \mathcal{L}$ with $Q \leq O_p(L)$, then $L \leq \widetilde{C}$. In particular, if $1 \neq X \leq Z(Q)$ then $N_G(X) \leq \widetilde{C}$.

(c) If $Q_1, Q_2 \in Q^G$ with $Z(Q_1) \cap Z(Q_2) \neq 1$, then $Q_1 = Q_2$.

(d) Let $L \in \mathcal{L}$ with $Q \leq L$. Then

  (a) $L^\circ = \langle Q^{L^\circ} \rangle$

  (b) $L = L^\circ(L \cap \widetilde{C})$.

  (c) $[C_L(Y_L), L^\circ] \leq O_p(L)$.

  (d) If L acts transitively on $Y_L^\sharp$, then $L^\circ = N_G(Y_L)^\circ$.

  (e) If $L^\circ \neq Q$, then $C_{Y_L}(L^\circ) = 1$.

To state our first Structure Theorem we need a few more definitions.

A finite group L is p-**minimal** if a Sylow p-subgroup of L is contained in a unique maximal subgroup of L but is not normal in L.

$P \in \mathcal{L}$ is a **minimal parabolic subgroup** if P is parabolic and p-minimal.

$\mathcal{P}$ denotes the set of minimal parabolics of G.

For $\mathcal{T} \subseteq \mathcal{L}$ let $\mathcal{T}^\circ = \{T \in \mathcal{T} \mid O^p(T) \leq T^\circ\}$.

It is an easy consequence of the definitions that if $P \in \mathcal{P}(S)$, then $P \in \mathcal{P}^\circ$ if and only if $P \not\leq \widetilde{C}$.

Let $P \in \mathcal{P}^\circ(S)$. We say that $gb(P) > 1$ if $Y_M \leq Q$ for all $M \in \mathcal{L}(P)$. Otherwise we say $gb(P) = 1$.

The Structure Theorem for $Y_M \leq Q$

Theorem Suppose that $Q!$ holds and that $P \in \mathcal{P}^\circ(S)$ with $gb(P) > 1$. Let $M \in \mathcal{L}(P)$ with M° maximal. Then one of the following two cases holds for $\overline{M} := M/C_M(Y_M)$ and $M_0 := M^\circ C_S(Y_M)$:

1.

(a) $\overline{M_0} \cong SL_n(p^k)$ or $Sp_{2n}(p^k)$ and $C_{\overline{M}}(\overline{M_0}) \cong C_r$, $r | p^k - 1$, or $\overline{M} \cong Sp_4(2)$ and $\overline{M_0} \cong Sp_4(2)'$ (and $p = 2$),

(b) $[Y_M, M_0]$ is the corresponding natural module for $\overline{M_0}$,

(c) $C_{M_0}(Y_M) = O_p(M_0)$, or $p = 2$ and $M_0/O_2(M_0) \cong 3Sp_4(2)'$.

2.

(a) $P = M_0 S$, $Y_M = Y_P$, and there exists a unique normal subgroup P^* of P containing $O_p(P)$ such that

(b) $\overline{P^*} = K_1 \times \cdots \times K_r$, $K_i \cong SL_2(p^k)$, $Y_M = V_1 \times \cdots \times V_r$, where $V_i := [Y_M, K_i]$ is a natural K_i-module,

(c) Q permutes the subgroups K_i of (b) transitively,

(d) $O^p(P) = O^p(P^*) = O^p(M_0)$, and $P^* C_M(Y_P)$ is normal in M,

(e) either $C_{M^\circ}(Y_P) = O_p(M_0)$, or $p = 2$, $r > 1$, $K_i \cong SL_2(2)$, and $C_{M_0}(Y_P)/O_2(M_0) = Z(M_0/O_2(M_0))$ is a 3-group.

The Structure Theorem for $Y_M \not\leq Q$.

Theorem Let $M \in \mathcal{L}(S)$ with M° maximal. Assume that $Y_M \not\leq Q$. Set $K = F^*(M^\circ S/C_{M^\circ S}(Y_M))$. Then one of the following holds:

1. K is quasisimple and isomorphic to $SL(n,q)$, $Sp(2n,q)'$, $\Omega^{\pm}(n,q)$, or $E_6(q)$, q a power of p. In case of $K \cong SL_n(q)$ or $E_6(q)$ no element in $M^\circ S$ induces diagram automorphisms.

2. $K \cong SL_n(q)' * SL_m(q)'$, q a power of p. Further Y_M is the tensor product module.

3. $p = 2$ and $K \cong 3A_6$, M_{22} or M_{24}.

4. $p = 3$ and $K \cong M_{11}$ or $2M_{12}$.

5. $M^\circ S$ is a minimal parabolic.

Further Y_M is a near FF-module, and except for case 5, Y_M contains a $M^\circ S$ submodule V as described on the next slide.

K	prime	module	example
$SL_n(q)$	p	ext. square	$\Omega_{2n}(q)$
$SL_n(q)$	p	sym. square	$Sp_{2n}(q)$
$SL_n(q^2)$	p	$V(\lambda_1) \otimes V(\lambda_1^\sigma)$	$SU_{2n}(q)$
A_6	2	natural	Suz
$3A_6$	2	6-dim	M_{24}
$Sp_8(2)$	2	8-dim	B
$\Omega_n^\pm(q)$	p	natural	$\Omega_{n+2}^\pm(q)$
$\Omega_{10}^\pm(q)$	2	half spin	$E_6(q)$
$E_6(q)$	p	$V(\lambda_1)$	$E_7(q)$
M_{11}	3	5-dim	Co_3
$2M_{12}$	3	6-dim	Co_2
M_{22}	2	10-dim	$M(22)$
M_{24}	2	11-dim	$M(24)$

The $P!$-Theorems

The P!-Theorem,I Suppose that $Q!$ holds and $\langle \mathcal{P}^\circ(S) \rangle \notin \mathcal{L}$. Then

(a) p is odd.

(b) $Q = \mathsf{B}(S)$, $\widetilde{C} = N_G(\mathsf{B}(S))$ and Q has order q^3, q a power of p.

(c) $P^\circ \sim q^2 SL_2(q)$ for all $P \in \mathcal{P}^\circ(S)$.

We say that P-Uniqueness ($P!$) holds in G provided that:

(P!-1) There exists a unique $P \in \mathcal{P}^\circ(S)$.

(P!-2) $P^\circ / O_p(P^\circ) \cong SL_2(q)$, q a power of p.

(P!-3) Y_P is a natural module for P°.

(P!-4) $C_{Y_P}(S \cap P^\circ)$ is normal in $\widetilde{C}$.

The P!-Theorem,II Suppose that

(i) $Q!$ holds.

(ii) There exists $P \in \mathcal{P}^\circ(S)$ with $gb(P) > 1$.

(iii) $M := \langle \mathcal{P}^\circ(S) \rangle \in \mathcal{L}$

Then $P!$ holds in G.

The $\widetilde{P}$! Theorem

Suppose Q! and P! and let P be the unique member of $\mathcal{P}^\circ(S)$. We say that $\widetilde{P}$! holds in G provided that

($\widetilde{P}$!-1) There exists at most one $\widetilde{P} \in \mathcal{P}(S)$ such that $\widetilde{P}$ does not normalize P° and $M := \langle P, \widetilde{P} \rangle \in \mathcal{L}$.

($\widetilde{P}$!-2) If such a $\widetilde{P}$ exists then,

(a) $M \in \mathcal{L}^\circ$.

(b) $M^\circ / C_{M^\circ}(Y_M) \cong SL_3(q), Sp_4(q)$ or $Sp_4(2)'$

(c) Y_M is a corresponding natural module.

The $\widetilde{P}$! Theorem Suppose Q! and that $gb(P) > 1$ for some $P \in \mathcal{P}^\circ(S)$. Then one of the following is true:

1. G fulfills $\widetilde{P}$!.

2. Let $\widetilde{P} \in \mathcal{P}(S)$ with $\widetilde{P} \not\leq N_G(P^\circ)$ and $M := \langle P, \widetilde{P} \rangle \in \mathcal{L}$. Then

   (a) $p = 3$ or 5.

   (b) $M/O_p(M) \cong SL_3(p)$.

   (c) $O_p(M)/Z(O_p(M))$ and $Z(O_p(M))$ are natural $SL_3(p)$-modules for $M/O_p(M)$, dual to each other.

Define the rank of G to be the minimal size of a non-empty subset Σ of $\mathcal{P}(S)$ with $\langle \Sigma \rangle \notin \mathcal{L}$. If no such subset exists we define the rank to be 1. Note that $\operatorname{rank} G = 1$ if and only if $|\mathcal{M}(S)| = 1$, which is impossible under our current assumption that G is equal to its p-core.

Elementary consequences of $P!$ and $\widetilde{P}!$

Lemma Suppose $E!$, $P!$, $\widetilde{P}!$ and that G has rank at least three. Let $L = N_G(P^\circ)$ and $H = (L \cap \widetilde{C})E$. Then

(a) There exists a unique $\widetilde{P} \in \mathcal{P}_H(S)$ with $\widetilde{P} \not\leq L$. Moreover, $\widetilde{P} \leq ES$.

(b) $\widetilde{P}/O_p(\widetilde{P}) \sim SL_2(q).p^k$.

(c) H has a unique p-component K.

(d) $H = K(L \cap H)$, $L \cap H$ is a maximal subgroup of H and $O_p(H \cap L) \neq O_p(H)$.

(e) Let $D = C_H(K/O_p(K))$. Then $D/O_p(H)$ is isomorphic to a section of the Borel subgroup of Aut$(SL_2(q))$.

(f) Let $Z_0 = C_{Y_P}(S \cap P^\circ)$ and $V = \langle Y_P^H \rangle$. Then $Z_0 \trianglelefteq V$ and $V \leq Q \leq O_p(H)$.

(g) Let $\overline{V} = V/Z_0$. Then $H \cap L$ contains a point-stabilizer for H on $\overline{V}$.

(h) $\langle H, L \rangle \notin \mathcal{L}$.

The Small World Theorem

Suppose Q! and let $P \in \mathcal{P}^{\circ}(S)$. We say that $gb(P) = 2$ if $gb(P) > 1$ and $\langle (Y_P)^E \rangle$ is not abelian.

The Small World Theorem Suppose E! and let $P \in \mathcal{P}^{\circ}(S)$. Then one of the following holds:

1. G has rank 1 or 2.

2. $gb(P) = 1$ or $gb(P) = 2$.

3. Neither 1. nor 2. hold and

   (a) There exists a unique $M \in \mathcal{M}(S)$ with $\widetilde{C} \neq M \neq N_G(P^{\circ})$.

   (b) $M^{\circ}/C_{M^{\circ}}(Y_M) \cong SL_3(q)$ or $Sp_4(q)$.

   (c) $\widetilde{C}$ has a unique p-component K and $K/O_p(K) \cong SL_3(q)$, $Sp_4(q)$ or $G_2(q)$.

The Open Rank 3 Problem

Rule out Case 3 of the Small World Theorem.

The rank 2 Case

Rank 2 Theorem, I Suppose $E!$, $P!$, $\widetilde{P}!$ and that G has rank 2. Choose $\widetilde{P} \in \mathcal{P}(S)$ such that

(i) $\langle P, \widetilde{P} \rangle \notin \mathcal{L}$.

(ii) $H := \langle P \cap \widetilde{C}, \widetilde{P} \rangle$ is minimal with respect to (i).

(iii) $\widetilde{P}$ is minimal with respect to (**??**) and (**??**)

Then one of the following holds:

1. $Y_P \not\leq O_p(\widetilde{P})$.

2. $(P^\circ N_H(P^\circ), H)$ is a weak BN-pair.

3. The structure of P and $\widetilde{P}$ is as in one of the following groups.

   1. For $p = 2$: $U_4(3).2^e$, $G_2(3).2^e$, $D_4(3).2^e$, $HS.2^e$, F_3, $F_5.2^e$ or Ru.

   2. For $p = 3$: $D_4(3^n).3^e$, Fi_{23}, F_2.

   3. For $p = 5$: F_2.

   4. For $p = 7$: F_1.

In Case 2. one can apply the Delgado-Stellmacher Weak-BN Pair paper. Which leave us in the rank 2 Case with

The Open "rank 2, gb(P)=1" Problem

Suppose $E!$ holds and there exist $P \in \mathcal{P}^\circ(S)$ and $\widetilde{P} \in \mathcal{P}(S)$ such that $\langle P, \widetilde{P} \rangle \notin \mathcal{L}$ and $gb(P) = 1$. Determine the structure of P and $\widetilde{P}$.

gb(P)=2

The Open “gb(P)=2” Problem

Suppose E!, P!, $\widetilde{P}$! and that $\langle Y_P^E \rangle$ is not abelian. Determine the structure of P and E.

The “$gb(P) = 2$”-Problem is actually just a special case of the symplectic amalgams treated by Parker and Rowley. But since the assumptions of the “$gb(P) = 2$”-Problem are stronger than for symplectic amalgams, we believe that a significantly shorter proof should be possible.

gb(P)=1

The H-Structure Theorem (for p=2) Supppose $E!$, rank $G \geq 3$ and that there exists $M \in \mathcal{M}(S)$ with M° maximal and $Y_M \not\leq Q$. If $p = 2$, then there exists $M^\circ S \leq H \leq G$ with $O_p(H) = 1$ such that H is of parabolic type H^* where H^* is one of the following groups:

1. A group of Lie-Type in characteristic p with Lie-rank at least three.

2. M_{24}, He, Co_2, $M(22).2^e$, Co_1, J_4, $M(24)'.2^e$, Suz, F_2 or F_1.

3. $U_4(3).2^e$.

Moreover, $M^\circ S$ has the same structure as its corresponding group in H^*.