Hypersolvable Groups

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Abstract

Call a group G hypersolvable if it has an ascending series with $G/C_G(A)$ solvable for each factor A of the series. In this paper we establish some basic facts about hypersolvable groups. We also prove that if G is a perfect Fitting p-group such that every proper subgroup is contained in a proper normal subgroup, then G has a proper non-hypersolvable subgroup.

1 Introduction

Let $\mathcal{D}$ be a class of pairs (B, A) such that B is a group acting faithfully on the group A. Let G be a group acting on a group N. A G-invariant normal series $\mathcal{A}$ of N is called a *$\mathcal{D}$-series* for G on N if $(G/C_G(A), A) \in \mathcal{D}$ for all factors A of $\mathcal{A}$.

An ascending $\mathcal{D}$-series for G on N is called a *hyper-$\mathcal{D}$* series. If such a series exists we say that G acts hyper-$\mathcal{D}$ on N. G is hyper-$\mathcal{D}$ means that G acts hyper-$\mathcal{D}$ on G. If $\mathcal{G}_1$, $\mathcal{G}_2$ are classes of groups, then $(\mathcal{G}_1, \mathcal{G}_2)$ denotes the class of pairs (B, A) with $B \in \mathcal{G}_1$, $A \in \mathcal{G}_2$ and B acting faithfully on A. We denote the class of all groups with $*$. So $(*, *)$ denotes the class of all pairs of groups (B, A) with B acting faithfully on A.

Consider the case $N = G$. Observe that hyper-($*$,abelian) groups are the hyperabelian groups and hyper-(1,$*$) groups are the hypercentral groups. We say that G is *hypersolvable* if G is hyper-(solvable,$*$). This notation might be slightly misleading since one probably would be tempted to define a hypersolvable group to be a hyper-($*$,solvable) group. But as the hyper-($*$,solvable) groups are just the hyperabelian groups such a definition would not be of much use. Similarly we define a hypernilpotent group to be a hyper-(nilpotent,$*$)-group.

Unwinding the definitions we see that a group G is hypersolvable if and only if G has a normal ascending series $\mathcal{A}$ such that $G/C_G(A)$ is solvable for all factors A of $\mathcal{A}$.

We say that G acts strongly hyper-$\mathcal{D}$ on N if for all G-invariant $M \lhd N$ there exists a G-invariant $M < \tilde{M} \unlhd N$ with $(G/C_G(\tilde{M}/M), \tilde{M}/M) \in \mathcal{D}$.

In section 2 we establish some basic facts about hyper-$\mathcal{D}$ groups. In particular, we show that if $\mathcal{D}$ is closed under quotients, then G acts hyper-$\mathcal{D}$ on N if and only if G acts strongly hyper-$\mathcal{D}$ on N.

In section 3 we investigate hyper-$(\mathcal{G}, *)$-groups, where $\mathcal{G}$ is a countable union of group varieties.

In section 4 we apply Theorem 3.9 to obtain commutator conditions which characterize hypersolvable and hypernilpotent groups.

In section 5 it is shown that the class of hypersolvable groups lies strictly between the classes of hypercentral-by-solvable and hypercentral-by-(residually solvable) groups. Similarly we show that the class of hypernilpotent groups lies strictly between the classes of hypercentral-by-nilpotent and hypercentral-by-(residually nilpotent) groups.

Recall that a Fitting group is a locally (nilpotent and normal) group, that is a group in which every finitely generated subgroup lies in a nilpotent, normal subgroup. We say that a group G is NNC-*proper* if G is not the normal closure of a proper subgroup. NNC-proper Fitting p-groups are considered in [AÖ1] and given a criterion for these groups to be non-perfect. In Theorem 7.3 we prove that every NNC-proper, perfect, Fitting p-group has a proper non-hypersolvable subgroup.

As a supplement to Theorem 7.3, in section 8 we provide some conditions which ensure that a group is NNC-proper.

2 Basic Properties of hyper-$\mathcal{D}$ groups

Let $\mathcal{D} \subseteq (*, *)$ (that is a class of pairs (A, B) of groups A and B with A acting faithfully on B, which is closed under isomorphism). We say that $\mathcal{D}$ is closed under subgroups if for all $(A, B) \in \mathcal{D}$, all $D \leq A$ and all D-invariant $E \leq B$ we have $(D/C_D(E), E) \in \mathcal{D}$. We say that $\mathcal{D}$ is closed under quotients if for all $(A, B) \in \mathcal{D}$ and all A-invariant $E \unlhd B$, $(A/C_A(B/E), B/E) \in \mathcal{D}$. A group G is finitely hyper-$\mathcal{D}$ if it has a finite hyper-$\mathcal{D}$-series.

Lemma 2.1 *Let $\mathcal{D} \subseteq (*, *)$ and let G be acting hyper-$\mathcal{D}$ on N.*

(a) Suppose that $\mathcal{D}$ is closed under subgroups. Let $H \leq G$ and let M be an H-invariant subgroup of N. Then H acts hyper-$\mathcal{D}$ on M.

(b) Suppose that $\mathcal{D}$ is closed under quotients and that M is a G-invariant normal subgroup of N. Then G acts hyper-$\mathcal{D}$ on N/M.

Proof: Let $(N_\alpha)_\alpha$ be a hyper-$\mathcal{D}$ series for G on N.

(a) Just observe that $(M \cap N_\alpha)_\alpha$ is a hyper-$\mathcal{D}$ series for H on M.

(b) Since quotients of ascending series are again ascending series, $(N_\alpha M/M)_\alpha$ is a hyper-$\mathcal{D}$ series for G on N/M. □

Lemma 2.2 *Let $\mathcal{D} \subseteq (*, *)$ and let G be acting on N.*

(a) If G acts strongly hyper-$\mathcal{D}$ on N, then G acts hyper-$\mathcal{D}$ on N.

(b) If $\mathcal{D}$ is closed under quotients, then G acts strongly hyper-$\mathcal{D}$ on N if and only if it acts hyper-$\mathcal{D}$ on N.

Proof: (a) Define $N_0 = 1$. If α is a limit ordinal, put $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$. If $\alpha = \beta + 1$ and $N_\beta \neq N$, put $N_\alpha = \tilde{N}_\beta$. Then $(N_\alpha)_\alpha$ is a hyper-$\mathcal{D}$ series on N.

(b) Follows from (a) and 2.1(b). □

Lemma 2.3 *Let $\mathcal{D} \subseteq (*, *)$ and let G be acting on N. Suppose that there exists a G-invariant normal ascending series on N such that G acts hyper-$\mathcal{D}$ on each of the factors. Then G acts hyper-$\mathcal{D}$ on N. In particular, if $(N_i, i \in I)$ is a family of groups with G acting hyper-$\mathcal{D}$ on each N_i, then G acts hyper-$\mathcal{D}$ on $\bigoplus_{i\in I} N_i$.*

Proof: For the first statement use the series on the factors to refine the given series to a hyper-$\mathcal{D}$ series.

For the second statement well-order I such that I has a maximal element. For $i \in I$ define $N_i^+ = \bigoplus_{j\leq i} N_j$ and $N_i^- = \bigoplus_{j<i} N_j$. Then $\{N_i^-, N_i^+ \mid i \in I\}$ is G-invariant normal ascending series on $\bigoplus_{i\in I} N_i$ with factors $N_i^+/N_i^- \cong N_i$. So the second statement follows from the first. □

Proposition 2.4 *Let $\mathcal{G}$ be any class of groups.*

(a) Suppose $\mathcal{G}$ is closed under quotients. Then hypercentral-by-$\mathcal{G}$ groups are hyper-$(\mathcal{G}, *)$ and nilpotent-by-$\mathcal{G}$ groups are finitely hyper-$(\mathcal{G}, *)$.

(b) Hyper-$(\mathcal{G}, *)$ groups are hypercentral-by-(residually $\mathcal{G}$). If $\mathcal{G}$ is closed under finite subdirect products then finitely hyper-$(\mathcal{G}, *)$-groups are nilpotent-by-$\mathcal{G}$.

(c) If $\mathcal{G}$ is closed under quotients and finite subdirect products, then the nilpotent-by-$\mathcal{G}$-groups are exactly the finitely hyper-$(\mathcal{G}, *)$ groups.

Proof: (a) Let $H \trianglelefteq G$ such that H is hypercentral and $G/H \in \mathcal{G}$. Let $\mathcal{Z}$ be the hypercentral series for H. Then $\mathcal{Z}$ is G-invariant. If Z is a factor of $\mathcal{Z}$, then $[Z, H] = 1$ and so $G/C_G(Z)$ is a quotient of G/H. Thus $G/C_G(Z) \in \mathcal{G}$. Also $G/C_G(G/H)$ is a quotient of G/H and so $\mathcal{Z} \cup \{G\}$ is a hyper-$(\mathcal{G}, *)$ series for G. If H is nilpotent, $\mathcal{Z}$ is finite and (a) is proved.

(b) Let $\mathcal{A} = (A_\alpha)_\alpha$ be a hyper-$(\mathcal{G}, *)$-series for G and put

$$H = \bigcap\{C_G(A) \mid A \text{ a factor of } \mathcal{A}\}.$$

Since $G/C_G(A) \in \mathcal{G}$ for all factors A, G/H is subdirect product of members of $\mathcal{G}$ and so residually-$\mathcal{G}$. Moreover $(A_\alpha \cap H)_\alpha$ is a hypercentral series for H and so H is hypercentral. If $\mathcal{A}$ is finite and $\mathcal{G}$ is closed under finite subdirect products, then $G/H \in \mathcal{G}$ and H is nilpotent. So (b) holds.

(c) Follows from (a) and (b). □

3 Countable unions of group varieties

For $n \in \mathbb{N}$, $F(n)$ denotes the free group on n-generators $x_1, x_2, \ldots, x_n$. Let G be a group, $m \in \mathbb{N} \cup \{\infty\}$ with $m \geq n$ and $g = (g_i)_{i=1}^m \in G^m$. Then there exists a unique homomorphism $\phi_g : F(n) \to G$ with $x_i \to g_i$ for all $1 \leq i \leq n$. Given a word $w \in F(n)$ we write $w(g)$ for $\phi_g(w)$. So if $w = x_{i_1}x_{i_2}\dots x_{i_m}$ with $1 \le i_k \le n$, then $w(g) = g_{i_1}g_{i_2}\dots g_{i_m}$. If $m \le n$ we view $F(m)$ as a subgroup of $F(n)$. Let $m = m(w) \in \mathbb{N}$ be minimal with $w \in F(m)$. Let $F := \bigcup_{n=1}^{\infty} F(n)$ and let $\mathcal{W}$ be the set of subsets of F. So the elements of $\mathcal{W}$ are sets of words.

Put $G^w := \langle w(g) \mid g \in G^n \rangle$ and note that G^w is a normal subgroup of G. For a set $W \in \mathcal{W}$ let $G^W = \langle G^w \mid w \in W \rangle$. Let $\mathcal{G}(W)$ be the class of groups G with $G^W = 1$, that is $\mathcal{G}(W)$ is the variety defined by W.

Proposition 3.1 *Let $W \in \mathcal{W}$ and let G be a group. Then G is hyper-$(\mathcal{G}(W), *)$ if and only if G^W is hypercentral.*

Proof: Let $N \trianglelefteq G$. Then $G/N \in \mathcal{G}(W)$ if and only if $G^W \le N$ if and only if G/N is residually $\mathcal{G}(W)$. Thus the proposition follows from 2.4. □

Definition 3.2 *Let $W = (W_i)_{i=1}^{\infty} \in \mathcal{W}^{\infty}$ be a sequence of sets of words.*

(a) W is decreasing if $F^{W_{i+1}} \le F^{W_i}$ for all i.

(b) W is almost decreasing if for all $i, j \in \mathbb{Z}^+$ there exists $k \ge j$ with $F^{W_k} \le F^{W_i}$.

(c) $\mathcal{G}(W) = \bigcup_{i=1}^{\infty} \mathcal{G}(W_i)$.

Lemma 3.3 *Let G be group.*

(a) Let $V, W \in \mathcal{W}$ with $F^V \le F^W$. Then $G^V \le G^W$.

(b) Let $W \in \mathcal{W}^{\infty}$ be almost decreasing. Then $(G^{W_i})_{i=1}^{\infty}$ is almost decreasing, that is for $i, j \in \mathbb{Z}^+$ there exists $k \ge j$ with $G^{W_k} \le G^{W_i}$.

Proof: (a) Let $g \in G^V$. Then $g \in H^V$ for some finitely generated subgroup H of G. Let $\alpha : F \to H$ be an onto homomorphism. Then $H^V = \alpha(F^V) \le \alpha(F^W) = H^W$ and so $g \in H^W \le G^W$.

(b) follows from (a). □

Definition 3.4 *Let G be a group acting on a group N, $W \in \mathcal{W}^{\infty}$ and α an ordinal.*

(a) Define $H_\alpha = \mathrm{Hyp}_\alpha^W(G, N)$ inductively as follows:

$$\begin{array}{rcll} H_\alpha = & & 1 & \text{if } \alpha = 0 \\ H_\alpha = & & \bigcup_{\beta<\alpha} H_\beta & \text{if } 0 \ne \alpha \text{ is a limit ordinal} \\ H_\alpha/H_{\alpha-1} = & & C_{N/H_{\alpha-1}}([N, G^{W_k}]G^{W_k}) & \text{if } \alpha = \beta + k \text{ with} \\ & & & \beta \text{ a limit ordinal and } k \in \mathbb{Z}^+. \end{array}$$

(b) $\delta = \delta^W(G, N)$ is the least ordinal such that $H_\delta = H_\beta$ for all $\beta \ge \delta$. Moreover, $\mathrm{Hyp}^W(G, N) := H_\delta$

(c) A hyper-W series is a hyper-($\mathcal{G}(W)$,$*$) series and a hyper-W group is a hyper-($\mathcal{G}(W)$,$*$) group.

If $\alpha = \beta + k$, β a limit ordinal and $k \in \mathbb{Z}^+$, then $H_\alpha/H_{\alpha-1}$ is the largest N-invariant subgroup of $N/H_{\alpha-1}$ centralized by G^{W_k}.

Define $\mathrm{Hyp}_\alpha^W(G) = \mathrm{Hyp}_\alpha^W(G, G)$ and $\mathrm{Hyp}^W(G) = \mathrm{Hyp}^W(G, G)$. If there is no doubt about the group G and the sequence W in question define $H_\alpha = \mathrm{Hyp}_\alpha^W(G)$.

Proposition 3.5 *Let G be a group and $W \in \mathcal{W}^\infty$.*

(a) *$(H_\alpha)_\alpha$ is a hyper-W series for G on $\mathrm{Hyp}^W(G)$.*

(b) *Let $A \trianglelefteq G$ and $(A_\alpha)_\alpha$ be a hyper-W series for G on A.*

  (a) *For every ordinal α there exists an ordinal α^* with $A_\alpha \le H_{\alpha^*}$. In particular, $A \le \mathrm{Hyp}^W(G)$.*

  (b) *If W is almost decreasing we can choose α^* such that $\alpha^* = \alpha + n_\alpha$ for some $n_\alpha \in \mathbb{N}$ and $n_\alpha = 0$ if α is a limit ordinal.*

(c) *G is hyper-W if and only if $G = \mathrm{Hyp}^W(G)$.*

Proof: (a) Let $\alpha = \beta + k$ for some limit ordinal β and some $k \in \mathbb{Z}^+$. Then G^{W_k} centralizes $H_\alpha/H_{\alpha-1}$. Hence $G/C_G(H_\alpha/H_{\alpha-1}) \in \mathcal{G}(W_k) \subseteq \mathcal{G}(W)$ and (a) holds.

(b) By induction we may assume that for all $\beta < \alpha$ there exists β^* with $A_\beta \le H_{\beta^*}$.

Suppose first that α is a limit ordinal. Let α^* be the least ordinal with $\alpha \le \alpha^*$ and $H_{\alpha^*} = \bigcup_{\beta<\alpha} H_{\beta^*}$. Then

$$A_\alpha = \bigcup_{\beta<\alpha} A_\beta \subseteq \bigcup_{\beta<\alpha} H_{\beta^*} = H_{\alpha^*}$$

Moreover, if for all $\beta < \alpha$, $\beta^* = \beta + n_\beta$ for some $n_\beta \in \mathbb{N}$ then $\alpha^* = \alpha$. So (b:a) and (b:b) hold for α.

Suppose next that $\alpha = \beta + k$ for some limit ordinal β and some $k \in \mathbb{Z}^+$. Since $(A_\alpha)_\alpha$ is hyper-W there exists $i \in \mathbb{Z}^+$ with $[A_\alpha, G^{W^i}] \le A_{\alpha-1}$.

Assume that W is almost decreasing. By induction $A_{\alpha-1} \le H_{\alpha-1+n_{\alpha-1}}$ for some $n_{\alpha-1} \in \mathbb{Z}^+$. Since W is almost decreasing there exists $n \in \mathbb{Z}^+$ with $n \ge k + n_{\alpha-1}$ and $G^{W_n} \le G^{W_i}$. Then

$$[A_\alpha, G^{W_n}] \le [A_\alpha, G^{W_i}] \le A_{\alpha-1} \le H_{\alpha-1+n_{\alpha-1}} = H_{\beta+k-1+n_{\alpha-1}} \le H_{\beta+n-1}.$$

Thus $A_\alpha \le H_{\beta+n} = H_{\alpha+n-k}$ and (b:b) holds with $n_\alpha = n - k$.

Assume next that W is not almost decreasing. Let γ be the smallest limit ordinal with $(\alpha-1)^* \le \gamma$. Then

$$[A_\alpha, G^{W_i}] \le A_{\alpha-1} \le H_{(\alpha-1)^*} \le H_\gamma \le H_{\gamma+i-1}$$

and so $A_\alpha \le H_{\gamma+i}$. Thus (b:a) holds.

(c) Follows from (a) and (b). □

Definition 3.6 (a) *For $i = 1, 2$ let w_i be a word and $m_i = m(w_i)$. Put*

$$\lceil w_1, w_2 \rceil := [\, w_1(\, (x_i)_{i=1}^{m_1}\,)\,,\, w_2(\,(x_{m_1+i})_{i=1}^{m_2}\,)\,] \in F(m_1 + m_2)$$

$\lceil w_1, w_2 \rceil$ is called the outer commutator of w_1 and w_2.

(b) *Following Möhres [M3, (3) Definition],* outer commutator words *are inductively defined as follows:*

  (a) *$w = x_1$ is the only outer commutator word with $m(w) = 1$.*

  (b) *If $m(w) > 1$ then w is an outer commutator word provided that there exist outer commutator words w_1, w_2 with $m(w_i) < m(w)$ and $w = \lceil w_1, w_2 \rceil$.*

(c) Let $w \in F^n$, $n \in \mathbb{N} \cup \{\infty\}$. Then $\check{w} \in F^{n+1}$ is inductively defined as follows: $\check{w}_1 = x_1$ and $\check{w}_{i+1} = \lceil \check{w}_i, w_i \rceil$.

(d) Let $W \in \mathcal{W}^n$, $n \in \mathbb{N} \cup \{\infty\}$. Then $\check{W} \in \mathcal{W}^{n+1}$ is inductively defined as follows: $\check{W}_1 = \{x_1\}$ and $\check{W}_{i+1} = \{\lceil v, w \rceil \mid v \in \check{W}_i, w \in W_i\}$.

For example, $\lceil x_1x_2^3, x_1x_2^2 \rceil = [x_1x_2^3, x_3x_4^2]$. Note that $m(\lceil w_1, w_2 \rceil) = m_1 + m_2$. Also $\check{W}_{i+1} = \{\check{w}_{i+1} \mid w \in \bigtimes_{j=1}^{i} W_j\}$. To improve readability we sometimes write $w\check{}$ for $\check{w}$.

Lemma 3.7 *Let G be a group, $w \in F^\infty$, $g \in G^\infty$ and $i \in \mathbb{Z}^+$.*

(a) Put $n = m(\check{w}_i)$ and $m = m(w_i)$. Then

$$\check{w}_{i+1}(g) = [\check{w}_i(g), w_i(\,(g_{n+j})_{j=1}^{m}\,)].$$

(b) Let $N \trianglelefteq G$. If $\check{w}_i(g) \in N$ then also $\check{w}_j(g) \in N$ for all $j \geq i$.

(c) Let $W \in \mathcal{W}^\infty$. Then $G^{\check{W}_{i+1}} = [G^{\check{W}_i}, G^{W_i}] \leq G^{\check{W}_i} \cap G^{W_i}$. In particular, $\check{W}$ is decreasing.

Proof: (a) By definition $\check{w}_{i+1} = \lceil \check{w}_i, w_i \rceil$. So (a) follows from the definition of the outer commutator.

(b) and (c) follow from (a). □

Definition 3.8 *(a) Let $W \in \mathcal{W}^\infty$. Then $\mathcal{H}(W)$ is the class of groups G such that for all $g \in G^\infty$ and all $w \in \bigtimes_{i=1}^{\infty} W_i$ there exists $n \in \mathbb{Z}^+$ with $w_n(g) = 1$ (or equivalently for all $g \in G^\infty$, there exists $n \in \mathbb{Z}^+$ with $w_n(g) = 1$ for all $w_n \in W_n$.)*

(b) Let $\mathcal{D} \subseteq (*, *)$. Then $\mathcal{HD}$ is the class of hyper-$\mathcal{D}$-groups. $\mathcal{FD}$ is the class of finitely hyper-$\mathcal{D}$-groups.

Observe that $\mathcal{G}(W)$ is the class of groups G for which there exists $n \in \mathbb{Z}^+$ with $w_n(g) = 1$ for all $g \in G^\infty$ and all $w_n \in W_n$. Thus $\mathcal{G}(W) \subseteq \mathcal{H}(W)$.

Theorem 3.9 *Let $W \in \mathcal{W}^\infty$. Then*

(a) $\mathcal{G}(\check{W}) \subseteq \mathcal{F}(\mathcal{G}(W), *)$ with equality if W is almost decreasing.

(b) $\mathcal{H}(\check{W}) \subseteq \mathcal{H}(\mathcal{G}(W), *)$ with equality if W is almost decreasing.

Proof: Suppose that $G^{\check{W}_n} = 1$ for some n. Then by 3.7(c)

$$1 = G^{\check{W}_n} \leq G^{\check{W}_{n-1}} \leq \ldots \leq G^{\check{W}_2} \leq G^{\check{W}_1} = G$$

is a finite hyper-W series on G. Thus $\mathcal{G}(\check{W}) \subseteq \mathcal{F}(\mathcal{G}(W), *)$.

Let G be a group which is not hyper-W. We will show that G is also not contained in $\mathcal{H}(\check{W})$. By 2.2 there exists $N \triangleleft G$ such that

$$(*) \qquad C_{G/N}(G^{W_n}) = 1 \text{ for all } n \in \mathbb{Z}^+.$$

Let $g_1 \in G \setminus N$. Note that $x_1(g_1) = g_1 \notin N$. Suppose inductively that we already found $(g_i)_{i=1}^{n_k} \in G^{n_k}$ and $w_i \in W_i, 1 \leq i < k$ with $\check{w}_k(\,(g_i)_{i=1}^{n_k}\,) \notin N$. Then by (*)

$[\check{w}_k(\,(g_i)_{i=1}^{n_k}\,), G^{W_k}] \nleq N$ and there exist $w_k \in W_k$ and $(g_{n_k+j})_{j=1}^{m(w_k)} \in G^{m(w_k)}$ with $[\check{w}_k(g_i)_{i=1}^{n_k}, w_k(\,(g_{n_k+j})_{j=1}^{m(w_k)}\,)] \notin N$. Put $n_{k+1} = n_k + m(w_k)$. Then by 3.7(a),

$$\check{w}_{k+1}(\,(g_i)_{i=1}^{n_{k+1}}\,) \notin N.$$

Put $g = (g_i)_{i=1}^{\infty}$ and $w = (w_i)_{i=1}^{\infty}$. Then $\check{w}_k(g) \neq 1$ for all k and so $G \notin \mathcal{H}(\check{W})$. Thus $\mathcal{H}(\check{W}) \subseteq \mathcal{H}(\mathcal{G}(W), *)$.

Suppose next that W is almost decreasing. We will prove the second assertions in (a) and (b) simultaneously. Let G be hyper-W and let $(A_\alpha)_{\alpha \leq \rho}$ be any hyper-W series on G. Let $i \in \mathbb{Z}^+$. If ρ is finite let $V_i = W_i$ and $H_i = G_i$. If ρ is infinite pick $w_i \in W_i$ and $g_i \in G$ and put $H_i = \{g_i\}$ and $V_i = \{w_i\}$

Let $g \in \bigtimes_{i=1}^{\infty} H_i$ and $w \in \bigtimes_{i=1}^{\infty} V_i$. Then $\check{w}_1(g_1) = g_1 \in G = A_\rho$. So we can choose an ordinal α minimal such that there exists $n \in \mathbb{Z}^+$ with $\check{w}_n(g) \in G_\alpha$ for all $w \in \bigtimes_{i=1}^{\infty} V_i$ and $g \in \bigtimes_{i=1}^{\infty} H_i$.

We will show that $\alpha = 0$. Suppose that $\alpha = \beta + 1$ for some ordinal β. Since $G/C_G(A_\alpha/A_\beta) \in \mathcal{G}(W)$, there exists $m \in \mathbb{Z}^+$ with $[A_\alpha, G^{W_m}] \leq A_\beta$. Since W is almost decreasing we may assume $m \geq n$. Let $w \in \bigtimes_{i=1}^{\infty} V_i$. Then $\check{w}_n(g) \in A_\alpha$ and $m \geq n$. So by 3.7(b), $\check{w}_m(g) \in A_\alpha$. Hence

$$\check{w}_{m+1}(g) \in [\check{w}_m(g), G^{W_m}] \leq [A_\alpha, G^{W_m}] \leq A_\beta$$

for all $w \in \bigtimes_{i=1}^{\infty} V_i$ and $g \in \bigtimes_{i=1}^{\infty} H_i$, a contradiction to the minimal choice of α. Thus α is a limit ordinal.

Suppose that $\alpha \neq 0$. Then ρ is infinite and so by our choice of V_i, $|V_i| = 1 = |H_i$ and there exist a unique $w \in \bigtimes_{i=1}^{\infty} V_i$ and a unique $g \in \bigtimes_{i \in I} H_i$. Since $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ there exists $\beta < \alpha$ with $\check{w}_n(g) \in A_\beta$, a contradiction to the choice of α.

Thus $\alpha = 0$ and so $\check{w}_n(g) = 1$ for all $w \in \bigtimes_{i=1}^{\infty} V_i$.

If ρ is finite, $V_i = W_i$ and $H_i = G_i$. Thus $G^{\check{W}_n} = 1$ and $G \in \mathcal{G}(\check{W})$. So (a) is proved.

In any case, $w_n(g) = 1$ shows that $G \in \mathcal{H}(\check{W})$ and (b) holds. □

The following example shows that the inclusions in 3.9 may be proper if W is not almost decreasing:

Let $G = \mathrm{Sym}(3)$, $x = x_1$, $W_1 = \{x^2\}$ and $W_i = \{x\}$ for $i \geq 2$. Then $w = (x^2, x, x, x, \ldots)$ is the unique element in $\bigtimes_{i=1}^{\infty} W_i$. Also $1 \leq \mathrm{Alt}(3) \leq \mathrm{Sym}(3)$ is a finite hyper-$(\mathcal{G}(W), *)$ series. Thus $\mathrm{Sym}(3) \in \mathcal{F}(\mathcal{G}(W), *) \subseteq \mathcal{H}(\mathcal{G}(W), *)$.

Put $g = ((12), (123), (12), (12), (12), \ldots)$. Then $\check{w}_1(g) = g_1 = (12)$, $\check{w}_2(g) = [(12), (123)^2] = (123)$, $\check{w}_3(g) = [(123), (12)] = (123)$ and so for all $n \geq 2$, $\check{w}_n(g) = (123)$. Thus $w_n(g) \neq 1$ for all n and $\mathrm{Sym}(3) \notin \mathcal{H}(\check{W})$. Since $\mathcal{G}(\check{W}) \subseteq \mathcal{H}(\check{W})$ we see that $\mathcal{G}(\check{W}) \neq \mathcal{F}(\mathcal{G}(W), *)$ and $\mathcal{H}(\check{W}) \neq \mathcal{H}(\mathcal{G}(W, *)$.

On the other hand, given an arbitrary $W \in \mathcal{W}^\infty$ define

$$V = (W_1, W_1, W_2, W_1, W_2, W_3, W_1, W_2, W_3, W_4, W_1, \ldots).$$

Then clearly V is almost decreasing. For any group G, $\mathcal{G}(W)$ only depends on $\{W_i \mid i \in \mathbb{Z}^+\}$ and so $\mathcal{G}(W) = \mathcal{G}(V)$. Thus by 3.9

$$\mathcal{G}(\check{V}) = \mathcal{F}(\mathcal{G}(W), *) \text{ and } \mathcal{H}(\check{V}) = \mathcal{H}(\mathcal{G}(W), *).$$

4 Hypersolvable and hypernilpotent groups

Definition 4.1 *(a)* $\tau(0) = (x_1)_{i=1}^{\infty}$ *and inductively* $\tau(i+1) = \tau(i)\check{}$.

(b) ϕ *is the unique sequence of words with* $\phi = \check{\phi}$. *So* $\phi_1 = x_1$ *and inductively* $\phi_{i+1} = \lceil \phi_i, \phi_i \rceil$.

It might be worthwhile to list the first few terms of the above sequence of words:

$\tau(0):$	x_1	x_1	x_1	x_1
$\tau(1):$	x_1	$[x_1, x_2]$	$[[x_1, x_2], x_3]$	$[[[x_1, x_2], x_3], x_4]$
$\tau(2):$	x_1	$[x_1, x_2]$	$[[x_1, x_2], [x_3, x_4]]$	$[[[x_1, x_2], [x_3, x_4]], [[x_5, x_6], x_7]]]$
$\phi:$	x_1	$[x_1, x_2]$	$[[x_1, x_2], [x_3, x_4]]$	$[[[x_1, x_2], [x_3, x_4]], [[x_5, x_6], [x_7, x_8]]]$

Lemma 4.2 *(a) Let* $\mathcal{N}(0)$ *be the class of trivial groups and inductively let* $\mathcal{N}(n+1)$ *be the class of nilpotent-by-*$\mathcal{N}(n)$ *groups. Then* $\mathcal{G}(\tau(n)) = \mathcal{N}(n)$. *In particular,* $\mathcal{G}(\tau(1))$ *the class of nilpotent groups.*

(b) $\mathcal{G}(\phi)$ *is the class of solvable groups.*

(c) $\mathcal{H}(\tau(1))$ *is the class of hypercentral groups and* $\mathcal{H}(\tau(2))$ *is the class of hypernilpotent groups.*

(d) $\mathcal{H}(\phi)$ *is the class of hypersolvable groups.*

Proof: (a) Let $w \in F^{\infty}$ be decreasing. By 3.9(a), $\mathcal{G}(\check{w}) = \mathcal{F}(\mathcal{G}(w), *)$ and so by 2.4(c):

$(*)$ $\mathcal{G}(\check{w})$ is the class of nilpotent-by-$\mathcal{G}(w)$groups.

Clearly $\mathcal{G}(\tau(0)))$ is the class of trivial groups. Since $\tau(1) = \tau(0)\check{}$, (*) says that $\mathcal{G}(\tau(1))$ is the class of nilpotent-by-trivial groups and so the class of nilpotent groups. Hence $\mathcal{G}(\tau(1)) = \mathcal{N}(1)$. Inductively suppose that $\mathcal{G}(\tau(n)) = \mathcal{N}(n)$. So (*) implies that $\mathcal{G}(\tau(n+1))$ is the class of nilpotent-by-$\mathcal{N}(n)$ groups. Thus $\mathcal{G}(\tau(n+1)) = \mathcal{N}(n+1)$ and (a) holds.

(b) We have $G = G^{\phi_1} = G^{(0)}$ and so inductively

$$G^{\phi_{i+1}} = [G^{\phi_i}, G^{\phi_i}] = [G^{(i-1)}, G^{(i-1)}] = G^{(i)}.$$

Hence $\mathcal{G}(\phi_i)$ is the class of solvable groups of derived length less than i and (b) holds.

(c) and (d) follow from (a), (b) and 3.9(b). □

Lemma 4.3 *Let* G *be a group and* w *an outer commutator word. Put* $m = m(w)$. *Then* $G^{\phi_m} \le G^w$.

Proof: For $m = 1$ we have $w = x_1 = \phi_1$. If $m > 1$, then $w = \lceil w_1, w_2 \rceil$ where w_1, w_2 are outer commutator words with $m_i := m(w_i) < m$. So

$$G^{\phi_{m-1}} \le G^{\phi_{m_i}} \le G^{w_i}$$

and thus

$$G^{\phi_m} = [G^{\phi_{m-1}}, G^{\phi_{m-1}}] \le [G^{w_1}, G^{w_2}] = G^w.$$

□

Corollary 4.4 *Let w be a sequence of outer commutator words. Then*

$$\mathcal{H}(\check{w}) \subseteq \mathcal{H}(\mathcal{G}(w), *) \subseteq \mathcal{H}(\phi).$$

Proof: The first statement follows from 3.9(b). Now let G be a group with a hyper-$(\mathcal{G}(w), *)$ series and T a factor of that series. Then $[T, G^{w_k}] = 1$ for some k. By 4.3 $[T, G^{(m)}] = 1$ for some m and so G is hypersolvable. Thus by 4.2(d), $G \in \mathcal{H}(\phi)$. □

5 Examples

In this section we construct various examples of groups which are hyper-$(\mathcal{G}, *)$ for some class $\mathcal{G}$ of groups. By 2.4 we know that any such group is hypercentral-by-(residually $\mathcal{G}$). The next proposition gives a partial converse:

Example 5.1 *Let $\mathcal{G}$ be a class of groups, $(H_i, i \in I)$ a family of members of $\mathcal{G}$ and H a subdirect product of $(H_i, i \in I)$. For $i \in I$ let A_i be a group with H_i acting on A_i. Suppose that*

(i) H is hyper-$(\mathcal{G}, *)$.

(ii) For each $i \in I$, A_i is abelian and H_i acts faithfully on A_i.

(iii) For each $1 \neq N \trianglelefteq H$, there exists $i \in I$ such that N does not act hypercentrally on A_i.

Put $A = \bigoplus A_i$. Note that H acts on A_i via its projection onto H_i and so also acts on A. Put $G = AH$. Then G is hyper-$(\mathcal{G}, *)$. Moreover, any hypercentral normal subgroup of G is contained in A.

Proof: Since $G/C_G(A_i) \cong H_i \in \mathcal{G}$, G acts hyper-$(\mathcal{G}, *)$ on A_i. So by 2.3, G is hyper-$(\mathcal{G}, *)$ on A. Also $G/A \cong H$ is hyper-$(\mathcal{G}, *)$ and hence by 2.3 G is hyper-$(\mathcal{G}, *)$.

Let $M \trianglelefteq G$ with $M \not\leq A$. Then $AM = AN$ for some $1 \neq N \trianglelefteq H$. By (iii) there exists $i \in I$ such that N does not act hypercentrally on A_i. So N also does not act hypercentrally on $[A_i, N]$. Since A is abelian, $[A_i, N] = [A_i, M] \leq M$ and M does not act hypercentrally on $[A_i, M]$. Thus M is not hypercentral. □

Example 5.2 *Let $\mathcal{G}$ be a class of groups and H a group. Suppose H is residually-$\mathcal{G}$ and hyper-$(\mathcal{G}, *)$. Then there exists a hyper-$(\mathcal{G}, *)$ group G and an abelian normal subgroup A of G such that $G/A \cong H$ and such that every hypercentral normal subgroup of G is contained in A.*

Proof: Let $\mathcal{M} = \{M \trianglelefteq H \mid G/M \in \mathcal{G}\}$. Since H is residually-$\mathcal{G}$, $\bigcap \mathcal{M} = 1$. In particular, H is a subdirect product of $(G/M \mid M \in \mathcal{M})$. For $M \in \mathcal{M}$ put $A_M = \mathbb{Z}[G/M]$. Then A_M is an abelian group with G/M acting faithfully on A_M by right multiplication. Let $1 \neq N \trianglelefteq H$ and choose $M \in \mathcal{M}$ with $N \not\leq M$. Then N does not act hypercentrally on A_M (indeed if NM/M is infinite, $C_{A_M}(N) = 0$ and if NM/M is finite, choose a prime p with $p \nmid |NM/M|$ and observe that N does not act hypercentrally on A_M/pA_M.)

So 5.1 completes the proof. □

Example 5.3 *For each prime p there exists a locally finite, hypersolvable p-group which is not hypercentral-by-solvable.*

Proof: For $1 < k \in \mathbb{N}$ let H_k be a solvable p-group of derived length k with $Z(H_k) = 1$. Let $A_k = \mathbb{F}_p H_k$ and $H = \bigoplus_{k=2}^{\infty} H_k$. Let $1 \neq N \trianglelefteq H$ and choose k such that the projection N_k of N in H_k is not trivial. If N_k is finite, $H_k/C_{N_k}(H_k)$ is a finite p-group acting on the finite p-group N_k and so $C_{N_k}(H_k) \neq 1$, contrary to $Z(H_k) = 1$. So N_k is infinite. Hence $C_{A_k}(N) = 1$ and N does not act hypercentrally on A_k. Put $A = \bigoplus A_k$ and $G = AH$. 5.1 now completes the proof. □

Example 5.4 *For each prime p there exists a hypernilpotent, 3-step elementary abelian, p-group G which is not hypercentral-by-hypercentral.*

Proof: Let $\mathbb{F}$ be an infinite field of characteristic p.

Let W be a vector space over $\mathbb{F}$ with basis $(w_i, i \in \mathbb{N})$. For $i \in \mathbb{N}$ let $i = \sum_{j=0}^{\infty} b_{ij} 2^j$ with $b_{ij} \in \{0, 1\}$. For $j \in \mathbb{N}$ and $f \in \mathbb{F}$ define $t_{jf} \in GL_{\mathbb{F}}(W)$ by $t_{jf}(w_i) = w_i + f w_{i+2^j}$ if $b_{ij} = 0$ and $t_{jf}(w_i) = w_i$ if $b_{ij} = 1$.

Let $T_j = \{t_{jf} \mid f \in \mathbb{F}\}$. Then T_j is an infinite elementary abelian p-group isomorphic to $(\mathbb{F}, +)$. Also $[T_j, T_k] = 1$ for all j, k and so also $T =: \langle T_j \mid j \in \mathbb{Z}^+ \rangle$ is an elementary abelian p-subgroup of $GL(W)$.

Define $W_i = \langle \mathbb{F} w_k \mid k \geq 2^i \rangle$. Then clearly W_i is an $\mathbb{F}T$-submodule of W and so W_i is a normal subgroup of the semidirect product $H = WT$. Moreover, W/W_i is finite dimensional and H/W_i is nilpotent. Since $\bigcap_{i=1}^{\infty} W_i = 1$, H is residually nilpotent.

Let $1 \neq N \trianglelefteq H$. We prove next that

(*) there exists k such that NW_k/W_k is infinite.

Since $C_H(W) = W$ either $[N, W] \neq 1$ or $N \leq W$. In either case there exists $1 \neq n \in N \cap W$. Let $n = \sum_{i=0}^{l} k_i w_i$ with $k_i \in F$ and pick $j \in \mathbb{N}$ with $2^j > l$. Put $m = \sum_{i=0}^{l} k_i w_{i+2^j}$. Then $t_{jf}(n) = n + fm$. Since F is infinite and $fm \notin W_{j+1}$ for all $0 \neq f \in \mathbb{F}$ we conclude that (*) holds for $k = j + 1$.

Since H is a p-group, (*) implies $Z(H) = 1$ and so H is not hypercentral. Since $H/C_H(W) \cong T$ is abelian and $H/C_H(H/W) = 1$ we conclude that $1 \leq W \leq H$ is a hypernilpotent series on H. Therefore H is hypernilpotent.

Let $A_i = \mathbb{F}_p[H/W_i]$ and put $A = \bigoplus_{i=1}^{\infty} A_i$. Then A is an elementary abelian p-group. Choose k as in (*). Then $C_{A_k}(N) = 1$ and so N does not act hypercentrally on A_k. Therefore the assumptions of 5.1 are fulfilled. Thus $G = AH = AWT$ is hypernilpotent and every hypercentral normal subgroup of G is contained in A. Since $G/A \cong H$ is not hypercentral, G is not hypercentral-by-hypercentral. □

Many thanks to Jon Hall who simplified the description of the action of T on W in the preceding lemma.

6 Möhres' Lemma

Fix a group G and let $\mathcal{F}$ be the set of finitely generated subgroups of G. For $H, K \leq G$ let $\mathcal{F}(H) = \{E \in \mathcal{F} \mid H \leq E\}$ and $\mathcal{F}(H, K) = \{E \in \mathcal{F}(H) \mid E \nleq K\}$. Put $\mathrm{D}(H, K) = G$ if $\mathcal{F}(H, K) = \emptyset$, and $\mathrm{D}(H, K) = \bigcap \mathcal{F}(H, K)$, otherwise. If the group G in question needs to be emphasized, we will also use the notations $\mathcal{F}_G$, $\mathrm{D}_G(H,K),\dots$ in place of $\mathcal{F}$, $\mathrm{D}(H,K),\dots$.

For $K \le G$ let $K^\circ \le G$ be such that $\langle K^G\rangle \le K^\circ$ and $K^\circ/\langle K^G\rangle$ is the hypercenter of $G/\langle K^G\rangle$.

Lemma 6.1 *Let G be a group and $K \le G$.*

(a) Let $E \le G$ and put $D = \mathrm{D}(E,K)$. Then $E \le D$, $\mathcal{F}(E,K) = \mathcal{F}(D,K)$ and $D = \mathrm{D}(D,K)$.

(b) $K = K^\circ$ if and only if $K \trianglelefteq G$ and $Z(G/K) = 1$.

(c) Let $K \le L \le K^\circ$. Then $L^\circ = K^\circ$. In particular $\langle K^G\rangle^\circ = K^\circ = (K^\circ)^\circ$.

(d) Suppose G is perfect. Then $[K^\circ, G] \le \langle K^G\rangle$. Moreover, $G = \langle K^G\rangle$ if and only if $G = K^\circ$.

Proof: (a) Clearly $E \le D$. Let $E \le H \in \mathcal{F}$ with $H \not\le K$. Then by definition of D, $D \le H$ and so $\mathcal{F}(E,K) \subseteq \mathcal{F}(D,K)$. Clearly $\mathcal{F}(D,K) \subseteq \mathcal{F}(E,K)$ and so (a) holds.

(b) is obvious.

(c) Clearly $\langle K^G\rangle^\circ = K^\circ$ and $\langle L^G\rangle \le K^\circ$. So we may assume that both K and L are normal in G. Since K°/K is hypercentral for G also K°/L is hypercentral for G. Thus $K^\circ \le L^\circ$. Since L°/K° and K°/K are hypercentral for G, L°/K is hypercentral for G and so $L^\circ \le K^\circ$.

(d) The first statement holds since the hypercenter of a perfect group is its center. The second follows from the first. □

The following lemma and its corollary have been abstracted from the proof of [M3, (4)Lemma].

Lemma 6.2 (Möhres' Lemma) *Let G be an* NNC*-proper, perfect group. Let $U \in \mathcal{F}$ and $a \in G \setminus U$. Then one of the following holds.*

1. There exists $N \lhd G$ and $a \notin V \in \mathcal{F}(U)$ with $a \in \mathrm{D}(V,N)$.

2. Let α be any outer commutator word and $a \notin V \in \mathcal{F}(U)$. Then

$$G = \langle H^\alpha \mid a \notin H \in \mathcal{F}(V)\rangle.$$

Proof: We assume that (1) and (2) are both false. Since (1) is false:

(*) $a \notin \mathrm{D}(V,N)$ for all $N \lhd G$ and all $a \notin V \in \mathcal{F}(U)$.

Since (2) is false, there exist an outer commutator word α with $m(\alpha)$ minimal and $a \notin V \in \mathcal{F}(U)$ such that $K := \langle H^\alpha \mid a \notin H \in \mathcal{F}(V)\rangle \ne G$. Let $N = K^\circ$. Since G is NNC-proper, $G \ne \langle K^G\rangle$ and so by 6.1(d), $N \ne G$.

Suppose that $m(\alpha) = 1$, that is $\alpha = x_1$. From (*), $a \notin \mathrm{D}(V,N)$ and so there exists $H \in \mathcal{F}(V)$ with $a \notin H$ and $H \not\le N$, a contradiction to $H = H^{x_1} = H^\alpha \le K \le N$.

Thus $m(\alpha) \ne 1$ and so there exist outer commutator words β and γ with $\alpha = \lceil \beta, \gamma \rceil$. By the minimal choice of $m(\alpha)$, $G = \langle H^\beta \mid a \notin H \in \mathcal{F}(V)\rangle$ and so there exists $a \notin H \in \mathcal{F}(V)$ with $H^\beta \not\le N$. Since $Z(G/N) = 1$, $[H^\beta, G] \not\le N$. Again by the minimal choice of $m(\alpha)$, $G = \langle R^\gamma \mid a \notin R \in \mathcal{F}(H)\rangle$ and thus there exists $a \notin R \in \mathcal{F}(H)$ with $[H^\beta, R^\gamma] \not\le N$. Since $H \le R$, $H^\beta \le R^\beta$ and so $R^\alpha = [R^\beta, R^\gamma] \not\le N$, a contradiction to $R^\alpha \le K \le N$. □

Corollary 6.3 *Let G be an* NNC*-proper, perfect group. Let $U \in \mathcal{F}$ and $a \in G \setminus U$. Then one of the following holds:*

1. *There exist $N \lhd G$ and $a \notin V \in \mathcal{F}(U)$ with $a \in \mathrm{D}(V, N)$.*
2. *Let $w = (w_i)_{i=1}^{\infty}$ be any sequence of outer commutator words. Then there exists $g \in G^{\infty}$ such that $a \notin \langle U, g\rangle$ and $\breve{w}_k(g) \neq 1$ for all $k \in \mathbb{Z}^+$. In particular, there exists a non-hypersolvable $H \leq G$ with $a \notin H$ and $U \leq H$.*

Proof: Suppose that (1) is false. Then 6.2(2) holds. In particular, there exists $g_1 \in G \setminus Z(G)$ with $a \notin \langle U, g_1\rangle$. Put $m_k = m(w_k)$ and $n_k = m(\breve{w}_k)$. Let $k \in \mathbb{Z}^+$ and suppose inductively that we have found

(*) $g_i \in G_i, 1 \leq i \leq n_k$ such that $a \notin U_k := \langle U, g_i, 1 \leq i \leq n_k\rangle$ and $h_k := \breve{w}_k(\,(g_i)_{i=1}^{n_k}\,) \notin Z(G)$.

Note that (*) holds for $k = 1$. Since G is perfect and $h_k \notin Z(G)$, $[h_k, G] \nleq Z(G)$. So by 6.2(2), applied with $\alpha = w_k$ and $V = U_k$ there exists $H_k \in \mathcal{F}(U_k)$ such that $a \notin H_k$ and $[h_k, H_k^{w_k}] \nleq Z(G)$. Hence we can choose $g_{n_k+i} \in H_k$, $1 \leq i \leq m_k$ with with $[h_k, w_k(\,(g_{n_k+i})_{i=1}^{m_k}\,)] \notin Z(G)$. Thus $h_{k+1} \notin Z(G)$. Moreover $U_{k+1} \leq H_k$ and so $a \notin U_{k+1}$.

By induction (*) holds for all $k \in \mathbb{N}$. Put $g = (g_i)_{i=1}^{\infty}$. Then $\breve{w}_k(g) \neq 1$ for all $k \in \mathbb{Z}^+$.

In the special case $w_i = \phi_i$, 4.2(d) shows that $\langle U, g\rangle$ is not hypersolvable. □

7 Perfect NNC-proper Fitting p-groups

In this section we prove that every perfect, NNC-proper, Fitting p-group has a proper non-hypersolvable subgroup.

Lemma 7.1 *Let G be a nilpotent p-group. Let H be a normal subgroup of G such that G/H is an infinite elementary abelian p-group. Let U be a finite subgroup of G and let $a \in G \setminus U$. Then there exists a subgroup V of G such that $U \leq V$, $a \notin V$ and $V/V \cap H$ is infinite.*

Proof: [M2, (6)Satz]. □

Corollary 7.2 *Let G be a perfect Fitting p-group. Then $U = \mathrm{D}(U, N)$ for all finite subgroups U of G and all $N \lhd G$.*

Proof: Suppose $U \neq \mathrm{D}(U, N)$ for some finite $U \leq G$ and some $N \lhd G$. Let $U < D \leq \mathrm{D}(U, N)$ with D finite. Since G is perfect, G/N is not nilpotent. As G is a Fitting group, $\langle D^G\rangle N/N$ is nilpotent. Thus $G \neq \langle D^G\rangle N$ and we may assume that $D \leq N$. Also $G \neq N^{\circ}$ and so we may assume $N = N^{\circ}$. Since G/N is a Fitting group, there exists a non-trivial abelian normal subgroup E/N in G/N. Choose $g \in E \setminus N$ with $g^p \in N$ and put $M = \langle D^G, g^G\rangle$. Then M is nilpotent, $M/M \cap N \cong MN/N$ is elementary abelian and $U < D \leq \mathrm{D}_M(U, M \cap N)$.

Suppose that $M/M \cap N$ is infinite. Pick $a \in D \setminus U$. Then by 7.1 (applied with $G = M$ and $H = N \cap M$) there exists $U \leq V \leq M$ with $a \notin V$ and $V/V \cap (M \cap N)$ infinite.

Pick $v \in V \setminus (M \cap N)$. Then $\langle U, v\rangle \leq V$ and so $a \notin \langle U, v\rangle$. Hence $a \notin \mathrm{D}(U, M \cap N)$, a contradiction to $a \in D$.

Thus $M/M \cap N$ is finite. So also MN/N is finite. Since G is perfect, we get $[M, G] \leq N$ and $M \leq N^\circ = N$, a contradiction to $g \in M \setminus N$. □

Proposition 7.3 *Let G be a* NNC*-proper, perfect, Fitting p-group. Let U be a finite subgroup of G and $a \in G \setminus U$. Then there exists a non-hypersolvable subgroup H of G with $a \notin H$ and $U \leq H$.*

Proof: From 7.2 $V = \mathrm{D}(V, N)$ for all finite subgroups V of G. Thus 6.3(1) does not hold and so 6.3(2) does. □

8 Normal closure of subgroups

Let $\mathcal{S}$ be a set of subgroups of a group G. We say that G is NNC-$\mathcal{S}$ if $G \neq \langle S^G\rangle$ for all $S \in \mathcal{S}$ (here NNC stands for "not normal closure"). Note that this is the case if and only if every member of $\mathcal{S}$ lies in a proper normal subgroup of G. If $\mathcal{G}$ is a class of groups, we say that G is NNC-$\mathcal{G}$ if G is NNC-$\mathcal{S}$, where $\mathcal{S} = \{S \leq G \mid S \in \mathcal{G}\}$. So G is NNC-abelian if G is not the normal closure of an abelian subgroup. G is strongly NNC-$\mathcal{G}$ if each non-trivial quotient of G is NNC-$\mathcal{G}$. We say that G is NNC-proper if G is NNC-$\mathcal{P}$ where $\mathcal{P}$ is the set of proper subgroups of G. We say G is NNC-centralizers if G is NNC-$\mathcal{C}$ where $\mathcal{C} = \{C_G(x) \mid 1 \neq x \in G\}$. Note that G is NNC-centralizer if and only if $G \neq \langle H^G\rangle$ for all $H \leq G$ with $Z(H) \neq 1$.

The goal of this section is to prove Proposition 8.4, which provides conditions which imply that G is NNC-proper.

Lemma 8.1 *Let G be a group and $i \in \mathbb{Z}^+$. Then the following are equivalent:*

(a) G is strongly NNC*-abelian.*

(b) G is strongly NNC*-solvable.*

(c) Let $K \leq G$. Then $G = \langle K^G\rangle$ if and only if $G = \langle (K^{(i)})^G\rangle$.

Proof:

(a) implies (b): Let H be a non-trivial quotient of G and let S be a solvable subgroup of H. By induction on the derived length of S, $N := \langle S'^H\rangle \neq H$. Since SN/N is abelian, $H/N \neq \langle SN/N^H\rangle$ and so also $H \neq \langle S^H\rangle$.

(b) implies (c): Put $N := \langle (K^{(i)})^G\rangle$. Clearly $G \neq \langle K^G\rangle$ implies $N \neq G$. Now suppose $N \neq G$. Since KN/N is solvable, (b) implies $G/N \neq \langle KN/N^G\rangle$ and so $\langle K^G\rangle \neq G$.

(c) implies (a): Let $H = G/N$ be a non-trivial quotient of G and $A = K/N$ an abelian subgroup of G/N. Then $K^{(i)} \leq N < G$ and so $G \neq \langle (K^{(i)})^G\rangle$. Thus by (c) $G \neq \langle K^G\rangle$ and so also $H \neq \langle A^H\rangle$. □

Definition 8.2 *Let G be a group. Then* $\mathrm{Sol}^*(H) = \mathrm{Hyp}^\phi(G)$.

Observe that by 4.2 and 3.5 $\mathrm{Sol}^*(H)$ is the largest normal subgroup of G on which G acts hypersolvablely.

Lemma 8.3 *Let G be an* NNC*-centralizer and strongly* NNC*-abelian group. Then $G \neq \langle K^G \rangle$ for all $K \leq H$ such that* $\mathrm{Sol}^*(K) \neq 1$*. In particular, G is* NNC*-hypersolvable.*

Proof: Let $K \leq G$ with $\mathrm{Sol}^*(K) \neq 1$. Then there exists a non-trivial normal subgroup A of K such that $K/C_K(A)$ is solvable. Let $1 \neq x \in A$. Then $C_K(A) \leq C_G(x)$ and since G is NNC-centralizer we get $N := \langle C_K(A)^G \rangle \neq G$. Then KN/N is solvable. Since G is strongly NNC-abelian, 8.1 implies that G is strongly NNC-solvable. Thus $G/N \neq \langle KN/N^G \rangle$ and $G \neq \langle K^G \rangle$. □

Proposition 8.4 *Suppose G is* NNC*-centralizer and that one of the following holds:*

(i) G is minimal non-hypercentral.

(ii) G is minimal non-hypersolvable and strongly NNC*-abelian.*

Then G is NNC*-proper.*

Proof: Let K be a proper subgroup of G.

If (i) holds, K is hypercentral. Hence $Z(K) \neq 1$ and since G is NNC, $G \neq \langle K^G \rangle$.

If (ii) holds, then K is hypersolvable and so $\mathrm{Sol}^*(K) = K \neq 1$. Thus by 8.3, $G \neq \langle K^G \rangle$. □

Corollary 8.5 *Every non-trivial,* NNC*-centralizer, strongly* NNC*-abelian, perfect Fitting p-group has a proper non-hypersolvable subgroup.*

Proof: Suppose G is a counterexample. Since G is non-trivial and perfect, G is not hypersolvable. So G is minimal non-hypersolvable. Thus 8.4 implies that G is NNC-proper. But then the assumption but not the conclusion of 7.3 are fulfilled, contradiction. □

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