A computer free construction of J_4

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Abstract

Define a finite simple group J to be of J_4-type (or simply J_4) provided that J contains an involution z with

$$C_J(z) \sim 2_+^{1+12} 3Aut\ Mat_{22}.$$

The purpose of this paper is to give the first computer free construction of a group of J_4-type. In addition we achieve yet another uniqueness proof for groups of J_4-type via the simple connectedness of the 2-local geometry of such a group.

1 Introduction

Initial evidence for the existence of groups of J_4-type was given by Z. Janko in [10]. He has shown that the order $o(J_4)$ of such a group is

$$86,775,571,046,077,562,880 = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43,$$

determined its conjugacy classes and much of the p-local structure. J. Conway, S. Norton, J. Thompson and D. Hunt used this information to determine the character table of J_4 and, in particular, proved the existence of an irreducible (irrational) complex character of degree 1333. Looking at the 2-modular reduction of this character J. Thompson conjectured the existence of an irreducible 112-dimensional representation of J_4 over $GF(2)$. Based on this conjecture S. Norton with the help of D. Benson, J. Conway, R. Parker and J. Thackray constructed J_4 as a subgroup of $GL_{112}(2)$. Their construction is outlined in [13], discussed in more detail in [3] and depends on the use of a computer. In [12], W. Lempken gave explicit generators for J_4 as a subgroup of $GL_{1333}(11)$. The proof that the group generated is in fact J_4 relies on its existence.

Definition 1.1 *Let I be a set of size n. A (finite) amalgam of rank n (over I) is a tuple $(\mathcal{A}; M_i, i \in I; *_i, i \in I)$ where $\mathcal{A}$ is a finite set, M_i is subset of $\mathcal{A}$ and $*_i$ is binary operation defined on M_i so that the following conditions hold:*

(i) $(M_i, *_i)$ *is a group for every* $i \in I$;

(ii) $\mathcal{A} = \cup_{i \in I} M_i$;

(iii) $\cap_{i \in I} M_i \neq \emptyset$;

(iv) *if* $x, y \in M_i \cap M_j$ *for* $i, j \in I$ *then* $x *_i y = x *_j y$.

We will write $(M_i \mid i \in I)$ for the amalgam $\mathcal{A}$ as above (since $\mathcal{A} = \cup_{i \in I} M_i$, there is no need to refer to $\mathcal{A}$ explicitly). Whenever x and y are in the same M_i there product $x *_i y$ is defined and it is independent of the choice of i. We will normally write this product simply by xy. Since $B := \cap_{i \in I} M_i$ is non-empty, one can easily see that B contains the identity element of $(M_i, *_i)$ for every $i \in I$. Moreover, these identity elements must be equal. The reader may notice that a more common definition of amalgams in terms of morphisms is essentially equivalent to the above one. For $J \subseteq I$ we put $M_J = \cap_{i \in J} M_i$. We will write, for instance M_{ijk} instead of $M_{\{i,j,k\}}$ and consider M_{ij} as a subgroup in M_i and M_{ji} as a subgroup of M_j. An amalgam of rank 3 will also be called a *triangle of groups*. The isomorphism of amalgams is defined in the obvious way. Let $(M, *)$ be a group, $\{M_i \mid i \in I\}$ be a family of subgroups in M and $*_i$ be the restriction of $*$ to M_i. Then $(M_i \mid i \in I)$ is an amalgam. This is the most important example of amalgam but it is not difficult to construct an amalgam which is not isomorphic to a family of subgroups in a group.

Definition 1.2 *A group M is said to be a completion of an amalgam $(M_i \mid i \in I)$ if there is a mapping φ of $\cup_{i \in I} M_i$ into M such that*

(i) *M is generated by the image of φ;*

(ii) *for every $i \in I$ the restriction of φ to M_i is a group homomorphism with respect to $*_i$ and the group operation in M.*

If φ is injective then the completion M is said to be faithful.

Definition 1.3 *A triangle (M_1, M_2, M_3) of groups is called a J_4-triangle provided that*

(i) *M_1 is the semidirect product of the Mathieu group Mat_{24} of degree 24 and the 11-dimensional Todd module;*

(ii) *M_2 is the semidirect product of $L_5(2)$ and the exterior square of a natural module of $L_5(2)$;*

(iii) *$|O_2(M_3)| = 2^{15}$ and $M_3/O_2(M_3) \cong Sym(5) \times L_3(2)$;*

(iv) *$|M_2/M_{21}| = 31, |M_3/M_{31}| = 5, |M_3/M_{32}| = 10$ and $|M_{23}/B| = 3$.*

It was shown in [10] (cf. Theorem A (4), (6), (9)) that every group of J_4-type is a faithful completion of a J_4-triangle of groups. This and the existence of the complex character of degree 1333 serve as motivation for our construction of J_4. The principal steps are as follows:

Step 1: Show that there exists a J_4-triangle of groups.

Step 2: Show that $GL_{1333}(\mathbf{C})$ contains a faithful completion of a J_4-triangle of groups.

Step 3: Show that any faithful completion of a J_4-triangle of groups is a group of J_4-type.

Steps 1 and 2 are realized in Lemma 5.9 and Theorem 7.1, respectively. Step 3 was done in [2] (as the main step in the uniqueness proof for J_4) and independently in [8]. Both these proofs were achieved by establishing the simple connectedness of the 2-local geometry of J_4; hence rely on the existence of J_4 and do not suit our purposes. In order to establish the existence of J_4 we need to carry out Step 3 without assuming that J_4 exists. In this form Step 3 is realized in Section 8 (cf. Theorem 8.26) and the proof is necessarily more complicated than the proofs in [2] and [8]. In particular our proof uses extremely detailed information about the 2-local geometries of Mat_{24} and Mat_{22}.

Although we do not need the uniqueness of J_4 to establish its existence, we include a uniqueness proof since it can be achieved with only little extra effort. Namely, we prove in Lemma 5.7 that any two J_4-triangles are isomorphic and within the realization of Step 3 (cf. Theorem 8.26) we show that every faithful completion of a J_4-triangle is finite and that its order is equal to $o(J_4)$. Since every completion of an amalgam is a quotient of the universal completion, this immediately implies that the unique J_4-triangle of groups has a unique faithful completion, namely the universal one.

2 Preliminaries

Our notation concerning groups is mostly standard. The symmetric, alternating and Mathieu group of degree n are denoted by $Sym(n)$, $Alt(n)$ and Mat_n, respectively. By writing $G \sim A_1A_2...A_n$ we mean that G has a normal series

$$1 \lhd G_1 \lhd ... \lhd G_n = G$$

such that $G_i/G_{i-1} \cong A_i$. We write p^a for a p-group of order p^a; $p^{a_1+a_2+\dots+a_n}$ for $p^{a_1}p^{a_2}...p^{a_n}$ and 2_ε^{1+2n} for the extraspecial group of order 2^{1+2n} and of type $\varepsilon \in \{+,-\}$. Throughout the paper $3 \cdot Alt(6)$ denotes the non-split extension of $Alt(6)$ by a centre of order 3 and $3 \cdot Sym(6)$ stands for the extension of $3 \cdot Alt(6)$ by an outer automorphism which induces a transposition on the $Alt(6)$-quotient. Given a subgroup H in a group G we denote by G/H the set of right cosets of H in G.

Definition 2.1 *Let G be a group, $K \trianglelefteq G$, $\bar{G} = G/K$ and V a $GF(2)G$-module of dimension n with kernel K. Then*

(i) *if $\bar{G} \cong L_n(2)$ then V is called a natural $L_n(2)$-module for G;*

(ii) *if $\bar{G} \cong \Omega_n(2)$ and G fixes a non-degenerate quadratic form of plus type on V then V is called a natural $\Omega_n^+(2)$-module for G;*

(iii) *if $\bar{G} \cong Sym(5)$, $n = 4$ and G' preserves a $GF(4)$-structure on V, then V is called a natural $\Gamma L_2(4)$-module for G;*

(iv) *if $\bar{G}' \cong 3 \cdot Alt(6)$ and $n = 6$, then V is called a hexacode module for G.*

The module dual to V will be denoted by V^*.

Notation 2.2 *Let $(M_i \mid i \in I)$ be an amalgam. Then for $i, j \in I$ we put $Q_i = O_2(M_i)$, $Q_i^* = O_{2,3}(M_i)$, $Z_i = Z(Q_i)$ and $T_{ij} = O_2(M_{ij})$.*

Definition 2.3 *Let I be a set of size n and let Γ be an undirected n-partite graph without loops, whose parts are Γ_i, $i \in I$. This means that if $a \in \Gamma_i$ is adjacent to $b \in \Gamma_j$ then $i \neq j$. Let d the usual distance function on Γ. Then*

(i) *if $a \in \Gamma_i$, then a is said to be of type i;*

(ii) *a path of type $n_1 - n_2 - \dots - n_k$ is a tuple $(a_1, a_2, \dots, a_k)$ of vertices in Γ such that a_i is of type n_i and a_i is adjacent to a_{i+1}; we denote such a path by*

$$\overset{a_1}{n_1} - \overset{a_2}{n_2} - \dots - \overset{a_k}{n_k};$$

(iii) *a non-degenerate path (or nd-path) is a path $(a_1, a_2, \dots, a_k)$ such that a_{i-1} is neither equal nor adjacent to a_{i+1};*

(iv) *let $\Lambda \subseteq \Gamma$ and $a_1, a_2, \dots, a_n \in \Gamma$, then*

$$\Lambda(a_1a_2\dots a_n) = \{b \in \Lambda \mid b \text{ is adjacent to } a_i \text{ for all } 1 \leq i \leq n\}.$$

Definition 2.4 *Let M be a group and $(M_i \mid i \in I)$ a tuple of subgroups of M.*

(i) *The coset graph* $\Gamma = \Gamma(M; M_i \mid i \in I)$ *is the graph with vertex set the disjoint union of the sets* $\Gamma_i = M/M_i, i \in I$ *and where two vertices are adjacent if they are distinct and have non-empty intersection. Note that the* Γ_i *are parts of* Γ *and that* M *acts on* Γ *by right multiplication.*

(ii) *A flag in* Γ *is a set of pairwise adjacent vertices. The type of a flag is the set of types of its elements.*

(iii) *Let* $a \in \Gamma$*. Then* $\Gamma(a)$ *is the graph whose vertices are the neighbours of* a *in* Γ *and two vertices are adjacent in* $\Gamma(a)$ *if and only if they are adjacent in* Γ*.*

(iv) *Let* $a \in \Gamma$*. Then the graph* $\Gamma^*(a)$ *on the neighbours of* a *in* Γ *is defined as follows. Assume without loss that* $a = M_i$*. Let* b, c *be adjacent to* a*, where* $b = M_j r$ *and* $c = M_k s$ *with* $r, s \in M_i$*. Then* b *is adjacent to* c *in* $\Gamma^*(a)$ *if and only if* $j \neq k$ *and* $M_{ij}r \cap M_{ik}s \neq \emptyset$*.*

(v) Γ *is called geometric if for all* $a \in \Gamma$ *the graphs* $\Gamma(a)$ *and* $\Gamma^*(a)$ *are equal.*

(vi) *If* $a, b, c, \ldots$ *are vertices of* Γ*, then* $M_{abc\ldots}$ *denotes their elementwise stabilizer in* M*. If* $a \in \Gamma$ *then* $Q_a = O_2(M_a)$*,* $Q_a^* = O_{2,3}(M_a)$ *and* $Z_a = Z(Q_a)$*.*

(vii) *Let* $a, b, c \in \Gamma$*. Then* $\angle\, abc = |c^{M_{ab}}|$*.*

We remark that if b, c are adjacent in $\Gamma^*(a)$ then they are also adjacent in $\Gamma(a)$. Furthermore, $\Gamma^*(M_i)$ is isomorphic to $\Gamma(M_i; M_{ij} \mid j \in I \setminus \{i\})$.

Lemma 2.5 *Let* Γ *be as in* 2.4*.*

(i) *Let* $\{a_i, a_j, a_k\}$ *be a flag in* Γ *where* a_l *is of type* l*. Then the following statements are equivalent:*

  (a) a_x *and* a_y *are adjacent in* $\Gamma^*(a_z)$ *for every* $z \in \{i, j, k\}$ *with* $\{x, y, z\} = \{i, j, k\}$*;*

  (b) a_x *and* a_y *are adjacent in* $\Gamma^*(a_z)$ *for some* $z \in \{i, j, k\}$ *with* $\{x, y, z\} = \{i, j, k\}$*;*

  (c) $a_i \cap a_j \cap a_k \neq \emptyset$*;*

  (d) (a_i, a_j, a_k) *is conjugate to* (M_i, M_j, M_k)*.*

(ii) Γ *is geometric if and only if* M *acts transitively on each set of flags of size three of a given type, that is if and only if all flags* $\{a_i, a_j, a_k\}$ *in* Γ *fulfill the equivalent conditions in* (i)*.*

Proof. (i) Clearly (a) implies (b). To show that (b) implies (c) we assume without loss that $a_k = M_k$, $a_i = M_i r$ and $a_j = M_j s$ for some $r, s \in M_k$. Since a_i and a_j are adjacent in $\Gamma^*(a_k)$, $\emptyset \neq M_{ij}r \cap M_{jk}s \subseteq M_i \cap M_j r \cap M_k s = a_i \cap a_j \cap a_k$ and so (c) holds. Clearly (c) implies (d), and (d) implies (a).

(ii) By the remark preceding this lemma, Γ is geometric if and only if (a) holds. □

Throughout the paper we will refer to the following easy principle concerning a set of size five.

Lemma 2.6 *Let* Γ_1 *be a* 5*-element set and* Γ_2 *be the set of* 2*-element subsets in* Γ_1*. Let* Γ *be the bipartite graph on* $\Gamma_1 \cup \Gamma_2$ *where* $a \in \Gamma_1$ *is adjacent to* $b \in \Gamma_2$ *if and only if* a *is not contained in* b*. Suppose that* $a \in \Gamma_1$ *and* $b \in \Gamma_2$ *are adjacent. Put* $R_a(b) = \Gamma_1 \setminus (b \cup \{a\})$*. Then every* $c \in \Gamma_1 \setminus \{a\}$ *is adjacent to exactly one of* b *and* $R_a(b)$*.*

Proof. This is clear since $b, R_a(b)$ is a partition of $\Gamma_1 \setminus \{a\}$. □

The next lemma contains some information on cohomologies of some small modules.

Lemma 2.7 (i) *Let* $E \cong L_4(2)$ *and* V *be a natural* $\Omega_6^+(2)$*-module for* E. *Then* $|H^1(V)| = 2$, $H^2(V) = 0$.

(ii) *Let* $E \cong L_4(2)$ *and* V *be a natural* $L_4(2)$*-module for* E. *Then* $H^1(V) = 0$.

(iii) *Let* $E \cong L_3(2)$ *and* V *be a natural* $L_3(2)$*-module for* E. *Then* $|H^1(V)| = 2$.

Proof. [4] and [11]. □

The following lemma describes the actions of various subgroups of $L_5(2)$ on the vectors of the exterior square of a natural $L_5(2)$-module.

Lemma 2.8 *Let* V *be a* 5*-dimensional* $GF(2)$*-space and* $M = GL(V)$, *so that* $M \cong L_5(2)$. *Let* $V_1 < V_3 < V$ *where* $\dim V_1 = 1$, $\dim V_3 = 2$. *Let* $M_i = N_M(V_i)$, $i = 1$ *and* 3, $B = M_1 \cap M_3$ *and as usual let* V^* *denote the dual of* V. *Then*

(i) M *has precisely two orbits* $H(2)$ *and* $H(s)$ *on the set of hyperplanes in* $\bigwedge^2 V^*$; *the hyperplanes in* $H(2)$ *are indexed by the* 2*-spaces of* V *and the hyperplanes in* $H(s)$ *are indexed by the pairs* (W, s), *where* W *is a hyperplane in* V *and* s *is a non-degenerate symplectic form on* W;

(ii) *if* $H \in H(s)$ *then* $N_M(H) \sim 2^4 Sym(6)$;

(iii) *the orbits of* M_1; M_3; B *on* $H(s)$ *are of lengths* 420 *and* 448; 84, 112 *and* 672; 84, 112, 224 *and* 448, *respectively;*

(iv) *let* H *be a hyperplane from the orbit of length* 84 *of* B *on* $H(s)$; *if* N, N_1, N_3, *and* N_0 *are the normalizers of* H *in* M, M_1, M_3 *and* B, *respectively, then*

$$N = N' N_0 \quad \text{and} \quad N' \cap N_0 = (N_1' \cap N_0)(N_3' \cap N_0);$$

(v) *the orbits of* M_3 *on* $H(2)$ *are of lengths* 1, 42 *and* 112; *the action of* B *on the* M_3*-orbit of length* 42 *is intransitive.*

Proof. (i) By the definition of the exterior square $\bigwedge^2 V^*$, its hyperplanes are in one-to-one correspondence with the non-zero symplectic forms on V^*. Hence (i) follows from the following well known facts: (a) any two non-degenerate symplectic forms on a finite dimensional vector space are isomorphic, (b) any vector space with a non-degenerate symplectic form is even dimensional and (c) there is exactly one non-degenerate symplectic form on a vector space with 4 elements.

(ii) Let W be a hyperplane in V, s a non-degenerate symplectic form on W and $R = N_M(W, s)$. Then clearly $C_R(W) = C_M(W)$ is elementary abelian of order 2^4 and $R/C_R(W) \cong Sp_4(2) \cong Sym(6)$.

(iii) Note that R acts transitively on the set of 1-spaces in W and $C_R(W)$ acts regularly on the set of 1-spaces in $V \setminus W$. Thus $R \cap M_1 \sim 2^4(C_2 \times Sym(4))$ if $V_1 \leq W$ and $R \cap M_1 \cong Sym(6)$ otherwise. Moreover, R has three orbits on the set of 2-spaces in V, distinguished by $V_3 \leq W$ and V_3 is singular with respect to s; $V_3 \leq W$ and V_3 is non-degenerate with respect to s; and $V_3 \not\leq W$. The corresponding shapes of $R \cap M_3$ are $2^4(C_2 \times Sym(4))$, $2^4(Sym(3) \times Sym(3))$ and $2(C_2 \times Sym(4))$. Finally R has four orbits on the set of pairs of incident 1- and 2-spaces corresponding to the following four cases: $V_3 \leq W$ and V_3 is singular with respect to s; $V_3 \leq W$ and is non-degenerate with respect to s; $V_1 \leq W$ and $V_3 \not\leq W$; and $V_1 \not\leq W$. The corresponding shapes of $R \cap B$ are $2^4(C_2 \times D_8)$,

$2^4(C_2 \times Sym(3))$, $2(C_2 \times Sym(4))$ and $C_2 \times Sym(4)$. Thus we have described the orbits of R on 1- and 2-spaces in V and also on the incident pairs of such subspaces. This immediately gives us all the orbits of M_1, M_3 and B on $H(s)$ and the corresponding stabilizers. That the lengths of the orbits are as given in (iii) is now a trivial computation.

(iv) Let H correspond to (W, s). Then $V_3 \leq W$ and V_3 is singular with respect to s. Notice that $C_N(W) \cong W$ as N-module and in particular, $C_N(W) = [C_N(W), N_3] \leq N_3' \leq N'$. Thus (iv) holds if and only if it holds modulo $C_N(W)$. But $N_i/C_N(W) \cong C_2 \times Sym(4)$ for $i = 1, 3$ and $N_0/C_N(W)$ is a Sylow 2-subgroup of $N/O_2(N) \cong Sym(6)$. Now (iv) is readily verified.

(v) There is one 2-space equal to V_3, $42 = 3 \cdot 14$ 2-spaces intersecting V_3 in a 1-space and $112 = 7 \cdot 16$ 2-spaces which intersect V_3 trivially. Since some 2-spaces in the orbit of length 42 contain V_1 while others do not, B does not act transitively on the M_3-orbit of length 42. $\square$

3 Mat_{24}

We assume that the reader is familiar with the basic properties of the unique Steiner system $\mathcal{S}$ of type (5,8,24) (see for instance [1] or [9]). Let Ω be the set of size 24 underlying $\mathcal{S}$ and let Γ_2 denote the block set of $\mathcal{S}$. This means that Γ_2 is a collection of 8-element subsets of Ω called *octads* such that every 5-element subset of Ω is in a unique octad. In particular $|\Gamma_2| = \binom{24}{5}/\binom{8}{5} = 759$. A triple of pairwise disjoint octads is called a *trio*. Every 4-element subset T of Ω is contained in a unique *sextet*, which is a partition of Ω into six 4-element subsets $T_1 = T, T_2, ..., T_6$ called *tetrads* such that $T_i \cup T_j \in \Gamma_2$ for all $1 \leq i < j \leq 6$.

Let Γ_3 denote the set of trios, let Γ_4 denote the set of sextets and let $\Gamma = \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. Define a graph on Γ as follows: a trio is adjacent to an octad if it contains the octad; a sextet is adjacent to an octad if the octad is the union of two of the tetrads in the sextet; and a sextet is adjacent to a trio if it is adjacent to all of the three octads in the trio.

Throughout this section M will stand for the automorphism group of $\mathcal{S}$ which is the Mathieu group Mat_{24} of degree 24. Let α, β and γ be pairwise adjacent octad, trio and sextet respectively, i.e. a maximal flag in Γ. If $\gamma = \{T_1, T_2, ..., T_6\}$ we can put $\alpha = T_1 \cup T_2$ and $\beta = \{T_1 \cup T_2, T_3 \cup T_4, T_5 \cup T_6\}$. Let $M_2 = M_\alpha$, $M_3 = M_\beta$ and $M_4 = M_\gamma$ (the stabilizers in M of α, β and γ, respectively). Then (M_2, M_3, M_4) is a triangle of groups and M is a faithful completion of this triangle. Since M is flag transitive on Γ, $\Gamma \cong \Gamma(M; M_2, M_3, M_4)$. We have chosen the index set $\{2, 3, 4\}$ rather then $\{1, 2, 3\}$ since M_2, M_3, M_4 will correspond to M_{12}, M_{13} and M_{14} in later sections.

We will need the following information on classes of elements in M of order 2 and 3 which can be deduced either from Section 21 in [1] together with the permutational characters of M on Γ_2, Γ_3 and Γ_4 given in [5] or from Sections 2.12 - 2.14 in [9].

Lemma 3.1 (i) *M has two classes, $2a$ and $2b$ of involutions and two classes, $3a$ and $3b$ of elements of order 3.*

(ii) *If $t \in 2a$ then t is 2-central, $C_M(t) \sim 2_+^{1+6} L_3(2)$, t fixes: 8 elements of Ω forming an octad, 71 octads, 99 trios and 91 sextets.*

(iii) *If $s \in 2b$ then s is non-2-central, $C_M(s) \sim 2^{1+1+4} Sym(5)$, $C_M(s)$ fixes a unique sextet, s fixes: 15 octads, 75 trios and 51 sextets.*

(iv) *If $x \in 3a$ then $C_M(x) \cong 3 \cdot Alt(6)$, x does not commute with a $2b$-involution and fixes: 21 octads, 15 trios and 16 sextets.*

(v) *If $y \in 3b$ then $C_M(y) \cong C_3 \times L_3(2)$, y commutes with a $2b$-involution, acts fixed-points freely on the set of octads and fixes* 15 *trios and* 7 *sextets.* □

The basic properties of the triangle (M_2, M_3, M_4) and of its completion M are given in the following lemma (cf. Section 19 in [1] or Section 2.10 in [9]).

Lemma 3.2 (i) $|M/M_2| = |\Gamma_2| = 759$, *$M_2 \sim 2^4 L_4(2)$ and Q_2 is a natural $L_4(2)$-module for M_2, Q_2 is $2a$-pure, M_2 acts as $Alt(8) \cong L_4(2)$ on the elements in α and as the doubly transitive affine group $AGL_4(2)$ on the elements outside α, in particular M_2 splits over Q_2.*

(ii) $|M/M_3| = |\Gamma_3| = 3795$; *$M_3 \sim 2^6(Sym(3) \times L_3(2))$, $Q_3 \cong D_1 \otimes D_2$, where D_1 and D_2 are natural $L_2(2)$- and $L_3(2)$-modules for M_3, respectively; M_3 has two orbits on $Q_3^\#$ with lengths* 21 *and* 42, *consisting of involutions of type $2a$ and $2b$, respectively and if $y \in Q_3^* \setminus Q_3$ then y is of type $3b$ and acts fixed-point freely on Q_3.*

(iii) $|M/M_4| = |\Gamma_4| = 1771$, *$M_4 \sim 2^6 3 \cdot Sym(6)$, Q_4 is a hexacode module for M_4, M_4 has two orbits on $Q_4^\#$ with lengths* 45 *and* 18, *consisting of involutions of type $2a$ and $2b$, respectively, if $x \in Q_4^* \setminus Q_4$ then x is of type $3a$ and acts fixed-point freely on Q_4, M_4 induces $Sym(6)$ on the tetrads constituting γ and the kernel induces $Alt(4)$ on the elements in each tetrad.*

(iv) $|M_3/M_{34}| = 7$, $|M_3/M_{23}| = 3$, $|M_4/M_{24}| = 15$ *and* $|M_{34}/B| = 3$.

(v) $M_{24}/Q_2 \sim 2^4(Sym(3) \times Sym(3))$ *and* $M_{23}/Q_2 \sim 2^3 L_3(2)$.

(vi) $M_{34}/Q_3 \cong Sym(3) \times Sym(4)$ *and* $M_{23}/Q_3 \cong Sym(2) \times L_3(2)$.

(vii) $M_{24}/Q_4^* \cong M_{34}/Q_4^* \cong Sym(4) \times Sym(2)$.

(viii) $|Q_2 \cap Q_3| = 8$, $|Q_2 \cap Q_4| = 4$ *and* $|Q_3 \cap Q_4| = 16$. □

Comparing 3.1 and 3.2 one can observe the following. If t is an involution in Q_2, s is an involution in the orbit of length 18 of M_4 on $Q_4^\#$, x is an element of order 3 in Q_4^* and y is an element of order 3 in Q_3^*, then

$$C_M(t) = C_{M_2}(t),\ C_M(s) = C_{M_4}(s),\ C_M(x) = C_{M_4}(s),\ C_M(y) = C_{M_3}(y).$$

As a direct corollary of 3.2 we have the following.

Lemma 3.3 (i) *M_2 acts on $\Gamma_3(\alpha)$ and $\Gamma_4(\alpha)$ of size* 15 *and* 35 *as it acts on the* 3- *and* 2-*spaces in Q_2, respectively;*

(ii) *M_3 acts on $\Gamma_2(\beta)$ and $\Gamma_4(\beta)$ of size* 3 *and* 7 *as it acts on the* 1-*spaces in D_1 and on the* 2-*spaces in D_2, respectively;*

(iii) *M_4 acts on $\Gamma_2(\gamma)$ and $\Gamma_4(\gamma)$ of size* 15 *each as it acts on the* 2-*element subsets of γ (considered as the set of six tetrads) and on the partitions of γ into three pairs.* □

Lemma 3.4 *Let $B = M_2 \cap M_3 \cap M_4$ be the stabilizer in M of the flag $\mathcal{F} = \{\alpha, \beta, \gamma\}$. Let S be a Sylow* 2-*subgroup of B which is also a Sylow* 2-*subgroup of M.*

(i) *α is the unique octad, β is the unique tetrad and γ is the unique sextet stabilized by S.*

(ii) *Let H be a maximal subgroups of M containing S. Then $H = M_i$ for $i = 2$, 3 or 4.*

(iii) *Let P be any subgroup of M containing S and let z denote the unique non-trivial element in $Z(S)$. Then one of the following holds:*

(a) *P is the normalizer of a subflag in $\mathcal{F}$.*

(b) *$|P| = 2^{10} \cdot 3^2$ and $P = Q_3^* C_{M_3}(z)$.*

(c) *$|P| = 2^{10} \cdot 3 \cdot 7$ and $P = C_M(z)$.*

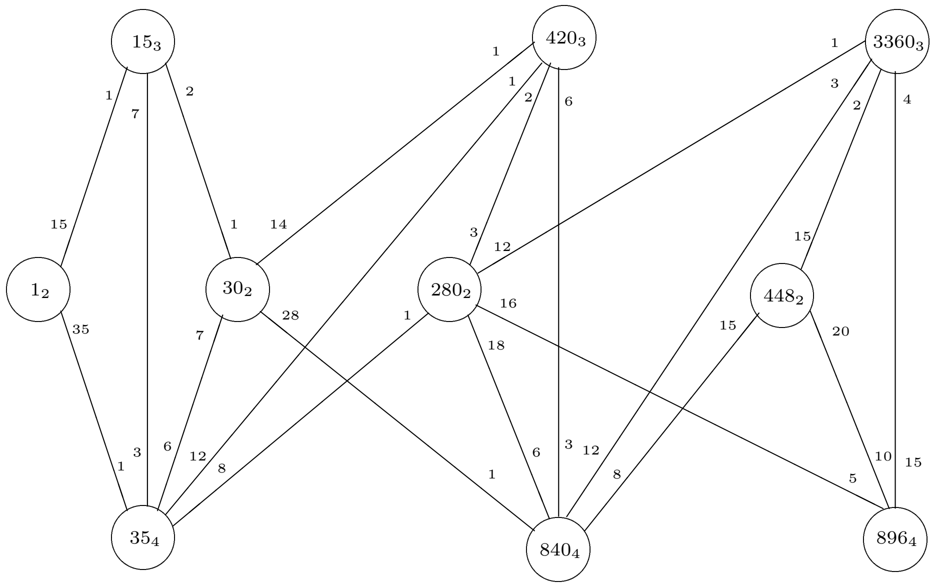
(d) *$|P| = 2^{10} \cdot 3$ and $P = Q_3^* S$, $C_{M_3}(z)$ or $C_{M_4}(z)$.*

(e) *$P = S$.*

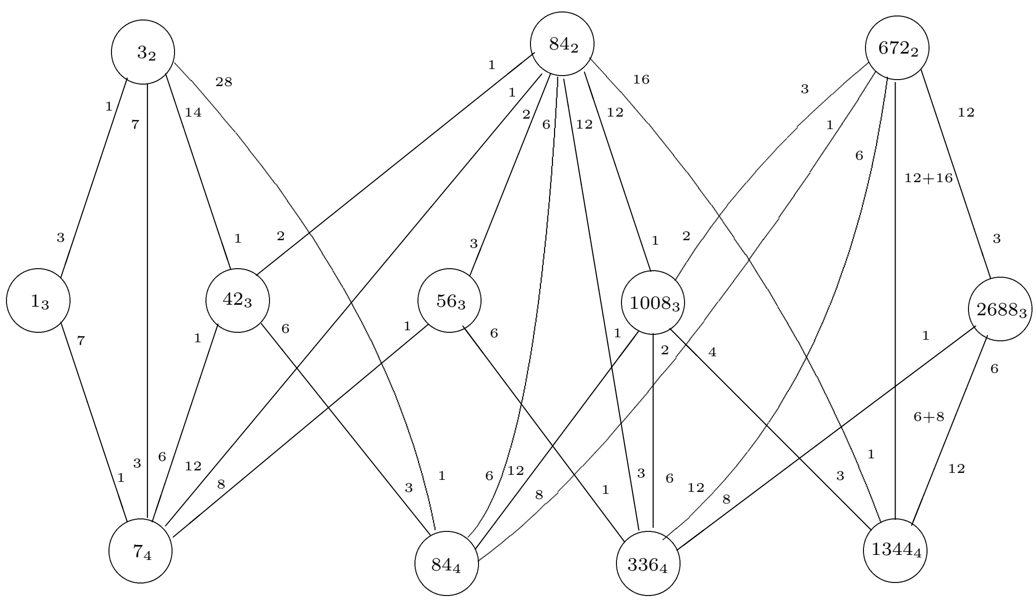
Proof. (ii) follows from [6], (i) follows from (ii) while (iii) follows from the structure of M_2, M_3 and M_4 as given in 3.2 (compare [15]). □

In subsequent sections we will need detailed information about the graph Γ and the action of M on this graph. For this purpose for every $i \in \{2, 3, 4\}$ we describe the orbits of M_i on the vertex set of Γ and for any two such orbits A and B we calculate the number $n_i(A, B)$ of vertices in B adjacent in Γ to a given vertex $a \in A$ and finally determine how these vertices split into orbits under the stabilizer of a in M_i. It is clear that $n_i(A, B)$ is zero unless $A \subseteq \Gamma_j$, $B \subseteq \Gamma_k$ for $j \neq k$ and that $|A| \cdot n_i(A, B) = |B| \cdot n_i(B, A)$. Finally for $i \neq j$ there is a natural correspondence between the orbits of M_i on Γ_j and the orbits of M_j on Γ_i.

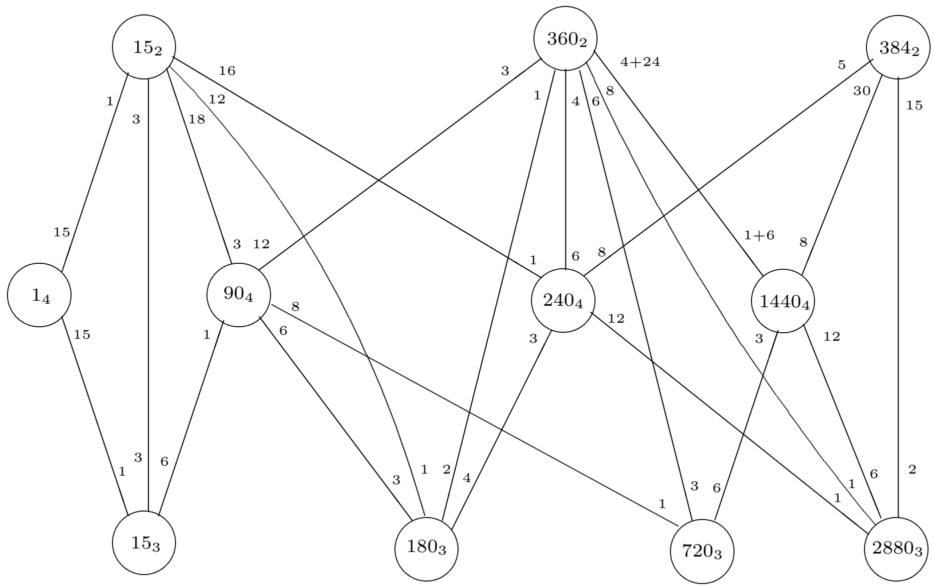
Let $\Gamma_j(m, i)$ denote an orbit of length m of M_i on Γ_j. It turns out that for every $i, j \in \{2, 3, 4\}$ the orbits of M_i on Γ_j all have different lengths so the orbit $\Gamma_j(m, i)$ is well defined. The information on the orbits of M_i on Γ is presented in the diagram $D_i(Mat_{24})$. In this diagram the orbit $\Gamma_j(m, i)$ is denoted by m_j and the numbers $n_i(A, B)$ and $n_i(B, A)$ are attached to the edge joining A with B. When such a number is presented as a sum this indicates that there is more than one orbit of $M_i \cap M_a$ (where $a \in A$) on the vertices in B adjacent to a. Moreover the summands give the lengths of these orbits. The complete proof of the diagrams (originally given in the early version of the present work) can be found in Section 3.7 of [9].



$D_2(Mat_{24})$



$D_3(Mat_{24})$



$D_4(Mat_{24})$

We will need the following refinement of the information given on the diagram $D_3(Mat_{24})$.

Lemma 3.5 *Let $b \in \Gamma_3(2688, 3)$. Then $M_{\beta b} \cap Q_b = 1$ and $M_{\beta b} \cong Sym(4)$.*

Proof. $|M_{\beta b}| = |M_3|/2688 = 24$ by direct calculation. By $D_3(Mat_{24})$ the subgroup $M_{\beta b}$ acts transitively on $\Gamma_2(b)$ and has two orbits in $\Gamma_4(b)$ with lengths 1 and 6. Since the action of $M_{\beta b}$ on $\Gamma_4(b)$ is a subgroup of $L_3(2)$, we conclude that the action is isomorphic either to $Sym(4)$ or to $Alt(4)$. Let K be the kernel of the action of $M_{\beta b}$ on $\Gamma_4(b)$. Then either $K = 1$ and $M_{\beta b} \cong Sym(4)$ or $|K| = 2$ and $M_{\beta b}/K \cong Alt(4)$. Assume the latter. Then $K = Z(M_{\beta b})$ and as $M_{\beta b}$ acts transitively on $\Gamma_2(b)$, $K \leq Q_b$. By symmetry we get $K = Q_\beta \cap Q_b$ and so $C_M(K)$ contains two elementary abelian groups of order 2^6 (namely Q_β and Q_b) intersecting in a group of order 2 (namely K). But this contradicts to the structure of $C_M(K)$ given by 3.1 (i) - (iii). □

Let $\mathcal{P}$ be the $GF(2)$-permutation module of M on Ω, that is the space of all the subsets of Ω with addition performed by the symmetric difference operator. The octads from Γ_2 generate in $\mathcal{P}$ a 12-dimensional subspace Y_0 known as the Golay code. The Golay code consists of: the empty set, the set Ω itself, 759 octads, 759 complements of octads and 2576 dodecads. The latter are 12-element subsets of Ω transitively permuted by M. The stabilizer of a dodecad is the Mathieu group Mat_{12} of degree 12 and it induces two non-equivalent 5-fold transitive actions on the dodecad and on its complement, which is also a dodecad. The setwise stabilizer of a pair of complementary dodecads is isomorphic to $Aut\ Mat_{12}$. The empty set together with the whole set Ω constitute the unique proper M-submodule in Y_0. The quotient Y of Y_0 over this submodule is called the *irreducible Golay code module* (of dimension 11). Let $\mathcal{P}_+$ denote the subspace in $\mathcal{P}$ of even subsets of Ω. Then $Y_0 \leq \mathcal{P}_+$ and $X = \mathcal{P}_+/Y_0$ is the module dual to Y which is called the *irreducible Todd module* (of dimension 11). The following information can be found for instance in [1, 19.8].

Lemma 3.6 (i) *The orbits of M on the non-zero vectors of Y (on the hyperplanes of X) are of length 759 and 1288. The vectors in these orbits are indexed by the octads and the complementary pairs of dodecads, respectively.*

(ii) *The orbits of M on the non-zero vectors of X (on the hyperplanes of Y) are of length* 276 *and* 1771. *The vectors in these orbits are indexed by the 2-element subsets of Ω and by the sextets, respectively.* □

Lemma 3.7 (i) *For $i = 2$, 3 and 4,*

$$1 < C_X(Q_i) < [X, Q_i] < X$$

is the unique composition series for M_i on X;

(ii) *$C_X(Q_2)$ is isomorphic the exterior square of Q_2, $[X, Q_2]/C_X(Q_2)$ is isomorphic to Q_2 and $|X/[X, Q_2]| = 2$.*

(iii) *Let D_1 and D_2 be as in* 3.2 (ii). *Then $C_X(Q_3)$ is dual to D_2, $[X, Q_3]/C_X(Q_3)$ is isomorphic to $D_1 \otimes D_2$ and $X/[X, Q_3]$ is isomorphic to D_1.*

(iv) *$|C_X(Q_4)| = 2$, $[X, Q_4]/C_X(Q_4)$ is isomorphic to the dual of Q_4 and $|X/[X, Q_4]| = 2^4$.*

Proof. The irreducible Todd module is dual to the irreducible Golay code module. Hence the result can be obtained by dualizing some of the information found in Sections 19 and 20 of [1]. □

Let D be the set of dodecads and H be the set of complementary pairs of dodecads. Recall that if $N(h)$ is the stabilizer of $h \in H$ in M then $N(h) \cong Aut\ Mat_{12}$ and $N(h)' \cong Mat_{12}$ is the subgroup of index 2 in $N(h)$ which preserves each of the dodecads constituting h. We are interested in the orbits on H of M_2, M_3 and M_{23}.

Lemma 3.8 *M_2 acting on the set D of dodecads has three orbits D_2, D_4 and D_6 with lengths* 448, 1680 *and* 448, *respectively. If $d_i \in D_i$ and K_i is the stabilizer of d_i in M_2 then $|d_i \cap \alpha| = i$, $K_2 \cong K_6 \cong Sym(6)$ and $K_4 \sim 2^5 Sym(3)$.*

Proof. [1, 19.6] □

Lemma 3.9 *Let $N \cong Aut\ Mat_{12}$ and N_2, N_3 and N_{23} subgroups of N such that $|N_2| = |N_3| = 2^7 \cdot 3$, $N_{23} = 2^7$ and $N_{23} = N_2 \cap N_3$. Then $N_{23} \cap N' = (N_{23} \cap N_2')(N_{23} \cap N_3')$.*

Proof. For $Z \leq N$ let $Z^* = Z \cap N'$. Then $N^* \cong Mat_{12}$, $|N/N^*| = 2$ and $|N^*|_2 = 2^6$. Thus N_{23}^* is a Sylow 2-subgroup of N^* and $|N_2^*| = |N_3^*| = 2^6 \cdot 3$. It follows (see for example [15]) that N_2^* and N_3^* are two maximal subgroups of N^* containing N_{23}^*. Choose notation such that $Z(N_2^*) \neq 1$. Thus by the structure of N_2^* and N_3^*, $O_2(N_2^*) \leq N_2^{*\prime}$, $O_2(N_3^*) \cap N_3^{*\prime} \not\leq O_2(N_2^*)$ and $|N_{23}^*/O_2(N_2^*)| = 2$. Thus $N_{23}^* = O_2(N_2^*)(O_2(N_3^*) \cap N_3^{*\prime}) = (N_{23} \cap N_2^{*\prime})(N_{23} \cap N_3^{*\prime})$. □

Lemma 3.10 *Let H be the set of complementary pairs of dodecads and for $h \in H$ let $N(h)$, $N_2(h)$, $N_3(h)$ and $N_{23}(h)$ denote the stabilizers of h in M, M_2, M_3 and M_{23}, respectively.*

(i) *M_2 has precisely two orbits $H_1(2)$ and $H_2(2)$ on H, where $|H_1(2)| = 840$ and $|H_2(2)| = 448$. M_3 has precisely three orbits $H_1(3)$, $H_2(3)$, and $H_3(3)$ on H, where $|H_1(3)| = 168$, $|H_2(3)| = 672$ and $|H_3(3)| = 448$. M_{23} has precisely four orbits H_1, H_2, H_3 and H_4 on H, where $|H_1| = 168$, $|H_2| = 224$, $|H_3| = 448$ and $|H_4| = 448$. Moreover, $H_1(2) = H_1 \cup H_2 \cup H_3$, $H_2(2) = H_4$, $H_1(3) = H_1$, $H_2(3) = H_2 \cup H_4$ and $H_3(3) = H_3$.*

(ii) *If $h \in H_1$, then $N(h) = N_{23}(h)N(h)'$ and $N_{23}(h) \cap N(h)' = (N_{23}(h) \cap N_2(h)')(N_{23}(h) \cap N_3(h)')$.*

(iii) *If* $h \in H_3$, *then* $N_3(h) = N_{23}(h)N_3(h)'$.

Proof. (i) The lengths of the orbits of M_2 on H follow directly from 3.8. Observe also that $N_2(h) \leq N(h)'$ (that is $N_2(h)$ fixes the two dodecads forming h) if and only if $h \in H_2(2)$.

Let Δ be the *octad graph*, that is a graph on Γ_2 in which two octads are adjacent if they are disjoint. For a vertex x of Δ let $\Delta^i(x)$ denote the set of vertices which are at distance i from x in Δ. It is well known and also easily seen from the diagram $D_2(Mat_{24})$ that

$$\Delta^1(\alpha) = \Gamma_2(30, 2),\ \Delta^2(\alpha) = \Gamma_2(280, 2),\ \Delta^3(\alpha) = \Gamma_2(448, 2)$$

and that

$$\mid \alpha \cap \delta \mid = 0,\ 4,\ 2 \ \text{ if } \ \delta \in \Delta^i(\alpha) \ \text{ for } \ i = 1,\ 2,\ 3.$$

By 3.6 we can and will identify $\Delta \cup H$ with the set of non-zero vectors in the irreducible Golay code module Y. Let $e \in \Delta^3(\alpha)$. Then as α and e intersect in 2 elements, the symmetric difference of α and e is a dodecad intersecting α in 6 elements. Thus $\alpha + e \in H_2(2)$ and since $|\Delta^3(\alpha)| = |H_2(2)| = 448$, we have a one-to-one correspondence between $H_2(2)$ and $\Delta^3(\alpha)$. By $D_2(Mat_{24})$, $M_{\alpha e}$ acts transitively on $\Gamma_3(\alpha)$. Thus M_{23} acts transitively on $\Delta^3(\alpha)$ and hence also on $H_4 \overset{def}{=} H_2(2)$.

Let $h \in H_1(2)$. Then $N_2(h)$ has order $2^7 \cdot 3$ and the intersections of α with the dodecads in h form a partition of α into two sets of sizes 4. Thus $N_2(h)Q_\alpha/Q_\alpha$is contained in a subgroup $2^4(Sym(3) \times Sym(3))$ of M_α/Q_α and so normalizes a 2-subspace U_2 in Q_α. Note that Q_α fixes 4 points in each of the two dodecads, $Q_\alpha \leq Mat_8 \cong Q_8$. As Q_α is elementary abelian, $Q_\alpha \cap N(h)$ has order at most two. It follows that $N_2(h)Q_\alpha/Q_\alpha$ has order $2^6 \cdot 3$ and $U_1 \overset{def}{=} Q_\alpha \cap N(h)$ has order 2, $U_1 \leq U_2$ and $Q_\alpha N_2(h) = N_{M_\alpha}(U_1) \cap N_{M_\alpha}(U_2)$. Thus the orbits of Q_α on $H_1(2)$ are in one-to-one correspondence with the pairs (U_1, U_2), where U_i is a i-space in Q_α and $U_1 \leq U_2$. This immediately implies that $M_{\alpha\beta}$ has three orbits H_1, H_2 and H_3 on $H_1(2)$ corresponding to the following three possibilities: (1) $U_2 \leq Q_\alpha \cap Q_\beta$, (2) $U_2 \not\leq Q_\alpha \cap Q_\beta$ and $U_1 \leq Q_\alpha \cap Q_\beta$ and (3) $U_1 \not\leq Q_\alpha \cap Q_\beta$. Now it is straightforward to calculate that $|H_1| = 7 \cdot 3 \cdot 8 = 168$, $|H_2| = 28 \cdot 1 \cdot 8 = 224$ and $|H_3| = 28 \cdot 2 \cdot 8 = 448$. (Notice that $\mid Q_\alpha \cap Q_\beta \mid = 8$ by 3.2 (i).)

Let L be the elementwise stabilizer in M of the octads in $\Delta(\beta)$. Then L is of index 2 in M_{23} and normal of index 6 in M_3. Hence each of the following holds (for the last statement note that $M = \langle M_2, M_3 \rangle$ acts transitively on H):

- For every i, L either acts transitively on H_i or has two orbits of the same length.
- Every M_3-orbit in H is the union of l of the orbits of equal lengths for L in H where $l \in \{1, 2, 3, 6\}$.
- There exists an M_3-orbit on H which has non-empty intersecting with both $H_1(2)$ and $H_2(2)$.

It is easy to check that these three conditions uniquely determine the fusion of the M_{23}-orbits into M_3-orbits.

(ii) and (iii): Let $h \in H_1$. As $N(h) \cong Aut\ Mat_{12}$, $|N(h)/N(h)'| = 2$. Moreover, $|N(h)/N_{23}(h)|$ is odd and so the first statement in (ii) holds. (iii) follows from a similar argument. By (i) we can apply 3.9 and so also the second part of (ii) holds. □

By [7] M has a 45-dimensional irreducible module V over the field $\mathbf{C}$ of complex numbers. Let χ be the corresponding character. Define $V_1(3) = C_V(Q_3)$ and $V_2(3) = [V, Q_3]$.

Lemma 3.11 (i) *Let* z *be a* 2*-central involution in* M. *Then* $\chi(z) = -3$, $C_V(z)$ *is* 21*-dimensional and* $[V, z]$ *is* 24*-dimensional.*

(ii) $C_V(Q_2) = 0$ *and* $C_V(H)$ *is* 3*-dimensional for each hyperplane* H *of* Q_2.

(iii) $V = V_1(3) \oplus V_2(3)$, $V_1(3)$ *is* 3*-dimensional and* $V_2(3)$ *is* 42*-dimensional.*

(iv) $C_{V_2(3)}(H) = 0$ *for any hyperplane of* Q_3 *containing* $Q_2 \cap Q_3$, *while* $C_{V_2(3)}(H)$ *is* 1*-dimensional for any hyperplane of* Q_3 *not containing any of the three conjugates of* $Q_2 \cap Q_3$ *under* M_3.

Proof. (i) The value for $\chi(z)$ is taken directly from the character table of Mat_{24} in [7]. Since $\dim C_V(z) + \dim[V, z] = 45$ and $\dim C_V(z) - \dim[V, z] = \chi(z)$ (i) holds.

(ii) Let $d = \dim[V, Q_2]$ and $e = \dim C_{[V,Q_2]}(H)$, where H is any hyperplane in Q_2. Since M_2 acts transitively on the fifteen hyperplanes in Q_2, e is well defined and $d = 15e$. Let $1 \neq z \in Q_2$. Then exactly eight of the hyperplanes in Q_2 do not contain z and so $24 = \dim[V, z] = 8e$. Thus $e = 3$, $d = 45$, $V = [V, Q_2]$ and (ii) holds.

(iii) and (iv) By 3.2 (iii) the orbits of M_3 on $Q_3^{\#}$ are of length 21 and 42 and by a dual argument the orbits of M_3 on the hyperplanes of Q_3 are of length 21 and 42. In particular, $\dim[V, Q_3]$ is divisible by 21. Moreover, by (ii) $C_V(Q_2 \cap Q_3)$ has dimension 3 and so $\dim C_V(Q_3) \leq 3$. Thus $\dim V_1(3) = 3$ and $\dim V_2(3) = 42$. Let H be a hyperplane in Q_3 with $f \stackrel{def}{=} \dim C_{V_2(3)}(H) \neq 0$. Then either $|H^{M_3}| = 42$ and $f = 1$; or $|H^{M_3}| = 21$ and $f = 2$. In particular H is unique up to conjugation. Suppose that $|H^{M_3}| = 21$. Let $1 \neq z \in Q_2 \cap Q_3$. Then it is easy to see that z lies in exactly $7 + 3 + 3 = 13$ of the elements of H^{M_3} and so $\dim[V, z] = f \cdot (21 - 13) = 16$, a contradiction to (i). Thus $|H^{M_3}| = 42$ and the lemma is proved. □

Lemma 3.12 (i) M_2 *acts irreducibly on* V *and as* M_2*-module* $V \cong V_1(3) \otimes_{\mathbf{C}M_{23}} \mathbf{C}M_2$.

(ii) $V_1(3)$ *and* $V_2(3)$ *are irreducible* M_3*-modules of dimension* 3 *and* 42*, respectively, and stay irreducible when restricted to* M_{23} *or* $O^2(M_{23})$.

(iii) $C_{M_3}(V_1(3)) = O_{2,3}(M_3)$ *and* M_3 *acts faithfully on* $V_2(3)$.

Proof. By 3.11 (ii), $V_1(3) = C_V(Q_2 \cap Q_3)$ is a Wedderburn component for Q_2 on V. Moreover, since M_{23} is maximal in M_2, $M_{23} = N_{M_2}(Q_2 \cap Q_3) = N_{M_2}(V_1(3))$ and so the second statement in (a) holds. Moreover, M_2 is irreducible on V if and only if M_{23} is irreducible on $V_1(3)$. Since $L \stackrel{def}{=} O^2(M_{23})$ acts transitively on the 42 hyperplanes in Q_3 which have fixed-points in $V_2(3)$, L acts irreducibly on $V_2(3)$. Suppose that L does not act irreducibly on $V_1(3)$. Since $V_1(3)$ has odd dimension and $|M_{23}/L| = 2$ we conclude that M_{23} does not act irreducibly on $V_1(3)$. Thus M_{23} has a 1- or 2-dimension submodule in $V_1(3)$ and M_2 has a 15- or 30-dimensional submodule in V. But this contradicts the fact that $V_2(3)$ is a 42-dimensional irreducible L-module. Hence L is irreducible on $V_1(3)$ and (i) and (ii) are proved.

To prove (iii) recall that $M_3 \sim 2^6(Sym(3) \times L_3(2))$. Note that $Q_2 \cap Q_3$ is a hyperplane in Q_2 and centralizes $V_1(3)$. Since Q_2 acts fixed-point freely on V we conclude that Q_2Q_3/Q_3 inverts $V_1(3)$. Furthermore, $O_{2,3}(M_3) = [M_3, Q_2]$ and so $O_{2,3}(M_3)$ centralizes $V_1(3)$. Hence either $C_{M_3}(V_1(3)) = O_{2,3}(M_3)$ or M_3' centralizes $V_1(3)$. But in the later case M_3 is not irreducible on $V_1(3)$. The second statement in (iii) holds since Q_3 is the unique minimal normal subgroup of M_3 and does not centralize $V_2(3)$. □

4 Mat_{22}

Definition 4.1 (i) *A* Mat_{22}*-triangle is a triangle of groups* (M_1, M_2, M_3) *such that*

(a) $M_1 \sim 2^4 Alt(6), M_2 \sim 2^3 L_3(2)$ *and* $M_3 \sim 2^4 Sym(5)$.

(b) $|M_2/M_{23}| = |M_2/M_{12}| = 7, |M_3/M_{13}| = 5$ *and* $|M_{23}/B| = 3$.

(ii) *An Aut* Mat_{22}*-triangle is a triangle of groups* $(\hat{M}_1, \hat{M}_2, \hat{M}_3)$ *such that*

(a) $\hat{M}_1 \sim 2^4 Sym(6), \hat{M}_2 \sim 2^4 L_3(2)$ *and* $\hat{M}_3 \sim 2^5 Sym(5)$.

(b) $|\hat{M}_2/\hat{M}_{23}| = |\hat{M}_2/\hat{M}_{12}| = 7, |\hat{M}_3/\hat{M}_{13}| = 5$ *and* $|\hat{M}_{23}/B| = 3$.

Lemma 4.2 *Let* (M_1, M_2, M_3) *be a* Mat_{22}*-triangle. Then the following assertions hold.*

(i) B *is a Sylow* 2*-subgroup of* M_2 *and* $B = (Q_1 \cap M_2)Q_2Q_3$.

(ii) $M_{13}/Q_3 \cong Sym(4)$ *and* $M_{23}/Q_3 \cong Sym(3) \times Sym(2)$.

(iii) $M_{12}/Q_2 \cong M_{23}/Q_2 \cong Sym(4)$.

(iv) $Q_1 \not\leq M_2$ *and* $M_{13}/Q_1 \cong M_{12}Q_1/Q_1 \cong Sym(4)$.

(v) $|Q_1 \cap Q_2| = 2$, $|Q_1 \cap Q_3| = 4$, $|Q_2 \cap Q_3| = 4$ *and* $Q_1 \cap Q_2 \leq Q_1 \cap Q_3$.

(vi) $T_{12} = (Q_1 \cap M_2)Q_2$, $T_{13} = Q_1Q_3$, $T_{23} = Q_2Q_3$.

Proof. Since M_3 has a unique class of subgroups of index 5, $M_{13}/Q_3 \cong Sym(4)$. Similarly $Sym(4) \cong M_{12}/Q_2 \cong M_{13}/Q_1 \cong M_{23}/Q_2$ and B is a Sylow 2-subgroup of M_2. Since $|M_{23}/B| = 3$, M_{23} has an orbit of length 3 on the cosets of M_{13} in M_3. Thus $M_{23}Q_3/Q_3$ is contained a subgroup $Sym(3) \times Sym(2)$ of M_3/Q_3. As M_{23} has index 10 in M_3 and $Sym(3) \times Sym(2)$ has index 10 in $Sym(5)$ we conclude that $Q_3 \leq M_{23}$ and $M_{23}/Q_3 \cong Sym(3) \times Sym(2)$. As $|Q_3| > |Q_2|$, $Q_3 \not\leq Q_2$ and since M_{23} acts irreducibly on T_{23}/Q_2, we conclude that $T_{23} = Q_2Q_3$ and $|Q_2 \cap Q_3| = 4$. Suppose that $Q_3 = Q_1$. Then $T_{23} = Q_2Q_3 = Q_2Q_1$ is normalized both by M_{12} and M_{23}. Since $M_{2i} = N_{M_2}(T_{2i})$ for $i = 1$ and 3, this means that $M_{12} = M_{23}$, a contradiction to $|M_{23}/B| = 3$. Thus $Q_3 \neq Q_1$, $T_{13} = Q_1Q_3$ and $|Q_1 \cap Q_3| = 4$. So $Q_1 \leq O_2(M_{13}) \leq M_3'Q_3$, $Q_1 \not\leq M_{12}$ and as no element of Q_1 acts as a 2-cycle on M_3/M_{13}, $Q_1 \cap M_{12} \not\leq Q_2$. Hence $T_{12} = Q_2(Q_1 \cap M_{12})$ and $|Q_2 \cap Q_1| = 2$. Since $B = T_{12}T_{23}$, the last statement in (i) holds and the proof is complete. □

We remark that a similar lemma holds for *Aut* Mat_{22}-triangles. Indeed the only changes necessary are that in part (iv), $Sym(4)$ has to be replaced by $Sym(2) \times Sym(4)$ and in part (v), $\hat{Q}_2 \cap \hat{Q}_3$ has order 8 and not 4.

As in the previous section let $\mathcal{S}$ be the Steiner system of type $(5, 8, 24)$ and let p, q be a pair of elements from the basic set Ω. In this section M and $\hat{M}$ will denote the elementwise and the setwise stabilizers of $\{p, q\}$ in the automorphism group Mat_{24} of $\mathcal{S}$, respectively. This means that M is the Mathieu group Mat_{22} of degree 22 with $|M| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ and $\hat{M}$ is the automorphism group of M.

Let γ be a sextet $T_1, T_2, ..., T_6$ in $\mathcal{S}$ such that p and q are in the same tetrad (say in T_1). Let α and β be disjoint octads adjacent to γ such that $\{p, q\} \subseteq \alpha$ (say $\alpha = T_1 \cup T_2$ and $\beta = T_3 \cup T_4$). Let M_α, M_β and M_γ be the stabilizers in M of α, β and γ, respectively. Similarly define $\hat{M}_\alpha, \hat{M}_\beta$ and $\hat{M}_\gamma$. The following lemma can be deduced directly from 3.2 (cf. Section 3.4 in [9]).

Lemma 4.3 (i) $(M_\alpha, M_\beta, M_\gamma)$ *is a* Mat_{22}*-triangle.*

(ii) $(\hat{M}_\alpha, \hat{M}_\beta, \hat{M}_\gamma)$ *is an Aut* Mat_{22}*-triangle.* □

It is easy to deduce from the main result in [16] that every *Aut* Mat_{22}-triangle with a faithful completion is isomorphic to $(\hat{M}_\alpha, \hat{M}_\beta, \hat{M}_\gamma)$ and that $\hat{M}$ is the unique completion of the triangle. In order to explain the deduction we need some definitions.

Recall that the *Petersen graph* has 2-element subsets of a fixed 5-set as vertices and two subsets are adjacent if they are disjoint. The Petersen graph has 10 vertices, 15 edges and $Sym(5)$ is its automorphism group.

Definition 4.4 *Let* $\Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3$ *be a* 3*-partite graph and suppose that for* $a_i \in \Xi_i$, $1 \leq i \leq 3$ *the following conditions hold.*

(i) $|\Xi_2(a_1)| = 2$, $|\Xi_3(a_1)| = 3$ *and every vertex from* $\Xi_2(a_1)$ *is adjacent to every vertex from* $\Xi_3(a_1)$.

(ii) $\Xi_1(a_2)$ *are the points and* $\Xi_3(a_2)$ *are the lines of a projective plane of order* 2 *with the natural adjacency relation.*

(iii) $\Xi_1(a_3)$ *are the edges and* $\Xi_2(a_3)$ *are the vertices of the Petersen graph with the natural adjacency relation.*

Then Ξ *is called a rank* 3 *Petersen type geometry.*

Theorem 4.5 *Up to isomorphism there are exactly two rank* 3 *flag-transitive Petersen type geometries:* $\Xi(Mat_{22})$ *and* $\Xi(3Mat_{22})$. *A flag-transitive automorphism group is isomorphic to* M *or* $\hat{M}$ *for* $\Xi(Mat_{22})$ *and to* $3M$ *or* $3\hat{M}$ *(non-split extensions) for* $\Xi(3Mat_{22})$. *The stabilizer of a vertex from* Ξ_3 *in* M *and* $3M$ *is* $2^4 Sym(5)$ *while in* $\hat{M}$ *and* $3\hat{M}$ *it is* $2^5 Sym(5)$.

Proof. [16]. □

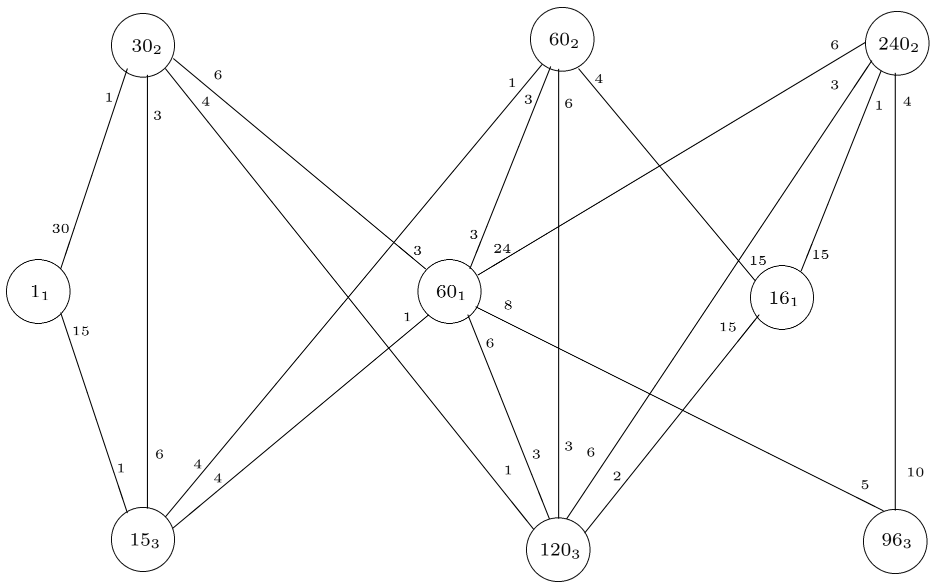
Lemma 4.6 (i) *Every* Mat_{22}*-triangle with a faithful completion is isomorphic to* $(M_\alpha, M_\beta, M_\gamma)$ *and* M *is the unique faithful completion of this triangle.*

(ii) *Every Aut* Mat_{22}*-triangle with a faithful completion is isomorphic to* $(\hat{M}_\alpha, \hat{M}_\beta, \hat{M}_\gamma)$ *and* $\hat{M}$ *is the unique faithful completion of this triangle.*

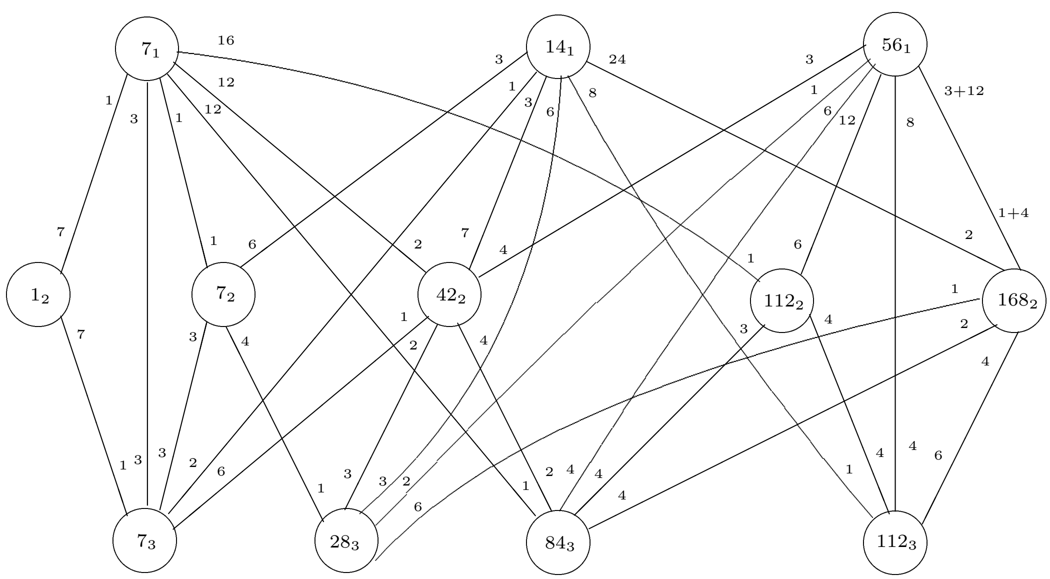
Proof. (i) Let (M_1, M_2, M_3) be a Mat_{22}-triangle with a faithful completion N. Define a triangle (N_1, N_2, N_3) by $N_1 = C_{M_1}(Q_1 \cap Q_2)$, $N_2 = M_2$ and $N_3 = M_3$. Then $N_1 \sim 2^4 Sym(4)$ by 4.2 and since $M_1 = \langle N_1, M_{13} \rangle$, N is also a faithful completion of (N_1, N_2, N_3). Let $\Xi = \Gamma(N; N_1, N_2, N_3)$. Then it is easy to check using the information in 4.2 that Γ is a rank 3 Petersen type geometry on which N acts flag-transitively. By 4.5 and since $N_3 = M_3 \sim 2^4 Sym(5)$ we have $N \cong M$ or $N \cong 3M$, but in the latter case $M_1 \sim 2^4 3 \cdot Alt(6)$. Hence $N \cong M$ and (M_1, M_2, M_3) is isomorphic to $(M_\alpha, M_\beta, M_\gamma)$.

(ii) is proved similarly. □

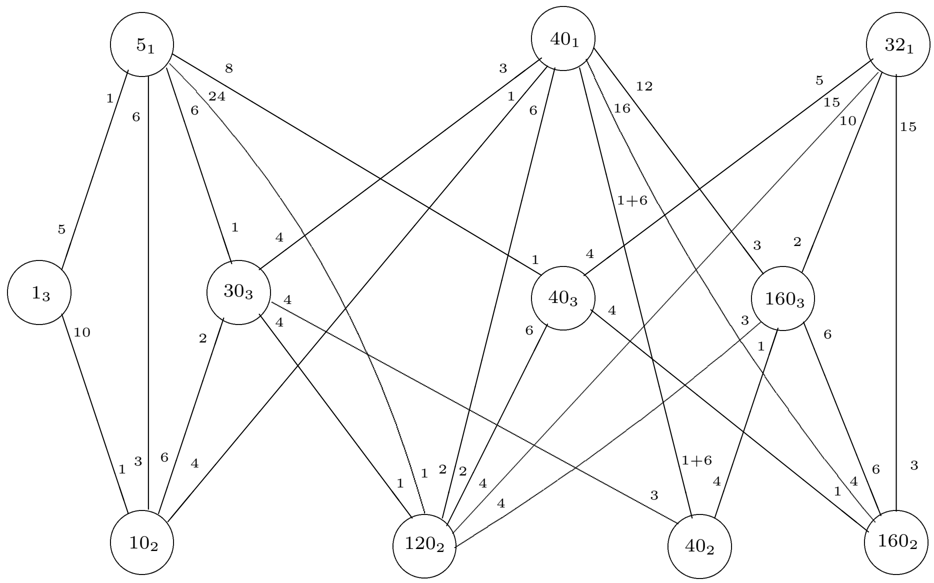
The coset graph $\Gamma = \Gamma(M; M_1, M_2, M_3)$ (which coincides with $\Gamma(\hat{M}; \hat{M}_1, \hat{M}_2, \hat{M}_3)$) possesses a natural description in terms of the Steiner system $\mathcal{S}$ and a pair p, q of distinguished elements from the basic set Ω. Namely, Γ_1 are the *hexads* which are octads containing $\{p, q\}$ with p and q removed; Γ_2 are the *octets* which are the octads disjoint from $\{p, q\}$ and Γ_3 are the *pairs* which are 2-element subsets of $\Omega \setminus \{p, q\}$. A hexad and an octet are adjacent if they are disjoint; the adjacency between the hexads and pairs is via inclusion, finally an octet is adjacent to a pair $\{r, s\}$ if it is the union of two tetrads from the sextet containing $\{p, q, r, s\}$. Below we present the diagrams $D_i(Mat_{22})$ describing the orbits of M_i on Γ and the adjacencies between the vertices in these orbits. These diagrams are analogous to the diagrams $D_i(Mat_{24})$. The proofs of the diagrams can be found in Section 3.9 in [9].



$D_1(Mat_{22})$



$D_2(Mat_{22})$



$D_3(Mat_{22})$

We need some further refinement of the information given on the above diagrams.

Lemma 4.7 (i) *Let $a \in \Gamma_1(32,3)$. View $\Gamma_3(a)$ as the set of points in a 4-dimensional symplectic space S over $GF(2)$ with $\hat{M}_a/Q_a$ acting as the full group of automorphisms. Then $\hat{M}_{\gamma a}$ fixes a non-degenerate quadratic form Q of minus type on S and $\Gamma_3(40,3)$ is the set of singular points of Q. In particular, for each $b \in \Gamma_2(a)$, there is a unique $c \in \Gamma_3(ab) \cap \Gamma_3(40,3)$.*

(ii) *$\hat{Q}_\gamma$ act regularly on $\Gamma_1(32,3)$.*

Proof. Note that any subgroup of index 32 in $\hat{M}_\gamma$ is isomorphic $Sym(5)$ and so in particular $\hat{M}_{\gamma a} \cong Sym(5)$ and $\hat{Q}_\gamma$ acts regularly on $\Gamma_1(32,3)$. Thus the lemma follows directly from the diagram $D_3(Mat_{22})$ and elementary properties of the 4-dimensional symplectic $GF(2)$-geometry. □

Lemma 4.8 *Let $c \in \Gamma_3(160,3)$. Then $\hat{M}_{\gamma c}\hat{Q}_c/\hat{Q}_c \cong Sym(3) \times C_2$, $Q_c \cap M_\gamma = 1$ and $\hat{Q}_c \cap \hat{M}_\gamma \cong C_2$.*

Proof. By $D_3(Mat_{22})$, $\gamma \cup c$ is not contained in a hexad. In particular γ and c are disjoint. Thus there exists exactly two hexads a_1 and a_2 such that $c \subset a_i$ and $\gamma \cap a_i \neq \emptyset$. Thus $\hat{M}_{\gamma c}$ normalizes a subset of size two of the five hexads adjacent to c. Thus $\hat{M}_{\gamma c}\hat{Q}_c/\hat{Q}_c$ is contained in a $Sym(3) \times C_2$ subgroup of M_c/Q_c. Let $t \in Q_c \cap M_\gamma$. Then t normalizes a_i and fixes $\gamma \cap a_i$ for $i = 1,2$ and also fixes the two elements in c. Thus t fixes three elements in a_1. Since $M_{a_1}/Q_{a_1} \cong Alt(6)$, t does not induce a 2-cycle on a_1 and thus fixes a_1 elementwise. Since t also fixes the point $a_2 \cap \gamma$ outside of a_1 we conclude $t = 1$ and $Q_c \cap M_\gamma = 1$. Thus $|\hat{Q}_c \cap \hat{M}_\gamma| \leq 2$. Since $|\hat{M}_{\gamma c}\hat{Q}_c/\hat{Q}_c| \leq 12$ and $|\hat{M}_{\gamma c}| = 24$, the lemma is established. □

5 J_4-triangles

In this section we establish the existence and uniqueness of a J_4-triangle of groups. We follow notation from Section 1.

Lemma 5.1 *Let (M_1, M_2, M_3) be a J_4-triangle. Let $K_1 \cong Mat_{24}$ be a complement to Q_1 in M_1; $K_2 \cong L_5(2)$ be a complement to Q_2 in M_2 and let L be the unique normal subgroup in M_3 such that $M_3/L \cong Sym_5$. Let $\mathcal{S}$ be the Steiner system of type $(5, 8, 24)$ such that Q_1 is the irreducible Todd module associated with the action of K_1 on $\mathcal{S}$ and $\Omega(3)$ be an M_3-set of size 5 such that $C_{M_3}(\Omega(3)) = L$. Then*

(i) *there are subsets $\Omega_1(3)$ and $\Omega_2(3)$ in $\Omega(3)$ of size 1 and 2 respectively with $\Omega_1(3) \not\subseteq \Omega_2(3)$ such that $M_{31} = N_{M_3}(\Omega_1(3))$ and $M_{32} = N_{M_3}(\Omega_2(3))$; in particular,*

$$M_{31}/Q_3 \cong Sym(4) \times L_3(2), \quad M_{32}/Q_3 \cong C_2 \times Sym(3) \times L_3(2) \quad \text{and} \quad B/Q_3 \cong C_2 \times C_2 \times L_3(2),$$

moreover, $Q_1 \cap Q_2 \leq Q_3$ and $T_{13} \not\leq M_2$;

(ii) *there is a natural $L_5(2)$-module $V(2)$ of M_2, a 1-space $V_1(2)$ and a 2-space $V_3(2)$ in $V(2)$ with $V_1(2) \leq V_3(2)$ such that $M_{23} = N_{M_2}(V_3(2))$ and $M_{21} = N_{M_2}(V_1(2))$; in particular,*

$$M_{23}/Q_2 \sim 2^6(Sym(3) \times L_3(2)), \quad M_{21}/Q_2 \sim 2^4 L_4(2) \quad \text{and} \quad B/Q_2 \sim 2^{3+3+1} L_3(2);$$

(iii) *there is an octad α and a trio β containing α such that $M_{12}Q_1 = N_{M_1}(\alpha)$ and $M_{13} = N_{M_1}(\beta)$; in particular,*

$$M_{13}/Q_1 \sim 2^6(Sym(3) \times L_3(2)), \quad M_{12}Q_1/Q_1 \sim 2^4 L_4(2) \quad \text{and} \quad BQ_1/Q_1 \sim 2^{3+3+1} L_3(2),$$

moreover, $|Q_1/Q_1 \cap M_2| = 2$;

(iv) *For all $i \neq j$, M_{ij} acts irreducibly on T_{ij}/Q_j.*

(v) *$T_{13} = Q_1 Q_3$, $T_{23} = Q_2 Q_3$ and $T_{12} = (Q_1 \cap M_2) Q_2$.*

(vi) *$|Q_2/Q_2 \cap Q_3| = 2$, $|Q_2/Q_1 \cap Q_2| = 2^4$ and Q_2 is isomorphic to $\bigwedge^2 V(2)^*$ where $V(2)$ is as in* (ii).

(vii) *$\Phi(Q_3) = Z(Q_3)$ is a natural $L_3(2)$-module for M_3 and $Q_3/\Phi(Q_3) \cong D_1 \otimes D_2$, where D_1 is a natural $\Gamma L_2(4)$-module for M_3 and D_2 is dual to $Z(Q_3)$.*

(viii) *$L = O^2(B)$.*

(ix) *$N_{M_i}(Q_i \cap Q_j) = M_{ij}$ if $(i, j) \neq (1, 2)$ and $N_{M_1}(Q_1 \cap Q_2) = Q_1 M_{12}$.*

Proof. Since $|M_3/M_{13}| = 5$ and M_3 has a unique class of subgroups of index 5, we can put $\Omega(3) = M_3/M_{31}$ so that $L = \cap_{g \in M_3} M_{31}^g$. Since $|M_{32}/B| = 3$, M_{32} has on orbit of length 3 on $\Omega(3)$. Thus $M_{32}L/L$ is contained in a $Sym(3) \times C_2$-subgroup of M_3/L. Since the index of M_{23} in M_3 and the index of $Sym(3) \times C_2$ in Sym_5 are both 10, we conclude that $L \leq M_{32}$ and $M_{32}/L \cong Sym(3) \times C_2$. In particular, $L \leq B$ and since $|M_{32}/B| = 3$ we have $B/L \cong C_2 \times C_2$. This implies that B/L contains 2-cycles and so $B/L \neq O_2(M_{31}/L)$. As $O_2(M_{31}/L) = T_{31}L/L$ we get $T_{31} \not\leq B$ and $T_{13} \not\leq M_2$ which gives (i).

For (ii) let $i \in \{1, 3\}$ and let $V(2)$ be some natural $L_5(2)$-module for M_2. Since $|M_2/M_{23}| = 155$ and $|M_2/M_{21}| = 31$, M_{2i} contains a Sylow 2-subgroup of M_2. In particular, $Q_2 \leq M_{2i}$ and M_{2i} is the normalizer of some flag in $V(2)$. Since $|M_2/M_{21}| = 31$, $M_{21} = N_{M_2}(V_1(2))$ for some 1- or 4-space $V_1(2)$ in $V(2)$. Replacing $V(2)$ by its dual if necessary we may assume that $V_1(2)$ is a 1-space. Since $|M_2/M_{23}| = 155$, $M_{23} = N_{M_2}(V_3(2))$ for some 2- or 3-space $V_3(2)$ in $V(2)$. Since $|M_{23}/M_{23} \cap M_{12}| = |M_{23}/B| = 3$ which is odd, $V_1(2) \leq V_3(2)$ and $V_3(2)$ is a 2-space. Thus (ii) holds.

(iii) Since $|M_1/M_{13}| = 3795$, M_{13} contains a Sylow 2-subgroup of M_1 and so by 3.4 $M_{13} = N_{M_1}(\beta)$ for some trio β in $\mathcal{S}$. Suppose that $Q_1 \leq Q_3$. Since $|Q_1| > |Q_2|$, $Q_1 \not\leq Q_2$. By (ii) M_{12} acts irreducibly on T_{12}/Q_2 and so $T_{12} = Q_1 Q_2$. Hence $T_{12} = Q_1 Q_2 \leq Q_3 Q_2 \leq T_{23}$ a contradiction since by (ii) T_{23} centralizes $V_3(2)$ but T_{12} does not. Thus $Q_1 \not\leq Q_3$ and so by (i), $Q_1 Q_3 = T_{13}$ and $Q_1 \not\leq M_2$. Hence $Q_1 \not\leq M_{12}$, $|Q_1/Q_1 \cap M_2| = 2$, $|M_1/M_{12}Q_1| = 759$ and by 3.4, $M_{12}Q_1 = N_{M_1}(\alpha)$ for some octad α. Also $|M_{13}/Q_1 B| = 3$. By 3.2, M_{13}/Q_1 has a unique class of subgroups of index 3 and so $BQ_1 = N_{M_{13}}(\alpha^*)$ for some octad α^* in β. By 3.4 (ii) BQ_1 fixes a unique octad, so $\alpha = \alpha^*$ and (iii) holds.

(iv) follows from (i), (ii) and (iii).

We already proved that $Q_1 Q_3 = T_{13}$. Now $(T_{13}L/L)^{\#}$ contains no 2-cycles and by (i) $Q_1 \cap M_2 \not\leq T_{23}$. Thus $Q_1 \cap M_2 \not\leq Q_2$ and by (iv), $(Q_1 \cap M_2)Q_2 = T_{12}$. Since $|Q_3| > |Q_2|$, $Q_3 \not\leq Q_2$ and by (iv) $Q_2 Q_3 = T_{23}$ and (v) holds.

By (v) and (i) $|Q_2/(Q_2 \cap Q_3)| = |Q_2 Q_3/Q_3| = |T_{23}/Q_3| = 2$, by (v) and (iii) $|Q_2/(Q_2 \cap Q_1)| = |T_{12}/Q_1| = 2^4$ and by (i) $Q_2 \cap Q_1 \leq Q_3$. By the definition of J_4-triangle Q_2 is isomorphic either to $\bigwedge^2 V(2)$ or to $\bigwedge^2 V(2)^*$. Since $M_{21} = N_{M_2}(V_1(2))$ for a 1-space $V_1(2)$ in $V(2)$, the only proper subspace in Q_2 normalized by M_{12} has dimension 4 in the former case and dimension 6 in the latter case. Since $Q_1 \cap Q_2$ is a 6-space (vi) follows.

Let $Z_3 = C_{Q_1}(Q_3)$. Since $T_{13} = Q_1 Q_3$, we have $Z_3 = C_{Q_1}(T_{13})$. Since Q_1 is the irreducible Todd module, by 3.7 Z_3 has order 2^3 and $Z_3 \leq Q_1 \cap Q_3$. By 3.7 (iii) $Z(Q_3) \leq Z_3$ and hence $Z_3 = Z(Q_3)$. By (iv) and (v) $\Phi(Q_3) \leq Q_1 \cap Q_2$. Since $Z_3 < Q_1 \cap Q_2 < Q_1 \cap Q_3$, since M_{13} acts irreducibly on $Q_1 \cap Q_3/Z_3$ and since M_{31} normalizes $\Phi(Q_3)$ we conclude that $\Phi(Q_3) \leq Z_3$. On the other hand by 3.7 $[Q_1 \cap Q_3, Q_3] = Z_3$ and so $\Phi(Q_3) = Z_3$. By 3.7, $(Q_1 \cap Q_3)/Z_3$ is the unique proper M_{31}-submodule in Q_3/Z_3. Moreover, all composition factors for L on Q_3/Z_3 are dual to Z_3 and the elements of order three in $C_{M_{31}}(Z_3)$ act fixed-point freely on Q_3/Z_3. By (ii) $Q_2 \cap Q_3/Z_3$ is the unique proper M_{23}-submodule in Q_3/Z_3 and since $Q_1 \cap Q_2 < Q_2 \cap Q_3$, $Q_1 \cap Q_3 \neq Q_2 \cap Q_3$. Thus M_3 acts irreducible on Q_3/Z_3 and (vii) holds.

(viii) By (i) $|B/L| = 4$ and by (vii) $O^2(L) = L$. Thus (viii) holds.

(ix) Clearly $Q_i \cap Q_j$ is normal in $Q_i M_{ij}$ and the latter is equal to M_{ij} unless $(i,j) = (1,2)$. On the other hand in each case $Q_i M_{ij}/Q_i$ is maximal in M_i/Q_i and hence the result follows. □

Our next result will be used as a characterization of M_{12}.

Lemma 5.2 *For $i = 1$ and 2 let X_i be a group generated by subgroups Z_i, A_i, B_i and R_i such that*

(i) *R_i is isomorphic to $L_4(2)$;*

(ii) *Z_i, A_i and B_i are elementary abelian 2-groups of order 2^6, 2^4 and 2^4, respectively;*

(iii) *R_i normalizes Z_i, A_i, and B_i, A_i and B_i are isomorphic natural $L_4(2)$-modules for R_i and Z_i is isomorphic to the exterior square of A_i, that is Z_i is a natural $\Omega_6^+(2)$-module for R_i;*

(iv) *Z_i centralizes A_i and B_i;*

(v) *$[A_i, B_i] = Z_i$.*

Then

(a) *there exists an isomorphism from X_1 onto X_2 mapping Y_1 to Y_2 for $Y = Z$, A, B and R;*

(b) *Out* X_i *is elementary abelian of order* 2^2.

Proof. Fix $i \in \{1,2\}$ and put $Y = Y_i$ for $Y \in \{X,R,A,B,Z\}$. Pick $1 \neq a \in A$ and put $P = C_R(a)$. Note that A, B and Z are absolutely irreducible $GF(2)R$-modules and so there exist unique $GF(2)R$-isomorphisms $\phi : A \to B$ and $\psi : \bigwedge^2 A \to Z$. Define $\xi : A \times A \to Z$ by $\xi(v,w) = [v, \phi(w)]$. Since A is irreducible and $[A,B] \neq 1$, $[a,B] \neq 1$. Note that $[a,B]$ and $B/C_B(a)$ are isomorphic as $GF(2)P$-modules. Moreover, P fixes no non-zero vector in Z and so $[a,\phi(a)] = 1$. Thus $\xi(a,a) = 1$ and so ξ extends to a $GF(2)R$-homomorphism $\Xi : \bigwedge^2 A \to Z$. Thus $\Xi = \psi$ and so $[v,\phi(w)] = \psi(v \wedge w)$ for all $v,w \in A$. It is now clear that (a) holds.

Put $Q = ABZ$. By 2.7 all complements to Q/Z in X/Z are conjugate in X/Z and ZR has two classes of complements to Z. Thus X has two classes of complements to Q and it follows easily from (a) that there exists an automorphism of X interchanging these two classes. Let α be an automorphism of X normalizing R. Since the module for R dual to A is not involved in Q, α induces an inner automorphism on R. So we may assume that α centralizes R.

Let C/Z be the unique irreducible R-submodule in Q/Z different from AZ/Z and BZ/Z. We claim that R does not normalize a complement to Z in C. First notice that if a complement in C exists, it should consist of the elements $b\phi(b), b \in A$, since $a\phi(a)$ is the only element in C invariant under the maximal parabolic P in R. However these elements are not closed under multiplication, since

$$a\phi(a)b\phi(b) = ab\phi(ab)[\phi(a), b]$$

and the factor $[\phi(a), b]$ is non-trivial when $a \neq b$. Thus the claim follows.

By the claim $A^\alpha \not\leq C$ and $\{A,B\} = \{A^\alpha, B^\alpha\}$. Again by (a) there exists an automorphism of X normalizing R and interchanging A and B. So we may assume that α normalizes A and B. Since α centralizes R and since A, B and Z are absolutely irreducible $GF(2)R$-modules, α centralizes A, B and Z and α is the identity automorphism. $\square$

Let $V(2)$ be a 5-dimensional $GF(2)$-space, $K_2^\circ = GL(V(2)) \cong L_5(2)$ and M_2° be the semidirect product of $Q_2^\circ := \bigwedge^2 V(2)^*$ and K_2° with respect to the natural action. Let $V_1(2)$ be a 1-space in $V(2)$, K_{21}° be the stabilizer of $V_1(2)$ in K_2° and M_{21}° be the subgroup in M_2° which is the semidirect product of Q_2° and K_{21}°. Let $\mathcal{S}$ be a Steiner system of type (5,8,24), $K_1^\circ = Aut\ \mathcal{S} \cong Mat_{24}$, Q_1° be the 11-dimensional Todd module associated with the action of K_1° on $\mathcal{S}$ and M_1° be the semidirect product of Q_1° and M_1°. Let α be an octad in $\mathcal{S}$, K_{12}° be the stabilizer of α in K_1°, H_1 be the unique hyperplane in Q_1° stabilized by K_{12}° (compare 3.2 and 3.7) and M_{12}° be the subgroup in M_1° which is the semidirect product of H_1 and K_{12}°.

Lemma 5.3 *Let* $X_i = Z_iA_iB_iR_i$ *the group introduced in* 5.2, *then*

(i) *there is an isomorphism of* M_{21}° *onto* X_i *which sends* K_{21}° *onto* A_iR_i*;*

(ii) *there is an isomorphism of* M_{12}° *onto* X_i *which sends* K_{12}° *onto* A_iR_i*.*

Proof. By 2.8 and the obvious duality there is an orbit $H(2)^*$ of $L_5(2)$ on the set of vectors in Q_2° indexed by the 3-spaces in $V(2)$. Let $A_2 = O_2(K_{21}^\circ)$, $Q_{21}^\circ = O_2(M_{21}^\circ)$ and R_2 a complement to A_2 in K_{21}° normalizing a complement U to $V_1(2)$ in $V(2)$. Then R_2 is isomorphic to $L_4(2)$ and A_2 is the kernel of the action of K_{21}° on the set of subspaces in $V(2)$ containing $V_2(1)$. This means that A_2 is dual to U and the latter is canonically isomorphic to $V(2)/V_1(2)$. The elements from $H(2)^*$ corresponding to 3-spaces containing $V_1(2)$ are centralized by A_2 and by a standard property of exterior squares they generate an R_2-submodule Z_{21} in Q_2° isomorphic to $\bigwedge^2 A_2$. The elements from

$H(2)^*$ corresponding to 3-spaces taken from U generate a complement B_2 to Z_{21} in Q_2° normalized by R_2 and isomorphic to A_2. In particular $Z_{21} = C_{Q_2^\circ}(A_2) = Z(Q_{21}^\circ)$. Moreover, $M_{21}^\circ = Z_{21}A_2B_2R_2$ and (i) follows.

Next, let $A_1 = O_2(K_{12}^\circ)$ and $Z_{12} = C_{Q_1^\circ}(A_1)$. Let $t \in Q_1^\circ \setminus H_1$. By 3.2 and 3.7 (i) A_1 acts regularly on the elements in tH_1/Z_{12}. Put $R_1 = N_{K_{12}^\circ}(tZ_{12})$. Then by the Frattini argument R_1 is a complement to A_1 in K_{12}°. In particular $R_1 \cong L_4(2)$. Put $B_1 = A_1^t$. Since t normalizes $Z_{12}R_1$ and $Z_{12}R_1$ normalizes A_1 we conclude that $Z_{12}R_1$ normalizes B_1. Thus B_1 is R_1-invariant. Clearly A_1 and B_1 are isomorphic as R_1-modules and by 3.7 Z_{12} is isomorphic to the exterior square of A_1. Moreover, $M_{12}^\circ = Z_{12}A_1B_1R_1$ and so by 5.2 we obtain (ii). □

Lemma 5.4 *With M_1°, M_2°, M_{12}° and M_{21}° as above there exists a unique amalgam (M_1°, M_2°) such that $M_1^\circ \cap M_2^\circ = M_{12}^\circ = M_{21}^\circ$ and $K_{12}^\circ = K_{21}^\circ$.*

Proof. By 5.3 there is an isomorphism of M_{12}° onto M_{21}° which sends K_{12}° onto K_{21}° and hence the existence follows. In order to prove the uniqueness it is sufficient to show that for every automorphism σ of M_{12}° there is an automorphism δ of M_1° which normalizes M_{12}° such that the restriction of δ to M_{12}° coincides with σ. This is certainly true if σ is an inner automorphism and by 5.2 (b) and 5.3 the outer automorphism group of M_{12}° is of order 2^2. Thus it is sufficient to present a subgroup $\hat{M}_1$ in the automorphism group of M_1°, containing the inner automorphisms such that M_{12}° (identified with a subgroup of inner automorphisms of M_1°) has trivial centralizer in $\hat{M}_1$ and $N_{\hat{M}_1}(M_{12}^\circ)/M_{12}^\circ \cong 2^2$. Let $\hat{Q}_1$ be the 12-dimensional $GF(2)M_1^\circ$-module obtained from the 24-dimensional permutational module on the element set Ω of the Steiner system $\mathcal{S}$ modulo the 12-dimensional Golay code. Let $\hat{M}_1$ be the semidirect product of $\hat{Q}_1$ and K_1°. Then $\hat{M}_1$ contains M_1° as a subgroup of index 2. It is well known (cf. [1] or [9]) that K_1° has four orbits on $\hat{Q}_1^\#$ with lengths 24, 276, 2024 and 1771 indexed by 1-, 2-, 3-element subsets of Ω and by the sextets, respectively. This shows that $C_{\hat{Q}_1}(K_{12}^\circ) = 1$ and hence $C_{\hat{M}_1}(M_{12}^\circ) = 1$. On the other hand it is clear that M_{12}° is a normal subgroup of index 2^2 in the subgroup in $\hat{M}_1$ which is the semidirect product of $\hat{Q}_1$ and K_{12}° and the result follows. □

In view of the preceding lemma we may and do identify M_{12}° with M_{21}° and K_{12}° with K_{21}°.

Lemma 5.5 *Let (M_1, M_2, M_3) be a J_4-triangle of groups. There exists an isomorphism κ of the amalgam (M_1°, M_2°) as in 5.4 onto the subamalgam (M_1, M_2).*

Proof. By 1.3 (i), (ii) there are isomorphisms $\kappa_1 : M_1^\circ \to M_1$ and $\kappa_2 : M_2^\circ \to M_2$. By 5.1 (ii) and (iii) these isomorphisms can be chosen in such a way that that $\kappa(M_{12}^\circ) = M_{12}$ and $\kappa(M_{21}^\circ) = M_{21}$. Now the uniqueness statement in 5.4 ensures existence of the isomorphism κ of amalgams. □

Notice that at this stage we do not know whether or not a J_4-triangle of groups exists but we do know that the rank two amalgam (M_1°, M_2°) exists.

Let β be a trio containing the octad α. Put $M_{13}^\circ = N_{M_1^\circ}(\beta)$, $B^\circ = M_{12}^\circ \cap M_{13}^\circ$, $L^\circ = O^2(B^\circ)$, $M_{23}^\circ = N_{M_2^\circ}(L^\circ)$, $Q_{13}^\circ = O_2(M_{13}^\circ)$, $Q_3^\circ = O_2(L^\circ)$ and $Z_3^\circ = Z(Q_3^\circ)$.

Lemma 5.6 (i) *$L^\circ = O^{2,3}(M_{13}^\circ)$, $M_{13}^\circ = N_{M_1^\circ}(L^\circ)$, $L^\circ/Q_3^\circ \cong L_3(2)$, L° splits over Q_3°, $Q_1^\circ \cap Q_3^\circ \leq H_1 \not\leq L^\circ$, $Q_1^\circ \cap Q_3^\circ = [Q_1^\circ, Q_{13}^\circ]$, $Q_{13}^\circ = Q_1^\circ Q_3^\circ$, $Z_3^\circ = \Phi(Q_3^\circ) = (Q_3^\circ)' = C_{Q_3^\circ}(Q_{13}^\circ)$, Z_3° is a natural $L_3(2)$-module for L° and Q_3°/Z_3° is the direct sum of four natural $L_3(2)$-modules dual to Z_3°.*

(ii) *$M_{23}^\circ = N_{M_2^\circ}(V_3(2))$ where $V_3(2)$ is some 2-space in $V(2)$ containing $V_1(2)$.*

(iii) *$M_{13}^\circ/L^\circ \cong Sym(4)$ and $M_{23}^\circ/L^\circ \cong Sym(3) \times C_2$.*

(iv) $B^\circ = M_{21}^\circ \cap M_{23}^\circ$, $B^\circ/L^\circ \cong C_2 \times C_2$ *and* B° *is not normal in* M_{13}°.

(v) $C_{M_1^\circ}(L^\circ) = 1 = C_{M_2^\circ}(L^\circ)$.

(vi) *the isomorphism* κ *in* 5.5 *can be chosen in such a way that* $\kappa(Y^\circ) = Y$ *for* $Y = B$, L, M_{13}, M_{23}, Q_3 *and* Z_3.

Proof. Let L' be the kernel of the action of M_{13}° on the three octads in β. Since $O^2(B^\circ)$ fixes the two octads in β different from α, $L^\circ \leq L' \leq B^\circ$ and so $L^\circ = O^2(L') \trianglelefteq M_{13}^\circ$. Since M_{13}° is maximal in M_1°, $M_{13}^\circ = N_{M_1^\circ}(L^\circ)$. The remaining statements in (i) now follow from 3.2 and 3.7 (i) - (iii).

Recall that we identified M_{12}° and M_{21}°. Since α is contained in 15 trios, B° has index 15 in M_{12}°. Thus $B^\circ = N_{M_{12}^\circ}(V_3(2))$ where $V_3(2)$ is 2- or 4-space in $V(2)$ containing $V_1(2)$. If $V_3(2)$ is a 4-space then B°/Q_2° is an extraspecial group of order 2^7 extended by $L_3(2)$. Since $L^\circ = O^2(L^\circ) = O^2(B^\circ)$ we conclude that L° has a chief factor isomorphic to C_2, a contradiction to (i). Thus $V_3(2)$ is a 2-space and $L^\circ = O^2(B^\circ) = O^2(C_{M_2^\circ}(V_3(2)))$. This means that L° is normal in $N_{M_2^\circ}(V_3(2))$ and since the latter is maximal in M_2° it must be equal to $M_{23}^\circ = N_{M_2^\circ}(L^\circ)$ and (ii) follows.

(iii) By 3.2, $M_{13}^\circ/L^\circ Q_1^\circ \cong Sym(3)$ and by 3.7 (iii) $Q_1^\circ/Q_1^\circ \cap Q_3^\circ$ is isomorphic to the natural $L_2(2)$-module for M_{13}°. Thus $M_{13}^\circ/L^\circ \cong Sym(4)$. In M_2° we compute that $M_{23}^\circ/L^\circ Q_2^\circ \cong Sym(3)$, M_{23}° splits over $L^\circ Q_2^\circ$ and $|Q_2^\circ/Q_2^\circ \cap Q_3^\circ| = 2$. Thus $M_{23}^\circ/L^\circ \cong Sym(3) \times C_2$.

(iv) Clearly $B^\circ = N_{M_{12}^\circ}(L^\circ) = M_{12}^\circ \cap M_{13}^\circ = M_{21}^\circ \cap M_{23}^\circ = N_{M_2^\circ}(V_1(2), V_3(2))$. Hence we compute in M_2° that $B^\circ/L^\circ \cong C_2 \times C_2$. Since $Q_1^\circ \not\leq M_{12}^\circ$, we have $B^\circ \neq Q_1^\circ L^\circ$. Since $Q_1^\circ L^\circ/L^\circ = O_2(M_{13}^\circ/L^\circ)$, B° is not normal in M_{13}°.

(v) is readily verified in M_1° (see 3.2) and M_2°.

Finally (vi) follows from (i) - (v) and 5.1. □

Let M_3° be the universal completion of the amalgam $(M_{13}^\circ, M_{23}^\circ)$ (which is the free amalgamated product of M_{13}° and M_{23}° over B°) and let $(M_1^\circ, M_2^\circ, M_3^\circ)$ be a triangle of groups where $M_i^\circ \cap M_j^\circ = M_{ij}^\circ$ for $1 \leq i < j \leq 3$. We are ready to prove the uniqueness statement for J_4-triangles.

Lemma 5.7 *Every* J_4*-triangle of groups is isomorphic to the triangle* $(M_1^\circ, M_2^\circ, M_3^\circ/N)$, *where* $N = C_{M_3^\circ}(L^\circ)$.

Proof. Let (M_1, M_2, M_3) be a J_4-triangle of groups, κ be an isomorphism of (M_1°, M_2°) onto (M_1, M_2) as in 5.5, satisfying the condition in 5.6 (vi). Since M_3 is generated by the subgroups M_{31}, M_{32} there is a mapping of $(M_1^\circ, M_2^\circ, M_3^\circ)$ onto (M_1, M_2, M_3) whose restriction to $M_1^\circ \cup M_2^\circ$ coincides with κ and whose restriction to M_3° is a homomorphism χ onto M_3. Thus the isomorphism type of (M_1, M_2, M_3) is uniquely determined by the kernel N of χ. We claim that $N = C_{M_3^\circ}(L^\circ)$. On the one hand, N and L° are normal subgroups in M_3° and $N \cap L^\circ = 1$ since the restriction of κ to L° is an isomorphism onto L, hence $N \leq C_{M_3^\circ}(L^\circ)$. On the other hand by 5.6 (v) and since $M_3/L \cong Sym(5)$ we have $C_{M_3}(L) = 1$ and hence $N \geq C_{M_3^\circ}(L^\circ)$. Thus the claim follows and implies the result. □

For the remainder of the section we identify (M_1, M_2, M_3) with $(M_1^\circ, M_2^\circ, M_3^\circ/N)$ where $N = C_{M_3^\circ}(L^\circ)$. In order to prove the existence we have to show that this is in fact a J_4-triangle of groups. For this we have to show that $M_3/L \cong Sym(5)$. By the definition M_3 is the subgroup in $Aut\ L$ generated by M_{13} and M_{23} (identified with their isomorphic images in $Aut\ L$). We need the following preliminary result.

Lemma 5.8 *Let* S *be the symmetric group* $Sym(6)$ *of degree* 6. *Let* H_1 *and* H_2 *be subgroups in* S *with* $H_1 \cong Sym(4)$, $H_2 \cong Sym(3) \times C_2$ *and* $H_1 \cap H_2 \cong C_2 \times C_2$. *Then* $\langle H_1, H_2 \rangle \cong Sym(5)$.

Proof. Let A_1 and A_2 be representatives of the conjugacy classes of $Sym(5)$ subgroups in S. Put $\Omega_i = S/A_i$, $i = 1,2$. We choose representatives k_1, k_2 and k_3 of the conjugacy classes of involutions in S so that k_i acts as a transposition on Ω_i for $i = 1,2$ and $k_3 \in S' \cong Alt(6)$. Then $C_S(k_1) \cong C_S(k_2) \cong Sym(4) \times C_2$ and $C_S(k_3) \cong D_8 \times C_2$. There are two conjugacy classes of $Sym(4)$ subgroups in S not contained in S'. We choose representatives B_1 and B_2 of these classes so that B_i is the elementwise stabilizer in S of a pair of cosets from Ω_i, or equivalently, that B_i contains a conjugate of k_i, $i = 1,2$. Applying the symmetry with respect to the full automorphism group of S, we assume that the central involution in H_2 is k_1. Then H_2 acting on Ω_1 fixes a coset, say α and $H_1 \cap H_2$ fixes two such cosets, say α and β. Since H_1 contains k_1, it is a conjugate of B_1 and hence fixes two cosets from Ω_1. Clearly these cosets must be α and β. This means that $\langle H_1, H_2\rangle$ fixes α and obviously it is the whole stabilizer of α in S, isomorphic to $Sym(5)$. □

Lemma 5.9 *Let M_3 be the subgroup of Aut L generated by M_{13} and M_{23}. Then $M_3/L \cong Sym(5)$. In particular, (M_1, M_2, M_3) is J_4-triangle of groups.*

Proof. We will use the information about M_{13} and M_{23} obtained in 5.6 without further reference. Let U be a complement to Q_3 in L and S a Sylow 7-subgroup of U. Since Z_3 is a natural module for L, M_3 induces only inner automorphism on L/Q_3 and so $M_3 = C_{M_3}(S)L$. Put $C = C_{M_3}(Q_3/Z_3) \cap C_{M_3}(Z_3)$ and $E = C_C(S)$. Then $C = Q_3E$. Let $e \in E$. Then the map

$$\xi : Q_3/Z_3 \to Z_3, \quad xZ_3 \mapsto [x, e]$$

is a $GF(2)S$-homomorphism. Since Q_3/Z_3 is the direct sum of four $L_3(2)$-modules dual to Z_3, none of the S-composition factors in Q_3/Z_3 are isomorphic to Z_3. Hence the image of ξ is the identity and E centralizes Q_3. In particular, $[U, E] \le C_L(Q_3) = Z_3$, E normalizes Z_3U and E acts faithfully on Z_3U. By 2.7 Z_3U has two classes of complements and so $|E| \le 2$ and $|C/Q_3| \le 2$.

Put $D = C_{M_3}(Z_3)$ and $\bar{M}_3 = M_3/C$. Then D centralizes L/Q_3 and $\bar{D}$ acts faithfully on Q_3/Z_3. Thus there exists a faithful four dimensional $GF(2)\bar{D}$-module R so that as D-module Q_3/Z_3 is isomorphic to the direct sum of three copies of R. Let $D_i = C_{M_{i3}}(Z_3) = M_{i3} \cap D$. Notice that $\bar{M}_3 = \bar{D} \times \bar{L}$, $\bar{M}_{i3} = \bar{D}_i \times \bar{L}$ and $M_3 = \langle M_{13}, M_{23}\rangle$. Thus $D = \langle D_1, D_2\rangle$, $\bar{D}_1 \cong Sym(4)$, $\bar{D}_2 \cong Sym(3) \times C_2$ and $\bar{D}_1 \cap \bar{D}_2 \cong C_2 \times C_2$. By 3.7 (iii) $Q_1 \cap Q_3$ is the only M_{13}-invariant subgroup between Z_3 and Q_3. Similarly, Q_2 is uniserial as $GF(2)M_{23}$-module and $Q_2 \cap Q_3$ is the only M_{23}-invariant subgroup between Z_3 and Q_3. In addition $Q_2 \cap Q_3$ has index 2 in Q_2 and $Q_1 \cap Q_2$ has index 2^4 in Q_2. Thus $Q_1 \cap Q_2 \cap Q_3 \ne Q_2 \cap Q_3$ and $Q_1 \cap Q_3 \ne Q_2 \cap Q_3$. Hence D acts irreducibly on R.

We claim that $\bar{D}$ preserves on R a non-degenerate symplectic form. Notice that Q_3 is non-abelian and D centralizes $Q_3' = Z_3$. Let $X \le Z_3$ with $|X| = 4$ and $Q_3' \not\le X$. Let Y be maximal in Q_3 with respect to the condition $[Q_3, Y] \le X$. Let W/Y be an irreducible D-submodule of Q_3/Y and let K be maximal in Q_3 with $[W, K] \le X$. Then we obtain a non-degenerate D-invariant bilinear map

$$\phi : W/Y \times Q_3/K \to Z_3/X \cong GF(2)$$

$$(wY, qK) \mapsto [w, q]X.$$

Hence by linear algebra, Q_3/K is isomorphic to the dual of W/Y and so irreducible. On the other hand all composition factors of D in Q_3/Z_3 are isomorphic to R. Hence ϕ induces a D-invariant non-degenerate bilinear map

$$\psi : R \times R \to GF(2).$$

It remains to show that we can choose ψ to be a symplectic form. Define $\psi^*(x,y) = \psi(x,y) + \psi(y,x)$. Then clearly $\psi^*(x,y)$ is symmetric and $\psi^*(x,x) = 0$. As D acts irreducibly on R, either $\psi^*(x,y) = 0$ for every $x, y \in R$ or ψ^* is non-degenerate D-invariant symplectic form. Suppose that ψ^* is trivial. In this case ψ is symmetric. In particular, $\{r \in R \mid \psi(r,r) = 0\}$ forms a D-invariant subspace of index at most 2 in R. As R is irreducible we conclude that $\psi(r,r) = 0$ for all $r \in R$ and so ψ is a symplectic form and the claim follows. Thus $\bar{D}$ is a subgroup in $Sp_4(2) \cong Sym(6)$ generated by $\bar{D_1} \cong Sym(4)$ and $\bar{D_2} \cong Sym(3) \times C_2$ with $\bar{D}_1 \cap \bar{D}_2 \cong C_2 \times C_2$. Hence $\bar{D} \cong Sym(5)$ by 5.8.

Notice that $M_3/C = DL/C$, and $|M_3/M_{13}C| = 5$. Since $M_{13}/L \cong Sym(4)$ does not contain normal subgroups of order 2, $M_{13} \cap C \leq Q_3$ and hence M_{13}/Q_3 is a complement to C/Q_3 in $M_{13}C/Q_3$. Thus by Gaschütz's theorem, M_3/Q_3 splits over C/Q_3. Since $D_i = D_i'(D_1 \cap D_2)Q_3$ and $|D_1 \cap D_2/(D_1 \cap D_2 \cap D_1')Q_3| = 2$, $|M_3/M_3'Q_3| \leq 2$. But $|Sym(5)/Sym(5)'| = 2$, hence $C/Q_3 = 1$ and the lemma is proved. □

Thus up to isomorphism there exists a unique J_4-triangle of groups.

6 Amalgams of Modules

In this section we prove a number of results to be used in the next section where a J_4-triangle of groups will be constructed inside $GL_{1333}(\mathbf{C})$. The following lemma is of crucial importance.

Lemma 6.1 *Let (M_1, M_2, M_3) be a triangle of groups, H be a group and A be a subgroup of $Aut\ H$. Suppose that for all $1 \leq i \leq 3$, there exist homomorphisms $\alpha_i : M_i \to H$ and elements $a_i \in A$ such that*

$$\alpha_1|_{M_{13}} a_2 = \alpha_3|_{M_{13}}, \alpha_2|_{M_{12}} a_3 = \alpha_1|_{M_{12}} \text{ and } \alpha_3|_{M_{23}} a_1 = \alpha_2|_{M_{23}}.$$

Put $M_{23}^* = M_{23}^{\alpha_3 a_2^{-1}}, M_{13}^* = M_{13}^{\alpha_1}, M_{12}^* = M_{12}^{\alpha_1}$ and $B^* = B^{\alpha_1}$. Then

(i) *The following two statements are equivalent:*

  (a1) *There exist $b_i \in A, 1 \leq i \leq 3$, such that*

$$\alpha_i b_i|_{M_{ij}} = \alpha_j b_j|_{M_{ij}}, \text{ for all } i \neq j.$$

  (a2) $a_2 a_1 a_3 \in C_A(M_{23}^*) C_A(M_{13}^*) C_A(M_{12}^*)$.

(ii) *$B^* \leq M_{12}^* \cap M_{13}^* \cap M_{23}^*$ and $a_2 a_1 a_3 \in C_A(B^*)$. In particular,* (a2) *and* (a1) *hold if*

$$(*) \quad C_A(B^*) = C_A(M_{23}^*) C_A(M_{13}^*) C_A(M_{12}^*).$$

(iii) *Assume that (a1) holds and that each $\alpha_i, 1 \leq i \leq 3$, is one to one. Put $M_i^* = M_i^{\alpha_i b_i}$. If $M_i^* \cap M_j^* = M_{ij}^{\alpha_i b_i}$ for all $1 \leq i < j \leq 3$, then (M_1^*, M_2^*, M_3^*) is a triangle of groups isomorphic to (M_1, M_2, M_3)*

Proof. Replacing α_2 by $\alpha_2 a_3, \alpha_3$ by $\alpha_3 a_2^{-1}$ and a_1 by $a_2 a_1 a_3$ we may assume that $a_2 = a_3 = 1$. (i) Replacing b_i by $b_i b_1^{-1}$, for all i, we see that (a1) is equivalent to :

(1) $\alpha_1|_{M_{13}} = \alpha_3 b_3|_{M_{13}}, \alpha_1|_{M_{12}} = \alpha_2 b_2|_{M_{12}}$ and $\alpha_2 b_2|_{M_{23}} = \alpha_3 b_3|_{M_{23}}$ for some $b_2, b_3 \in A$.

Since $\alpha_1|_{M_{13}} = \alpha_3|_{M_{13}}, \alpha_1|_{M_{12}} = \alpha_2|_{M_{12}}$ and $\alpha_2|_{M_{23}} = \alpha_3 a_1|_{M_{23}}$, (1) is equivalent to

(2) $b_3 \in C_A(M_{13}^*), b_2 \in C_A(M_{12}^*)$ and $a_1 b_2 b_3^{-1} \in C_A(M_{23}^*)$ for some $b_2, b_3 \in A$.

Now (2) is obviously equivalent to (a2).

(ii) Since $a_2 = a_3 = 1, \alpha_2|_B = \alpha_1|_B = \alpha_3|_B$ and so $B^* \leq M_{12}^* \cap M_{13}^* \cap M_{23}^*$. Moreover, since $\alpha_2|_{M_{23}} = \alpha_3|_{M_{23}} a_1$, we get $\alpha_1|_B = \alpha_1 a_1|_B$ and $a_1 \in C_A(B^*)$.

(iii) is obvious. □

Lemma 6.2 *Let K be a field, G be a group, H be a subgroup of finite index m in G, W be a finite dimensional KG-module and U be a non-zero finite dimensional KH-module. Suppose that each of the following statements holds:*

(i) *U is isomorphic to a KH-submodule of W;*

(ii) *$dim_K W = m \cdot dim_K U$;*

(iii) *At least one of W and $U \bigotimes_{KH} KG$ is irreducible as a KG-module.*

Then $W \cong U \bigotimes_{KH} KG$ as KG-modules.

Proof. By (i) and the universality property of induced modules, there exists a non-zero KH-homomorphism $\Phi : U \bigotimes_{KH} KG \to W$. By (iii) Φ is onto (in the first case) or one-to-one (in the second case). By (ii) $\dim_K W = m \cdot \dim_K U = \dim_K U \bigotimes_{KH} KG$ and so Φ is an isomorphism. □

Fundamental to our construction of a J_4-triangle inside $GL_{1333}(\mathbf{C})$ is the concept of "amalgam of modules". Amalgams of modules are a special case of sheaves (see for example [14]) and can be discussed in broad generality, but we will restrict ourselves to what is needed in this paper.

Definition 6.3 *Let H be a group and H_1 and H_2 subgroups of H with $H = \langle H_1, H_2 \rangle$. Put $H_0 = H_1 \cap H_2$ and let K be a field.*

(i) *An amalgam of K-modules for $H_1 \leftarrow H_0 \rightarrow H_2$ is a tuple $(W_0, W_1, W_2, \phi_1, \phi_2)$, where W_i is a KH_i-module, $0 \leq i \leq 2$ and $\phi_i : W_0 \to W_i$ is a KH_0-monomorphism, $1 \leq i \leq 2$. Such an amalgam of modules is denoted by*

$$W_1 \stackrel{\phi_1}{\leftarrow} W_0 \stackrel{\phi_2}{\rightarrow} W_2.$$

(ii) *A faithful KH-completion for $W_1 \stackrel{\phi_1}{\leftarrow} W_0 \stackrel{\phi_2}{\rightarrow} W_2$ is a tuple (W, ψ_1, ψ_2), where W is a KH-module and, for $1 \leq i \leq 2$, $\psi_i : W_i \to W$ are KH_i-monomorphisms with $\phi_1\psi_1 = \phi_2\psi_2$. Such a completion is denoted by*

$$W_1 \stackrel{\psi_1}{\rightarrow} W \stackrel{\psi_2}{\leftarrow} W_2.$$

Let W be as in part (ii) of the above definition. In abuse of notation, we will refer to W itself as a completion of the amalgam of modules.

We following elementary lemma is at the heart of the construction of J_4.

Lemma 6.4 *Let $W_1 \overset{\phi_1}{\leftarrow} W_0 \overset{\phi_2}{\rightarrow} W_2$ be an amalgam of K-modules for $H_1 \leftarrow H_0 \rightarrow H_2$. Assume that each of the following three statements holds:*

(1) *W_i is irreducible for $0 \leq i \leq 2$.*

(2) *There exists a normal elementary abelian subgroup Q of H contained in H_0 with $C_{W_0}(Q) = 0$ and a hyperplane A in Q such that $C_{W_i}(A)$ is one dimensional for $0 \leq i \leq 2$.*

(3) *Put $N_i = N_{H_i}(A)$ for $0 \leq i \leq 2$ and $N = N_H(A)$. Then $N_0 \cap N' = (N_0 \cap N_1')(N_0 \cap N_2')$ and $N = N_0 N'$.*

Then $W_1 \leftarrow W_0 \rightarrow W_2$ has a faithful and irreducible KH-completion W of dimension $|H/N|$. Moreover, the Wedderburn components for Q on W are 1-dimensional and the action of H on these Wedderburn components is isomorphic to the action on A^H.

Proof. Let $0 \leq i \leq 2$ and put $X_i = C_{W_i}(A)$. Then from (1) and (2), X_i is a Wedderburn component for Q on W_i and so $W_i \cong X_i \otimes_{KN_i} KH_i$. Since X_i is one dimensional, N_i' centralizes X_i. Let $1 \leq j \leq 2$. Clearly $X_0^{\phi_j} = X_j$ and $N_0 \cap N_j'$ centralizes X_j and X_0. By (3), $N_0 \cap N'$ centralizes X_0. Define the KN-module X by $X = X_0$ as K-vector space and $x^g = x^h$ whenever $x \in X, g \in N$ and $h \in N_0$ with $N'g = N'h$. Since $N = N_0 N'$ such h always exists and since $N' \cap N_0$ centralizes X_0 this is well defined. Put $W = X \otimes_{KN} KH$. As $W_i \cong X_i \otimes_{KN_i} KH_i$ we conclude that W is a faithful KH-completion of $W_1 \leftarrow W_0 \rightarrow W_2$. Clearly X is a Wedderburn component for Q on W, $N_H(A) = N$, W is irreducible and $\dim W = |H/N|$. □

7 A J_4-triangle in $GL_{1333}(\mathbf{C})$

In this section (M_1, M_2, M_3) is an arbitrary J_4-triangle of groups and $\mathbf{C}$ is the field of complex numbers. Our goal is to define a J_4-triangle inside $GL_{1333}(\mathbf{C})$.

The following notations will be used throughout this section. Let $1 \leq i, j \leq 3$ with $i \neq j$. If X is an $\mathbf{C}M_i$-module, then $R_{ij}(X)$ is the restriction of X to M_{ij}; if Y is an $\mathbf{C}M_{ij}$-module then $I^i(Y) = Y \otimes_{\mathbf{C}M_{ij}} \mathbf{C}M_i$ (the module for $\mathbf{C}M_i$ induced from Y) and $R_0(Y)$ is the restriction of Y to B; and if Z is an $\mathbf{C}B$-module, then $I^{ij}(Z) = Z \otimes_{\mathbf{C}B} \mathbf{C}M_{ij}$.

In what follows $X_t(i)$ will always denote an $\mathbf{C}M_i$-module, $Y_t(ij)$ an $\mathbf{C}M_{ij}$-module and Z_t an $\mathbf{C}B$-module. If G is a group, $H \leq G$, U is an $\mathbf{C}H$-module and W is an $\mathbf{C}G$-module we write $U \rightarrow W$ or $W \leftarrow U$ provided that U is isomorphic to a $\mathbf{C}H$-submodule of W. (We remark that in all cases below the $\mathbf{C}H$-submodule of W isomorphic to U will be unique).

Put $L = O^2(B)$.

Let $X_1(1)$ be an irreducible 45-dimensional $\mathbf{C}M_1/Q_1$-module given by [7] regarded as an $\mathbf{C}M_1$-module. Then clearly

(1) $X_1(1)$ is irreducible of dimension 45 and $C_{M_1}(X_1(1)) = Q_1$.

The next three statements follow from 3.12.

(2) Put $Y_1(12) = R_{12}(X_1(1))$. Then $Y_1(12)$ is irreducible of dimension 45 and $C_{M_{12}}(Y_1(12)) = Q_1 \cap M_{12}$.

(3) Restricted to M_{13}, $X_1(1)$ is the direct sum of irreducible $\mathbf{C}M_{13}$-modules $Y_1(13)$ and $Y_2(13)$, of dimension 3 and 42, respectively. Moreover, $C_{M_{13}}(Y_1(13)) = O_{2,3}(M_{13})$ and $C_{M_{13}}(Y_2(13)) = Q_1$.

(4) For $i = 1, 2$ put $Z_i = R_0(Y_i(13))$. Then Z_1 and Z_2 are irreducible of dimension 3 and 42, respectively. Moreover, restricted to B, $Y_1(12)$ is isomorphic to $Z_1 \oplus Z_2$.

Let A/Q_3 be the subgroup isomorphic to $Alt(5)$ in M_3/Q_3. By 5.1 $(A \cap M_{13})/Q_3 \cong Alt(4)$. Thus $A \cap M_{13} \leq O_{2,3}(M_{13})$ and so by (3), $A \cap M_{13}$ centralizes $Y_1(13)$. Moreover, $M_3 = AM_{13}$ and thus there exists an $\mathbf{C}M_3$-module $X_1(3)$ such that

(5) $X_1(3)$ is irreducible of dimension 3, $C_{M_3}(X_1(3))/Q_3 \cong Alt(5)$ and $X_1(3)$ is isomorphic to $Y_1(13)$ as an $\mathbf{C}M_{13}$-module.

By (4) and (5)

(6) Put $Y_1(23) = R_{23}(X_1(3))$. Then $Y_1(23)$ is irreducible of dimension 3 and restricted to B isomorphic to Z_1.

Put $X_1(2) = I^2(Y_1(23))$. There are 15 2-spaces of $V(2)$ containing $V_1(2)$ and $140 = 155 - 15$ 2-spaces of $V(2)$ which do not contain $V_1(2)$. Hence the orbits of M_{12} on M_2/M_{23} have length 15 and 140. Moreover, $15 = |M_{12}/B|$, $Y_1(12)$ is irreducible of dimension $45 = 15 \cdot 3 = |M_{12}/B| \cdot \dim Z_1$ and so by 6.2, $Y_1(12) \cong I^{12}(Z_1)$. Since $Z_1 = R_0(Y_1(23))$ the definition of $X_1(2)$ now implies

(7) $Y_1(23) \to X_1(2)$, $X_1(2)$ is 465-dimensional and is as an $\mathbf{C}M_{12}$-module isomorphic to the direct sum of $Y_1(12)$ and a 420 dimensional $\mathbf{C}M_{12}$-module $Y_2(12)$.

We remark that $X_1(2)$ and $Y_2(12)$ are irreducible. With some effort this could be proved directly at this stage, but we prefer to prove this later on (see (17) and (29)) in shorter but indirect way.

Put $X_2(3) = I^3(Y_2(13))$. By 3.12 the restriction of $Y_2(13)$ to L is an irreducible module U. Hence $X_2(3)$ restricted to L is the sum of five irreducible $\mathbf{C}L$-modules $U_1 = U$, $U_2, ..., U_5$. By (3) $C_L(U_1) = Q_1 \cap L$. Since $Z(Q_3) < Q_1 \cap L < Q_3$ and $Z(Q_3)$ is the only proper M_3-invariant subgroup properly contained in Q_3, we have $C_L(X_2(3)) \neq C_L(U_1)$. Also M_3 acts primitively on $\{U_1, ..., U_5\}$ and hence $C_L(U_i) \neq C_L(U_j)$ for $i \neq j$ and we conclude:

(8) $X_2(3)$ is the direct sum of five pairwise non-isomorphic 42-dimensional $\mathbf{C}L$-modules naturally permuted by $M_3/L \cong Sym(5)$.

By 5.1 (i) the orbits of M_{13}, M_{23} and B on M_3/M_{13} have lengths 1 and 4; 3 and 2; and 1, 2 and 2, respectively. Thus (8) and Clifford theory implies the following four statements:

(9) $X_2(3)$ is irreducible of dimension 210.

(10) Restricted to M_{13}, $X_2(3)$ is isomorphic to the direct sum of $Y_2(13)$ and $Y_3(13)$, where $Y_3(13)$ is an irreducible $\mathbf{C}M_{23}$-module of dimension 168.

(11) Restricted to M_{23}, $X_2(3)$ is the direct sum of irreducible $\mathbf{C}M_{23}$-modules $Y_2(23)$ and $Y_3(23)$ of dimension 126 and 84, respectively.

(12) Restricted to B, $Y_2(23)$ is isomorphic to the direct sum of Z_2 and an irreducible 84-dimensional $\mathbf{C}B$-module Z_3. Put $Z_4 = R_0(Y_3(23))$. Then Z_4 is an irreducible 84-dimensional $\mathbf{C}B$-module and $Z_4 \not\cong Z_3$. Moreover, restricted to B, $Y_3(13)$ is isomorphic to the direct sum of Z_3 and Z_4.

Note that by definition (see (4)), Z_2 is isomorphic to $Y_2(13)$ as an $\mathbf{C}B$-module, by (3) $Y_2(13) \to X_1(1)$, and by definition (see (2)) $Y_1(12)$ is isomorphic to $X_1(1)$ as an $\mathbf{C}M_{12}$-module. Moreover, by (7) $Y_1(12) \to X_1(2)$. Hence as $\mathbf{C}B$-modules

$$Z_2 \cong Y_2(13) \to X_1(1) \cong Y_1(12) \to X_1(2).$$

Hence by (12), $Y_2(23) \leftarrow Z_2 \to X_1(2)$. By (11) $Y_2(23)$ is irreducible of dimension $126 = 3 \cdot 42 = |M_{23}/B| \cdot \dim Z_2$ and we conclude from 6.2 that $Y_2(23) \cong I^{23}(Z_2)$. As $Z_2 \to X_1(2)$, the universal property of induced representations implies that there exists a non-zero $\mathbf{C}M_{23}$-homomorphism from $Y_2(23)$ ($\cong I^{23}(Z_2)$) to $X_1(2)$. As $Y_2(23)$ is irreducible, this homomorphism is one-to-one. So $Y_2(23) \to X_1(2)$. Then by (12) $Z_3 \to Y_2(23)$ and so $Z_3 \to X_1(2)$. Since dim $Z_3 >$ dim $Y_1(12)$ and Z_3 is irreducible, by (7) we get $Z_3 \to Y_2(12)$. We record:

(13) $Y_2(23) \to X_1(2)$ and $Z_3 \to Y_2(12)$.

By 5.1 we can pick $t \in Q_1 \setminus M_2$. Then clearly t normalizes B and M_{12}. So if T is one of B and M_{12} and W is an $\mathbf{C}T$-module, then T acts on W by $w \to w^{(g^t)}$ for all $w \in W, g \in T$ and we obtain a new $\mathbf{C}T$-module denoted by W^t. Put $\hat{B} = B\langle t\rangle$ and $\hat{M}_{12} = M_{12}\langle t\rangle$. Since $\hat{B}/L$ normalizes $B/L \cong C_2 \times C_2$ in $M_3/L \cong Sym(5)$, clearly $\hat{B}/L \cong D_8$ and has orbits of length 1 and 4 on M_3/M_{13}. In particular, $\hat{B}$ interchanges the two orbits of length 2 for B on M_3/M_{13}. Thus (8) - (12) imply

(14) $Z_4 \cong Z_3^t$ and $\hat{B}$ acts irreducibly on $Y_3(13)$.

Let $X = Y_2(12) \bigotimes_{\mathbf{C}M_{12}} \mathbf{C}\hat{M}_{12}$ and $Y = Z_3 \bigotimes_{\mathbf{C}B} \mathbf{C}\hat{B}$. By (13) $Z_3 \to Y_2(12)$ and by (12) $Z_3 \to Y_3(13)$. Hence the universal property of induced modules implies the first part of the following statement (the second part is still to be proved):

(15) $X \leftarrow Y \to Y_3(13)$ and $C_{Y_2(12)}(Q_1 \cap Q_2) = 0$.

Our nearest goal is to invoke 6.4 to find a faithful M_1-completion for the amalgam $X \leftarrow Y \to Y_3(13)$ of $\mathbf{C}$-modules for $\hat{M}_{12} \leftarrow \hat{B} \to M_{13}$. We start by proving the second part of (15) which is equivalent to the claim that $Q_1 \cap Q_2$ acts fixed-point freely on $Y_2(12)$ and immediately implies that $Q_1 = \langle Q_1 \cap M_2, t\rangle$ acts fixed-point freely on Y. For this notice that by 5.1 (vi) $Q_3 \cap Q_2$ is a hyperplane in Q_2. Furthermore, by definition (see (6)), $Y_1(23) = R_{23}(X_1(3))$ and so by (5) Q_3 and so also $Q_3 \cap Q_2$ centralize $Y_1(23)$. Since $X_1(2) = I^2(Y_1(23))$ and $N_{M_2}(Q_2 \cap Q_3) = M_{23}$ by 5.1, every hyperplane of Q_2 which centralizes a non-zero vector in $X_1(2)$ is $(Q_2 \cap Q_3)^m$ for some $m \in M_2$ and the vectors centralized by such a hyperplane form a 3-space in $X_1(2)$. By 5.1 (vi) $Q_1 \cap Q_2$ lies in 15 hyperplanes of Q_2 and by (1), (2) $Q_1 \cap Q_2$ centralizes the 45-dimensional space $Y_1(12)$ in $X_1(2)$. Since $45 = 15 \cdot 3$, this and (7) imply that $Q_1 \cap Q_2$ acts fixed-point freely on $Y_2(12)$ and the claim follows.

Recall that $Y_3(13)$ restricted to $\hat{B}$ is isomorphic to Y and Y restricted to B is the direct sum of two irreducible non-isomorphic $\mathbf{C}B$-modules Z_3 and Z_4. Hence both Y and $Y_3(13)$ are irreducible 168-dimensional modules. Let A be a hyperplane in Q_1 with $d \overset{def}{=} \dim C_Y(A) \neq 0$. Since Q_1

centralizes neither Y nor $Y_3(13)$, $C_Y(A)$ is a Wedderburn component for Q_1 on Y and $Y_3(13)$. Hence $d \cdot |A^{\hat{B}}| = 168 = d \cdot |A^{M_{13}}|$. By 3.6 there are two M_1-orbits on the set of hyperplanes in Q_1, one is indexed by the octads and the other one by complementary pairs of dodecads in $\mathcal{S}$. Suppose that A is from the former of the orbits. Recall that M_{13} is the stabilizer in M_1 of a trio β and $\hat{B}$ is the stabilizer of $\mathcal{T}$ and an octad α contained in $\mathcal{T}$. By $D_3(Mat_{24})$ there are exactly two orbits S_1 and S_2 of M_{13} on the octads with length less than or equal to 168. Here S_1 is the three octads in $\mathcal{T}$ and S_2 contains the octads which are disjoint from exactly one octad in $\mathcal{T}$. Since $\hat{B}$ acts transitively neither on S_1 nor on S_2, this is a contradiction. Thus A corresponds to a complementary pair of dodecads and by 3.10 (i), $|A^{M_{13}}| = |A^{\hat{B}}| = 168$, $d = 1$ and $|A^{\hat{M_{12}}}| = 840$. In particular, X is irreducible and $C_X(A)$ is 1-dimensional. By 3.10 (ii) we can apply 6.4 and obtain a $\mathbf{C}M_1$-module $X_2(1)$ such that

(16) $X \to X_2(1) \leftarrow Y_3(13)$, $X_2(1)$ is irreducible of dimension 1288, the Wedderburn components for Q_1 on $X_2(1)$ are 1-dimensional and the action of M_1/Q_1 on these Wedderburn components is isomorphic to the action of M_1/Q_1 on pairs of complementary dodecads.

Put $Y_3(12) = Y_2(12)^t$. Then by definition, $X \cong Y_2(12) \bigoplus Y_3(12)$ as $\mathbf{C}M_{12}$-module. Moreover, by (12) and (14) $Z_3 \not\cong Z_3^t$ and since X is irreducible we get

(17) $Y_3(12)$ and $Y_2(12)$ are irreducible of dimension 420, $Z_4 \to Y_3(12)$, $Y_3(12) \not\cong Y_2(12)$, $Y_2(12) \to X_2(1)$ and $Y_3(12) \to X_2(1)$.

(16) and 3.10 (i) imply the following two statements:

(18) Restricted to M_{13}, $X_2(1)$ is isomorphic to the direct sum of $Y_3(13), Y_4(13)$ and $Y_5(13)$, where $Y_4(13)$ and $Y_5(13)$ are irreducible $\mathbf{C}M_{13}$-modules of dimension 672 and 448, respectively.

(19) $\hat{B}$ acts irreducibly on $Y_5(13)$.

By (12) and (17) $Y_3(23) \leftarrow Z_4 \to Y_3(12)$ and we will use 6.4 to find a faithful M_2-completion for this amalgam of $\mathbf{C}$-modules for $M_{23} \leftarrow B \to M_{12}$. By (15), $C_{Y_2(12)}(Q_1 \cap Q_2) = 0$ and as t normalizes $Q_1 \cap Q_2$, $C_{Y_3(12)}(Q_1 \cap Q_2) = 0 = C_{Y_3(12)}(Q_2) = C_{Z_4}(Q_2)$. Let A be a hyperplane in Q_2 with $C_{Z_4}(A) \neq 0$.

The hyperplanes in Q_2 are described in 2.8. Suppose that A corresponds to a 2-space in $V(2)$. Then by 2.8e the orbits of M_{23} on A^{M_2} have lengths 1, 42 and 112. If A is normal in M_{23}, then since $Y_3(23)$ is irreducible, Q_2 inverts $Y_3(23)$. This is a contradiction, since by (8) and (11) Q_2 interchanges two of the three irreducible L-submodules in $Y_3(23)$. Moreover, $112 > \dim Y_3(23)$ and hence the only possibility to consider is that $|A^{M_{23}}| = 42$. In this case by 2.8 B does not act transitively on $A^{M_{23}}$, contradicting the irreducibility of Z_4.

So $A \in H(s)$. By 2.8 (iii) the orbits of M_{23} on $H(s)$ have lengths 84, 112 and 672. It follows that $|A^{M_{23}}| = 84$, $C_{Y_3(23)}(A)$ is 1-dimensional, $|A^B| = 84$, $|A^{M_{12}}| = 420$ and $C_{Y_3(12)}(A)$ is 1-dimensional. By 2.8 (iv) we can apply 6.4 and so there exists an $\mathbf{C}M_2$-module $X_2(2)$ such that

(20) $Y_3(23) \to X_2(2) \leftarrow Y_3(12)$, $X_2(2)$ is irreducible of dimension 868, the Wedderburn components for Q_2 on $X_2(2)$ are 1-dimensional and the action of M_2/Q_2 on these Wedderburn components is isomorphic to the action of M_2/Q_2 on $H(s)$.

In particular, 2.8 (iii) yields the following three statements:

(21) Restricted to M_{23}, $X_2(2)$ is isomorphic to the direct sum of $Y_3(23)$, $Y_4(23)$ and $Y_5(23)$, where $Y_4(23)$ and $Y_5(23)$ are irreducible $\mathbf{C}M_{23}$-modules of dimension 112 and 672, respectively.

(22) Restricted to M_{12}, $X_2(2)$ is isomorphic to the direct sum of $Y_3(12)$ and $Y_4(12)$, where $Y_4(12)$ is an irreducible $\mathbf{C}M_{12}$-module of dimension 448.

(23) Put $Z_5 = R_0(Y_4(23))$ and $Z_6 = R_0(Y_4(12))$. Then Z_5 and Z_6 are irreducible of dimension 112 and 448, respectively. Moreover, restricted to B; $X_2(2)$ is isomorphic to the direct sum of Z_4, Z_5, Z_6 and Z_7; $Y_5(23)$ is isomorphic to the direct sum of Z_6 and Z_7; and $Y_3(12)$ is isomorphic to the direct sum of Z_4, Z_5 and Z_7. Here Z_7 is an irreducible $\mathbf{C}B$-module of dimension 224.

Put $Z_8 = Z_5^t$ and $Z_9 = Z_7^t$. By (14), $Z_4 \cong Z_3^t$ and by definition (see after (17)) $Y_3(12) = Y_2(12)^t$. By (23) $Y_3(12) \cong Z_4 \bigoplus Z_5 \bigoplus Z_7$ as an $\mathbf{C}B$-module and since $t^2 = 1$ we conclude that

(24) Restricted to B, $Y_2(12)$ is isomorphic to the direct sum of Z_3, Z_8 and Z_9.

Put $X_3(3) = I^3(Y_4(23))$. Note that by (23) and (17) $Z_5 \to Y_3(12) \to X_2(1)$ and that by (22) and (23) $\dim Y_5(13) = 448 > 2 \cdot 112 = 2 \cdot \dim Z_5$. Thus by (19) and since $|\hat{B}/B| = 2$, $Z_5 \not\to Y_5(13)$. Further by (12) $Z_5 \not\to Y_3(13)$ and so by (18), $Z_5 \to Y_4(13)$. Since $Y_4(13)$ is irreducible of dimension $672 = 6 \cdot 112 = |M_{13}/B| \cdot \dim Z_5$, 6.2 implies $Y_4(13) \cong I^{13}(Z_5)$. Thus

(25) $Y_4(13) \to X_3(3)$.

We claim that L acts irreducibly on $Y_4(23)$. For this let A be a hyperplane in Q_2 with $C_{Y_4(23)}(A) \neq 0$. By (20) and (21), $|A^{M_{23}}| = 112$ and A corresponds to a pair (W, s), where W is a 4-space in $V(2)$ and s is a non-degenerate symplectic form on W. Let U be the 2-space in $V(2)$ normalized by M_{23}. Since $|A^{M_{23}}| = 112$ the proof of 2.8 (iii) implies $U \leq W$ and $s|_U$ is non-degenerate. Let $U = \langle u_1, u_2\rangle$ and $W = U + \langle v_1, v_2\rangle$ with $s(u_i, v_j) = 0$. Note that each hyperplane of Q_2 corresponds to a vector in $V(2) \wedge V(2)$ and, in particular, $Q_2 \cap L$ corresponds to $u_1 \wedge u_2$ and A corresponds to $u_1 \wedge u_2 + v_1 \wedge v_2$. Thus the third hyperplane of Q_2 containing $A \cap L$ corresponds to $v_1 \wedge v_2$ and A is the unique element of $H(s)$ containing $A \cap L$. Thus $N_L(A) = N_L(A \cap L)$ and $C_{Y_4(23)}(A \cap L) = C_{Y_4(23)}(A)$ is 1-dimensional. Since $N_{M_{23}}(A) = N_{M_2}(U, W, s)$ acts as $GL(U)$ on U and $C_{M_{23}}(U) = LQ_2$, $M_{23} = N_{M_{23}}(A)L$. Thus $|(A \cap L)^L| = |A^L| = 112$ and L acts irreducibly on $Y_4(23)$.

Since L is normal in M_3 we conclude from the definition of $X_3(3)$ that $X_3(3)$ is the direct sum of ten irreducible $\mathbf{C}L$-modules of dimension 112. Suppose these ten irreducibles are pairwise isomorphic. As $Y_4(13)$ has dimension $6 \cdot 112$, we conclude from (25) that $Y_4(13)$ is the direct sum of six isomorphic irreducible $\mathbf{C}L$-submodules. Let H be a hyperplane in $Q_1 \cap Q_3$ with $C_{Y_4(13)}(H) \neq 1$. Since $Q_1 \cap Q_3 \leq L$, $C_{Y_4(13)}(H)$ is at least 6-dimensional and H lies in at least six hyperplanes of Q_1 corresponding to complementary pairs of dodecads. On the other hand all three hyperplanes of Q_1 containing $Q_1 \cap Q_3$ correspond to octads. Hence H is contained in at least nine hyperplanes of Q_1, a contradiction to $|Q_1/H| = 8$.

Thus $X_3(3)$ is not the direct sum of isomorphic $\mathbf{C}L$-modules. Since M_{23} is maximal in M_2, M_3 acts primitively on the cosets of M_{23} in M_3 and we conclude

(26) $X_3(3)$ is irreducible of dimension 1120, $Y_4(23)$ is an irreducible Wedderburn component for L on $X_3(3)$ and $N_{M_3}(Y_4(23)) = M_{23}$.

Put $Y_6(23) = I^{23}(Z_8)$. By definition, $Z_5 \to Y_4(23) \to X_3(3)$ and $Z_8 = Z_5^t$. Thus 6.2 and (26) imply:

(27) $Y_6(23) \to X_3(3)$, $Y_6(23)$ is irreducible of dimension 336 and $Z_8 \ncong Z_5$.

By (24) and (7), $Z_8 \to X_1(2)$. So by (27) and 6.2, $Y_6(23) \to X_1(2)$. By (13) $Y_2(23) \to X_1(2)$ and by (7) $Y_1(23) \to X_1(2)$. Since $\dim X_1(2) = 465 = 3+126+336 = \dim Y_1(23)+\dim Y_2(23)+\dim Y_6(23)$ we conclude:

(28) Restricted to M_{23}, $X_1(2)$ is isomorphic to the direct sum of $Y_1(23), Y_2(23)$ and $Y_6(23)$.

Since $Y_1(23), Y_2(23), Y_6(23), Y_1(12)$ and $Y_2(12)$ are all irreducible, by (7) and (28) we get

(29) $X_1(2)$ is irreducible.

By (2), (3) and (4), $Y_1(12) \cong Z_1 \bigoplus Z_2$, by (24) $Y_2(12) \cong Z_3 \bigoplus Z_8 \bigoplus Z_9$ and by (12) $Y_2(23) \cong Z_2 \bigoplus Z_3$ as $\mathbf{C}B$-modules. Hence using (7) and (28) we get

(30) Restricted to B, $X_1(2)$ is isomorphic to the direct sum of Z_1, Z_2, Z_3, Z_8 and Z_9. Restricted to B, $Y_6(23)$ is isomorphic to the direct sum of Z_8 and Z_9.

In particular, $Z_9 \to Y_6(23) \to X_3(3)$ and since $Z_9 = Z_7^t$ we get $Z_7 \to X_3(3)$ and thus by (26), $Z_7 \ncong Z_9$. Moreover, by (23), $Z_7 \to Y_5(23)$ and by 6.2, $I^{23}(Z_7) \cong Y_5(23)$. Hence $Y_5(23) \to X_3(3)$. By definition of $X_3(3)$, $Y_4(23) \to X_3(3)$ and by (27), $Y_6(23) \to X_3(3)$. Therefore:

(31) $Z_7 \ncong Z_9$ and restricted to M_{23}, $X_3(3)$ is isomorphic to the direct sum of $Y_4(23), Y_5(23)$ and $Y_6(23)$.

Put $X(1) = X_1(1) \bigoplus X_2(1)$, $X(2) = X_1(2) \bigoplus X_2(2)$ and $X(3) = X_1(3) \bigoplus X_2(3) \bigoplus X_3(3)$. Then by the definition of $Y_1(23)$, (11), (21), (28) and (31):

(32) $X(2)$, $X(3)$ and $\bigoplus_{i=1}^{6} Y_i(23)$ are isomorphic as $\mathbf{C}M_{23}$-modules.

By (6), (12), (23) and (30) each of the $Y_i(23)'s$ can as $\mathbf{C}B$-module be decomposed into a direct sum of some of the $Z_j's$. Hence by (32):

(33) $X(2)$, $X(3)$ and $\bigoplus_{i=1}^{9} Z_i$ are isomorphic as $\mathbf{C}B$-modules.

Note that M_{13} has orbits of length 6 and 4 on M_3/M_{23}. Hence by (25) and (26), $X_3(3) \cong Y_4(13) \bigoplus Y_5^a(13)$ as $\mathbf{C}M_{13}$-modules, where $Y_5^a(13)$ is an irreducible 448-dimensional $\mathbf{C}M_{13}$-module. Let Z be the restriction of $Y_5^a(13)$ to $\hat{B}$. Then by (26) and (33), Z is irreducible and restricted to B isomorphic to $Z_7 \bigoplus Z_9$. Hence 6.2 implies that $Z \cong Z_9 \otimes_{\mathbf{C}B} \mathbf{C}\hat{B}$. By (24) and (17), $Z_9 \to Y_2(12) \to X_2(1)$ and hence Z is isomorphic to a $\mathbf{C}\hat{B}$-submodule of $X_2(1)$. From 3.10 (i) and (18) we conclude $Y_5(13), Y_5^a(13)$ and Z are isomorphic as $\mathbf{C}\hat{B}$-modules. Let H be a hyperplane in Q_1 with $C_Z(H) \neq 0$ and N and N_0 the normalizers of H in M_{13} and $\hat{B}$, respectively. By 3.10 (iii), $N = N_0N'$. Let D and D^a be the centralizers of H in $Y_5(13)$ and $Y_5^a(13)$, respectively. Then D and D^a are 1-dimensional and so N' centralizes D and D^a. Since D and D^a are isomorphic as $\mathbf{C}N_0$-modules, we conclude that D and D^a are isomorphic as $\mathbf{C}N$-modules. Thus $Y_5(13) \cong D \otimes_{\mathbf{C}N} \mathbf{C}M_{13} \cong D^a \otimes_{\mathbf{C}N} \mathbf{C}M_{13} \cong Y_5^a(13)$. We have proved:

(34) Restricted to M_{13}, $X_3(3)$ is isomorphic to the direct sum of $Y_4(13)$ and $Y_5(13)$. Restricted to B, $Y_5(13)$ is isomorphic to the direct sum of Z_7 and Z_9.

From (3), (5), (10), (18) and (34) we conclude that:

(35) $X(1)$, $X(3)$ and $\bigoplus_{i=1}^{5} Y_i(13)$ are isomorphic as $\mathbf{C}M_{13}$-modules.

From (33) and (35)

(36) $X(1)$, $X(2)$, $X(3)$ and $\bigoplus_{i=1}^{9} Z_i$ are isomorphic as $\mathbf{C}B$-modules.

By (17) and 3.10 (i), $X_2(1)$ restricted to M_{12} is isomorphic to the direct sum of $Y_2(12)$, $Y_3(12)$ and $Y_4^a(12)$, where $Y_4^a(12)$ is a 448-dimensional $\mathbf{C}M_{12}$-module. It follows from (36) that both $Y_4(12)$ and $Y_4^a(12)$ are isomorphic to Z_6 as $\mathbf{C}B$-modules. In particular, $Y_4^a(12)$ is irreducible. Let H be a hyperplane in Q_2 with $C_{Z_6}(H) \neq 0$. Let N and N_0 be the normalizers of H in M_{12} and B respectively. Then $N/Q_2 \cong Sym(6)$, $|N/N_0| = |M_{12}/B| = 15$, $N = N_0N'$ and as in the proof of (34) we get $Y_4(12) \cong Y_4^a(12)$. Thus

(37) $X_2(1)$ restricted to M_{12} is isomorphic to the direct sum of $Y_2(12), Y_3(12)$ and $Y_4(12)$.

Now (2), (7), (22) and (37) imply:

(38) $X(1)$, $X(2)$ and $\bigoplus_{i=1}^{4} Y_i(12)$ are isomorphic as $\mathbf{C}M_{12}$-modules.

We are now able to construct a completion of the J_4-triangle in $GL_{1333}(\mathbf{C})$. Let $\{i,j,k\} = \{1,2,3\}$. Then by (32), (35) and (38) $X(i)$ and $X(j)$ are isomorphic as $\mathbf{C}M_{ij}$-modules. Let X be a 1333-dimensional vector space over $\mathbf{C}$. Then there exist monomorphisms $\alpha_i : M_i \to GL(X)$, $1 \leq i \leq 3$, and inner automorphisms a_i of $GL(X)$ such that

$$\alpha_1 a_2|_{M_{13}} = \alpha_3|_{M_{13}}, \alpha_2 a_3|_{M_{12}} = \alpha_1|_{M_{12}} \text{ and } \alpha_3 a_1|_{M_{12}} = \alpha_2|_{M_{23}}.$$

Thus the assumptions of 6.1 are fulfilled with $H = GL(X)$ and $A = Inn(GL(X))$. Note that if Y is one of B^*, M_{23}^*, M_{13}^* and M_{12}^*, then X is the direct sum of pairwise non-isomorphic, absolutely irreducible $\mathbf{C}Y$-modules and so $C_{GL(X)}(Y)$ consists of exactly those linear transformations which act as non-zero scalars on each of the irreducible $\mathbf{C}Y$-submodules. By 6.1b (ii), $B^* \leq Y$ and so $C_{GL(X)}(Y) \leq C_{GL(X)}(B^*)$. It is now easy to verify that $C_A(B^*) = C_A(M_{23}^*)C_A(M_{13}^*)C_A(M_{12}^*)$. Thus by 6.1 (i), (a1) in 6.1 holds. Put $M_i^* = M_i^{\alpha_i b_i}$. Then by 6.1 (iii)

Theorem 7.1 *There exist subgroups M_1^*, M_2^*, M_3^* of $GL_{1333}(\mathbf{C})$ such that (M_1^*, M_2^*, M_3^*) is a J_4-triangle isomorphic to (M_1, M_2, M_3).*

8 Faithful Completions of J_4-triangles

This section is devoted to study completions of J_4-triangles. Let (M_1, M_2, M_3) be a J_4-triangle of groups with a faithful completion M. Let S be a Sylow 2-subgroup of B, $Z_4 = Z(S)$, $M_{i4} = C_{M_i}(Z_4)$ and M_4 the subgroup of M generated by M_{14}, M_{24} and M_{34}.

We will use the definitions introduced in 2.4 and 2.3 with respect to $I = \{1,2,3,4\}$ and $\Gamma_i = M/M_i$.

Let $R = C_B(Z_4)$, $Q_4^* = O^2(R)$, $Q_4 = O_2(Q_4^*)$ and $Z_3 = Z(Q_3)$. Let $V(2), V_1(2)$ and $V_3(2)$ be as in 5.1 (ii).

Lemma 8.1 (i) *$Q_4 \cong 2_+^{1+12}$, $Z(Q_4) = Z_4$, $Q_4^*/Q_4 \cong C_3$ and Q_4^*/Q_4 acts fixed-point freely on Q_4/Z_4.*

(ii) Q_4^* *is normal in* M_4 *and* $M_4/Q_4^* \cong Aut\ Mat_{22}$.

(iii) $M_{i4} = M_i \cap M_4$ *for all* $1 \leq i \leq 3$.

(iv) $M_{14} \sim 2^{1+6+4}2^6 3{\cdot}Sym(6)$, $M_{24} \sim 2^{1+6+3}(2^6(Sym(3) \times L_3(2)))$ *and* $M_{34} \sim 2^{1+2+8+4}(Sym(5) \times Sym(4))$.

Proof. First notice that Z_4 is non-trivial since it is the centre of S which is a 2-group. Since $Z_4 \leq Z(M_4)$, $M_{i4} \leq M_i \cap M_4 \leq C_{M_i}(Z_4) = M_{i4}$ and so (iii) holds. Put $Y_i = O_{2,3}(M_{i4})$ for $1 \leq i \leq 3$. Let us locate Z_4 in M_1 and determine M_{14}. Consider M_1 as the semidirect product of Q_1 and K_1 where $K_1 \cong Mat_{24}$ and Q_1 is the irreducible Todd module for K_1. One can see, for instance from 3.7 (i) that a Sylow 2-subgroup of K_1 acts faithfully on $Q_1 \cap M_2$ and hence $Z_4 \leq Q_1$. Let K_{14} be the stabilizer in K_1 of a sextet $\mathcal{H}$ and $R_{14} = O_2(K_{14})$. Since R_{14} stabilizes all octads and trios incident to $\mathcal{H}$, R_{14} is contained in S. On the other hand by 3.7 (iv) R_{14} centralizes a unique non-zero vector in Q_1 and clearly this is the vector which corresponds to $\mathcal{H}$ in the sense of 3.6 (ii). Thus $|Z_4| = 2$, $Z_4 = C_{Q_1}(R_{14})$ and $M_{14} = N_{M_1}(\mathcal{H})$. By 3.2 (iii) and 3.7 (iv) we have that $M_{14} \sim 2^{1+6+4}2^6 3 \cdot Sym(6)$. Now Y_1 normalizes all the trios and octads adjacent to $\mathcal{H}$ and so $Y_1 \leq RQ_1$. Also, RY_1/Y_1 is a Sylow 2-subgroup of M_{14}/Y_1 and so $Q_4^* = O^2(R) = O_2(O^2(Y_1)))$. One can see from 3.7 (iv) that $O_2(O^2(Y_1))) = \langle [Q_1, R_{14}], R_{14} \rangle$ is extraspecial of order 2^{13}, a Sylow 3-subgroup of Y_1 acts fixed-point freely on $O_2(O^2(Y_1)))/Z_4$ and so (i) follows.

Since Q_2 is isomorphic to $\bigwedge^2(V(2)^*)$, and S is a Sylow 2-subgroup of M_2, by 2.8 Z_4 corresponds to a 2-subspace in $V(2)^*$ and dually to a 3-space $V_4(2)$ in $V(2)$ with $V_3(2) \leq V_4(2)$. Thus $M_{24} = N_{M_2}(V_4(2))$ and from 3.2 (ii) together with standard properties of Q_2 we have $M_{24} \sim 2^{1+6+3}(2^6(Sym(3) \times L_3(2)))$. Note that $Y_2 \leq C_{M_2}(V_4(2)) \leq M_{123} = B$. Thus similar to the above, $Q_4^* = O^2(Y_2)$ and $M_{24}/Q_4^* \sim C_2 \times 2^3 L_3(2)$. Since $Q_3 \leq S$, $Z_4 \leq Z_3$ and so by 5.1 (vii), $M_{34} \sim 2^{1+2+8+4}(Sym(5) \times Sym(4))$. Thus $Y_3 \leq L \leq B$ (recall that $L = O^2(B)$). As above $Q_4^* = O^2(Y_3)$ and $M_{34}/Q_4^* \sim 2^{4+1}Sym(5)$. Since $Q_4^* = O^2(Y_i)$ for $1 \leq i \leq 3$, we conclude that Q_4^* is normal in M_{i4} for all $1 \leq i \leq 3$ and so Q_4^* is normal in M_4. In particular all statements but the last one in (ii) are proved. It is now straightforward to verify that $(M_{14}/Q_4^*, M_{24}/Q_4^*, M_{34}/Q_4^*)$ is an $Aut\ Mat_{22}$-triangle of groups (compare 4.1) and thus by 4.6 $M_4/Q_4^* \cong Aut\ Mat_{22}$, completing the proof of the lemma. □

Lemma 8.2 (i) $M_{12}Q_1$, M_{13} *and* M_{14} *are normalizers in* M_1 *of an octad, a trio and a sextet, which are pairwise adjacent.*

(ii) M_{12}, M_{23} *and* M_{24} *are the normalizers in* M_2 *of* 1*-space,* 2*-space and* 3*-space in a flag in* $V(2)$.

(iii) M_{13} *and* M_{23} *are the normalizers in* M_3 *of* 1*- and* 2*-subsets in the* 5*-set* $\Omega(3)$*, which are disjoint;* M_{34} *is the normalizer in* M_3 *of a* 1*-space in* Z_3.

(iv) M_{14}, M_{24} *and* M_{34} *are the normalizers in* M_4 *of a hexad, an octet and a pair, which form a flag.*

(v) Γ *is geometric.*

Proof. (i) - (iv) follow from 8.1 and 5.1.

(v) We will appeal to 2.5. Let $\{a_1, a_2, a_3\}$ be a flag of type $\{i, j, k\}$. If $i = 3$ and $j = 4$, then by (iii) $M_{43}M_{k3} = M_3$ and so any two vertices of type 4 and k in $\Gamma^*(M_3)$ are adjacent. Hence a_2 and a_3 are adjacent. So we may assume that $a_1 = M_1$ and $a_2 = M_2$. Since $V_1(2) = C_{V(2)}(Q_1 \cap M_2)$ is 1-dimensional, any $Q_1 \cap M_2$-invariant proper subspace of $V(2)$ contains $V_1(2)$. Now a_3 is adjacent to

M_1 and its type is different from 2. So $Q_1 \leq M_{a_3}$, $V_{a_3}(2)$ is $Q_1 \cap M_2$-invariant and $V_1(2) \leq V_{a_3}(2)$. Hence by (ii) a_1 and a_3 are adjacent in $\Gamma^*(a_2)$ and (v) follows from 2.5. $\square$

Recall that $\angle\, abc = |c^{M_{ba}}|$. We remark that for all nd-paths (a,b,c), $c^{M_{ba}}$ is completely determined by a,b, the type of c and $\angle\, abc$. Furthermore, if a and d are both adjacent to b and c, put $\angle\, a \overset{b}{c}\, d = |d^{M_{abc}}|$.

Recall also that $\overset{a_1}{n_1} - \overset{a_2}{n_2} - \ldots - \overset{a_k}{n_k}$ stands for a path $(a_1, a_2, \ldots, a_k)$ of type $n_1 - n_2 - \ldots - n_k$.

Given $\overset{a}{1} - \overset{b}{2}$, define $R_a(b)$ by $\{b, R_a(b)\} = b^{Q_a}$.

For b of type 2, let $V(b)$ be a natural 5-dimensional $GF(2)M_b/Q_b$-module such that Q_b is isomorphic to $\bigwedge^2 V(b)^*$, the exterior square of the dual of $V(b)$. For $c \in \Gamma(b)$ let $V_c(b) = C_{V(b)}(Q_c)$ and note that $V_c(b)$ is a 1-space in $V(b)$, if c is of type 1, a 2-space if c has type 3 and 3-space if c is of type 4. Similarly, put $V_c^*(b) = C_{V(b)^*}(Q_c) = \{\phi \in V(b)^* \mid \phi(V_c(b)) = 0\}$ and note that $V_c^*(b)$ is a 4-, 3- and 2-space, respectively.

For c of type 3 let $\Omega(c)$ (see 5.1) be the M_c-set of size 5 with $M_c/C_{M_c}(\Omega(c)) \cong Sym(5)$. For $a \in \Gamma_1(c)$ let $\Omega_a(c)$ be the element of $\Omega(c)$ fixed by M_{ac} and for $b \in \Gamma_2(c)$ let $\Omega_b(c)$ be the subset of size 2 in $\Omega(c)$ fixed by M_{bc}.

If a and b are adjacent, $D_a(b)$ denotes the orbit-diagram for the orbits of M_{ab} on $\Gamma(b)$. We will use these diagrams only for b of type 1 or 4, in which case they can be found in before 3.1 and 4.7.

Let a be a fixed vertex of type 1. For b, c of type 1, $b \neq c$, write $b \sim c$ if b and c are adjacent to a common vertex of type 3, and let $d(b,c)$ be the distance between b and c in $(\Gamma_1, \sim)$. Let $X_0(a) = \{a\}$ and $X_1(a) = \{b \in \Gamma_1 \mid a \sim b\}$.

Throughout this section we will often use 5.1 and 8.2 without further reference.

Lemma 8.3 *Let $a \in \Gamma_1$ and $a_i \in \Gamma_i(a), 2 \leq i \leq 4$. Then*

(i) *For $k = 3, 4$, a_2 is adjacent to a_k if and only if $Z_{a_k} \leq Q_{a_2} \cap Q_a$ and if and only if $Q_{a_k} \cap Q_a \leq M_{a_2} \cap Q_a$.*

(ii) *a_3 is adjacent to a_4 if and only if $Z_{a_4} \leq Z_{a_3}$ and if and only if $Q_{a_4} \cap Q_a \leq Q_{a_3} \cap Q_a$.*

(iii) *$Z_{a_4} \leq Q_{a_3}$ if and only if $\angle\, a_3 a a_4 \neq 1344$.*

Proof. (i) Without loss $a = M_1$ and $a_2 = M_2$. Suppose that $Z_{a_k} \leq Q_2 \cap Q_1$. Then Q_2 centralizes Z_{a_k}. Since M_{1k} is maximal in M_1, $N_{M_1}(Z_k) = M_{1k}$ and thus Q_2 fixes a_k. Now $Q_2 Q_1 = O_2(M_{12}Q_1)$ and so $O_2(M_{12}Q_1)$ fixes a_k. But this easily implies that a_k is adjacent to M_2. Suppose next that $Q_{a_k} \cap Q_1 \leq M_2 \cap Q_1$. Since $[Q_1, Q_{a_k}] \leq Q_1 \cap Q_{a_k}$ we conclude that Q_{a_k} normalizes $M_2 \cap Q_1$ and so $Q_{a_k} \leq M_{12}Q_1$. As above a_k is adjacent to M_2. Hence one direction of (i) is proved. The other direction is obvious.

(ii) is proved similar to (i).

For (iii) assume that $\angle\, a_3 a a_4 \neq 1344$. Then by $D_{a_3}(a)$ there exists a path $\overset{a_3}{3} - \overset{b}{4} - \overset{c}{3} - \overset{a_4}{4}$ in $\Gamma(a)$. Hence $Z_{a_4} \leq Z_c \leq Q_b \cap Q_1 \leq Q_{a_3} \cap Q_1$. $\square$

Lemma 8.4 (i) *Given a path $\overset{b}{3} - \overset{c}{1} - \overset{d}{3}$ with $b \neq d$. Then $(Q_b \cap Q_c)(Q_d \cap Q_c) = Q_c$ if $\Gamma_2(bcd) = \emptyset$, and $(Q_b \cap Q_c)(Q_d \cap Q_c) = Q_c \cap M_x$, if $x \in \Gamma_2(bcd)$.*

(ii) *Given a path $\overset{b}{4} - \overset{c}{1} - \overset{d}{3}$ with $\Gamma_2(bcd) = \emptyset$. Then $(Q_b \cap Q_c)(Q_d \cap Q_c) = Q_c$.*

(iii) *Given a path* $\overset{a}{1} - \overset{b}{4} - \overset{c}{3}$ *with* $\angle abc = 96$. *Then* $Q_a \cap M_c = Q_a \cap Q_b$, $M_{ac} = M_{abc}$, $(Q_a \cap Q_b)(Q_c \cap Q_b) = Q_b$, $Q_a \cap Z_c = Z_b$ *and* $|(Q_a \cap Q_c)Z_c/Z_c| = 2^4$.

Proof. (i) Since M_{bc} is maximal in M_c, $N_{M_c}(Q_b \cap Q_c) = M_{bc}$ and so $Q_b \cap Q_c \neq Q_d \cap Q_c$. Suppose that $(Q_b \cap Q_c)(Q_d \cap Q_c) \neq Q_c$. Since the three hyperplanes of Q_c containing $Q_b \cap Q_c$ are conjugated under M_{bc}, we conclude that $(Q_b \cap Q_c)(Q_d \cap Q_c) \leq Q_c \cap M_x$ for some $x \in \Gamma_2(bc)$. Thus by 8.3 $x \in \Gamma(bcd)$. Since $|Q_c/Q_b \cap Q_c| = 4$, $(Q_b \cap Q_c)(Q_d \cap Q_c) = Q_c \cap M_x$.

(ii) Similar to (i).

(iii) Since $\angle abc = 96$, it is easy to see that $Q_a \cap M_c \leq Q_b$. Since Q_a does not fix c, Q_a does not normalize $Q_b \cap Q_c$. Thus $Q_a \cap Q_b \not\leq Q_b \cap Q_c$. Since Q_b^* acts irreducibly on $Q_b/Q_b \cap Q_c$, $Q_b = (Q_a \cap Q_b)(Q_c \cap Q_b)$. In particular, $Q_a \cap Q_c$ has order $|Q_a \cap Q_b|/|Q_b/Q_c \cap Q_b| = 2^7/2^2 = 2^5$.

Suppose that $Q_a \cap Z_c \neq Z_b$. Since Q_b^* acts irreducibly on Z_c/Z_b we conclude that $Z_c \leq Q_a$. But then Q_a centralizes Z_c and $Q_a \leq M_c$, a contradiction. Hence $Q_a \cap Z_c = Z_b$, which implies that $M_{ac} = M_{abc}$ and also that $|(Q_a \cap Q_c)Z_c/Z_c| = |(Q_a \cap Q_c)/Z_b| = 2^5/2 = 2^4$. □

Lemma 8.5 *Given a path* $\overset{b}{1} - 3 - \overset{c}{4}$ *or* $\overset{b}{2} - 3 - \overset{c}{4}$. *Then b is adjacent to c.*

Proof. Let $i \in \{1, 2\}$. Then $M_{i3}M_{43} = M_3$. So if $g \in M_3$, then $M_{i3}g = M_{i3}h$ for some $h \in M_{43}$. Thus $M_i g$ is adjacent to M_4. □

Lemma 8.6 (i) *There exists a unique class of nd-paths* $\overset{a}{1} - \overset{b}{3} - \overset{c}{1}$. *Moreover, for any such path* $M_{abc}/Q_b \cong L_3(2) \times Sym(3)$, $Q_a M_{abc} = M_{ab}$, $M_{ac} = M_{abc}$ *and* $Q_a \cap Q_c = Z_b$.

(ii) *M_a acts transitively on $X_1(a)$ and $(\Gamma_1, \sim)$ is connected.*

(iii) $|X_1(a)| = 2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 23 = 15,180$.

Proof. (i) Q_a acts transitively on the four elements of $\Gamma_1(b) \setminus \{a\}$ and so all but the last two statements of (i) follow. Since M_{ab} is maximal in M_a, $N_{M_a}(Q_a \cap Q_b) = M_{ab}$. Since $Q_a \cap Q_b = Q_a \cap M_c$, $M_{ac} \leq N_{M_a}(Q_a \cap Q_b) = M_{ab}$. Since $\langle Q_a, Q_c\rangle Q_b/Q_b \cong Alt(5)$ and since $\langle Q_a, Q_c\rangle$ centralizes $Q_a \cap Q_c$, $Q_a \cap Q_c = Z_b$.

(ii) By (i) M_a is transitive on $X_1(a)$. Let $c \in X_1(a)$ and $b \in \Gamma_3(ac)$. Then $\langle M_{ab}, M_{bc}\rangle = M_b$ and so $\langle M_a, M_c\rangle = \langle M_a, M_b\rangle = M$ and hence $(\Gamma_1, \sim)$ is connected.

(iii) $|X_1(a)| = 4 \cdot |\Gamma_3(a)| = 2^2 \cdot 3 \cdot 5 \cdot 11 \cdot 23$. □

Lemma 8.7 *There exists a unique class of nd-paths* $\overset{a}{1} - \overset{b}{3} - \overset{c}{2}$. *Moreover for any such path,* $M_{abc}/Q_b \cong L_3(2) \times Sym(3)$, $M_{abc}Q_a = M_{ab}$, $M_{abc}Q_c = M_{bc}$, $Q_a \cap M_c = Q_a \cap Q_b$, $Q_c \cap M_a = Q_c \cap Q_b$, $Q_a \cap Q_c = Z_b$, $(Q_a \cap M_c)Q_c = O_2(M_{bc})$, $(Q_c \cap M_a)Q_a = O_2(M_{ab})$ *and* $M_{ac} = M_{abc}$.

Proof. Q_a acts transitively on the four elements of $\Gamma_2(b) \setminus \Gamma_2(a)$ and Q_c acts transitively on the two elements of $\Gamma_1(b) \setminus \Gamma_1(c)$. Furthermore $\langle Q_a, Q_c\rangle Q_b/Q_b \cong Sym(5)$ and since $Q_a \cap Q_c$ is centralized by $\langle Q_a, Q_c\rangle$ we conclude that $Q_a \cap Q_c = Z_b$. In particular, $M_{ac} \leq N_{M_a}(Z_b) = M_{ab}$. Moreover, by an order argument $Q_b = (Q_b \cap Q_a)(Q_b \cap Q_c)$ and the remaining statements are readily verified. □

Lemma 8.8 *Given a path* $\overset{b}{i} - \overset{c}{2} - \overset{d}{3}$ *with* $i = 3$ *or* 4. *Then* $Z_b \not\leq Q_d$ *if and only if* $V_b(c) \cap V_d(c) = 0$.

Proof. Note that $Q_c \not\leq Q_d$, $Q_c = \langle Z_x \mid x \in \Gamma_i(c)\rangle$ and M_{cd} acts transitively on $\{x \in \Gamma_i(c) \mid V_x(c) \cap V_d(c) = 0\}$. Hence it suffices to show that $Z_x \leq Q_d$ whenever $x \in \Gamma_i(c)$ with $V_x(c) \cap V_d(c) \neq 0$. Pick $y \in \Gamma_1(c)$ with $V_y(c) \leq V_x(c) \cap V_d(c)$, i.e. $y \in \Gamma_1(xcd)$. Then by 8.3 $Z_x \leq Q_y \cap Q_c \leq Q_d \cap Q_c$. □

Lemma 8.9 *There exists a unique class of nd-paths* $\overset{a}{1} - \overset{b}{3} - \overset{c}{2} - \overset{d}{3} - \overset{e}{1}$ *with* $V_b(c) \cap V_d(c) = 0$. *Moreover, for any such path* $|M_{ae}| = |M_{abcde}| = 2^{14} \cdot 3^2$, $|Z_b \cap M_e| = 4$, $Q_a \cap Q_e = 1$, $M_{ae}Q_a = N_{M_a}(Z_b \cap M_e)$, $M_{ae}Q_c = M_{bcd}$ *and* $M_{ae}Q_c/Q_c \cong Sym(4) \times Sym(4)$.

Proof. By 8.7 $M_{abc}Q_c = M_{bc}$ and so there exists a unique class of nd-paths $\overset{a}{1} - \overset{b}{3} - \overset{c}{2} - \overset{d}{3}$ with $V_b(c) \cap V_d(c) = 0$. By 8.8 $Z_b \not\leq Q_d$ and so Z_b acts transitively on the two elements of $\Gamma_1(d) \setminus \Gamma_1(c)$. Thus the uniqueness assertion in the lemma is proved and $Z_b \cap M_e = 8/2 = 4$. In particular, $|M_{abcde}| = |M_{abc}|/(\angle\, bcd \cdot \angle\, cde) = 2^{19} \cdot 3^2 \cdot 7/(2^4 \cdot 7 \cdot 2) = 2^{14} \cdot 3^2$. Moreover, $M_{abcde}Z_b = M_{abcd}$. By 8.7, $(Q_a \cap M_c)Q_c = O_2(M_{bc})$ and thus $Q_a \cap M_c$ acts transitively on the 2-spaces in $V_b(c) + V_d(c)$ which intersect $V_b(c)$ trivially (there are 16 such 2-spaces). Hence $M_{bcd} = M_{abcd}Q_c = M_{abcde}Q_c$, $|Q_a \cap M_c/Q_a \cap M_{cd}| = 2^4$, $|Q_a \cap M_{cd}| = 2^9/2^4 = 2^5$ and $|Q_a \cap M_{cde}| = 2^4$. It follows that $|M_{abcde}Q_a/Q_a| = 2^{10} \cdot 3^2$. Since $|M_{ab}/Q_a| = 2^{10} \cdot 3^2 \cdot 7$ and M_{ab} acts transitively on the 7 subgroups of order 4 in Z_b, $|N_{M_{ab}}(Z_b \cap M_e)/Q_a| = 2^{10} \cdot 3^2$. So $M_{abcde}Q_a = N_{M_{ab}}(Z_b \cap M_e)$.

Since $V_b(c) \cap V_d(c) = 0$, $\Gamma_4(bcd) = \emptyset$ and so $Z_b \cap Z_d = 1$. By 8.7, $Q_c \cap Q_e = Z_d$ and so $Z_b \cap Q_e = Z_b \cap Z_d = 1$. In particular, $Q_a \cap M_e \not\leq Q_e$ and since M_{ae} normalizes $Q_a \cap M_e$ we conclude $M_{ae}Q_e \neq M_e$. By symmetry $M_{ae}Q_a \neq M_a$. By 3.4 the only group between $N_{M_{ab}}(Z_b \cap M_e)$ and M_a is M_{ab}. Hence $M_{ae} \leq M_b$. By symmetry $M_{ae} \leq M_d$. Since $Z_bQ_d = Q_cQ_d$, $M_{bd} \leq N_{M_d}(Q_cQ_d) = M_{cd}$, $M_{bd} = M_{bcd}$ and $M_{ae} = M_{abcde}$. Hence $M_{ae}Q_c/Q_c = M_{abcde}Q_c/Q_c = M_{bcd}/Q_c \cong Sym(4) \times Sym(4)$. It remains to prove that $Q_a \cap Q_e = 1$. As $V_b(c) \cap V_d(c) = 0$, $Q_b \cap Q_d \leq Q_c$. By 8.7, $Q_a \cap M_c \leq Q_b$ and so

$$Q_a \cap Q_e \leq Q_a \cap Q_b \cap Q_d \cap Q_e \leq (Q_a \cap Q_c) \cap (Q_e \cap Q_c) = Z_b \cap Z_d = 1. \quad \square$$

Lemma 8.10 *Given an nd-path* $\overset{b}{4} - \overset{c}{3} - \overset{d}{4}$. *Then* Q_b^* *acts transitively on* $\Gamma_4(c) \setminus \{b\}$, $M_{bcd}Q_d^* = M_{cd}$, $|Q_b \cap Q_d/Z_c| = 2^4$, $Q_b \cap Q_d \leq Q_c$, $Q_c = (Q_b \cap Q_c)(Q_d \cap Q_c)$, $(Q_b \cap M_d)Q_d^*/Q_d^* = O_2(M_{dc}/Q_d^*)$ *and* $M_{bcd} = C_{M_b}(Z_d) = M_{bd}$.

Proof. Since $Z_b \leq Z_c \leq Q_b$ and Q_b^* acts fixed-point freely on Q_b/Z_b, Q_b^* acts transitively on Z_c/Z_b. Now $[Z_c, Q_b] = Z_b$ and so Q_b^* acts transitively on $Z_c \setminus Z_b$ and so also on $\Gamma_4(c) \setminus \{b\}$. In particular, $M_{bcd}Q_b^* = M_{bc}$ and by symmetry, $M_{bcd}Q_d^* = M_{cd}$. Since $|Q_b \cap Q_c/Z_c| = 2^{13}/2^{2+3} = 2^8$ and $|Q_c/Z_c| = 2^{12}$, $2^4 \leq |Q_b \cap Q_c \cap Q_d/Z_c| \leq 2^8$, where 2^4 occurs exactly then $Q_c = (Q_c \cap Q_b)(Q_c \cap Q_d)$. Since M_{bc} is a maximal subgroup of M_c, $Q_b \cap Q_c \neq Q_d \cap Q_c$. Moreover, the elements of order 5 in M_{bcd} act fixed-point freely on Q_c/Z_c and so also on $Q_b \cap Q_c \cap Q_d/Z_c$. Thus $|Q_b \cap Q_c \cap Q_d/Z_c| = 2^4$ and $(Q_b \cap Q_c)(Q_d \cap Q_c) = Q_c$. Since $Q_cQ_d = O_2(M_{cd})$ we conclude $(Q_b \cap Q_c)Q_d = O_2(M_{dc})$. Clearly, $(Q_b \cap M_d)Q_d^*/Q_d^* \leq O_2(M_{dc}/Q_d^*)$. Now since Q_b is extraspecial, $[Z_c, Q_b \cap M_d] = [Z_c, C_{Q_b}(Z_d)] = Z_b \not\leq Z_d$ and so $Q_b \cap M_d$ inverts Q_d^*/Q_d. Since $\langle Z_d^{Q_b^*} \rangle = Z_c$, $C_{M_b}(Z_d) \leq N_{M_b}(Z_c) = M_{bc}$ and so $M_{bd} \leq C_{M_b}(Z_d) \leq C_{M_{bc}}(Z_d) = M_{bcd}$. $\square$

Lemma 8.11 *There exists a unique class of nd-paths* $\overset{a}{1} - \overset{b}{4} - \overset{c}{3} - \overset{d}{4}$ *with* $\angle\, abc = 96$. *Moreover, for any such path* $M_{abcd}Q_d^* = M_{cd}$, $M_{abcd} = C_{M_a}(Z_d) = M_{ad}$, $M_{abcd}Q_a/Q_a = C_{M_a/Q_a}(Z_dQ_a/Q_a)$ *and* Z_dQ_a/Q_a *is in the class of non 2-central involutions of* M_a/Q_a.

Proof. Let $\overset{a}{1} - \overset{b}{4} - \overset{c}{3}$ be an nd-path with $\angle\, abc = 96$. By 8.10 Q_b^* acts transitively on $\Gamma_4(c) \setminus \{b\}$. In particular, the existence and uniqueness statements hold with $|M_{abcd}| = |M_{ab}|/(\angle\, abc \cdot \angle\, bcd) = 2^{21} \cdot 3^3 \cdot 5/(96 \cdot 6) = 2^{15} \cdot 3 \cdot 5$.

By 8.10 $(Q_b \cap M_d)Q_d^*/Q_d^* = O_2(M_{dc}/Q_d^*)$ and by 4.7 Q_bQ_c/Q_b acts regularly on $\{a \mid a \in \Gamma_1(b), \angle\, abc = 96\}$. Hence $M_{abcd}Q_d^* = M_{cd}$. Note that $Z_d \leq Z_c$ and $Z_d \neq Z_b$. By 8.4 (iii),

$Q_a \cap Z_c = Z_b$. Thus $Z_d \not\leq Q_a$ and d is not adjacent to a. Since M_{abcd} centralizes Z_d and has order divisible by 5, Z_dQ_a/Q_a is in the class of non 2-central involutions in $M_a/Q_a \cong Mat_{24}$ (see 3.1). Thus $|C_{M_a/Q_a}(Z_dQ_a/Q_a)| = 2^9 \cdot 3 \cdot 5$. Since $Q_a \cap M_c \leq Q_b$ by 8.4 (iii) and $C_{M_b}(Z_d) = M_{bcd}$ by 8.10 we have $C_{Q_a}(Z_d) = Q_a \cap M_{bcd} = C_{Q_a \cap Q_b}(Z_d)$ and as Q_b is extraspecial $|Q_a \cap M_{bcd}| = |Q_a \cap Q_b|/2 = 2^6$. Thus $|M_{abcd}Q_a/Q_a| = |M_{abcd}|/2^6 = 2^9 \cdot 3 \cdot 5$, $M_{abcd}Q_a/Q_a = C_{M_a/Q_a}(Z_dQ_a/Q_a)$ and $M_{abcd} = C_{M_a}(Z_d) = M_{ad}$. □

Lemma 8.12 *There exists a unique class of nd-paths* $\overset{a}{1} - \overset{b}{4} - \overset{c}{3} - \overset{d}{4} - \overset{e}{1}$ *with* $\angle\, abc = 96 = \angle\, edc$. *Moreover, for any such path* $M_{ae} = M_{abcde}$, $Q_a \cap Q_e = 1$, $M_{ae}/(Q_b \cap Q_d) \cong Sym(5)$, $|(Q_b \cap Q_d)/Z_c| = 2^4$ *and* $|M_{ae}| = 2^{10} \cdot 3 \cdot 5$.

Proof. The uniqueness statement follows from 8.11. By 8.10, $|(Q_b \cap Q_d)/Z_c| = 2^4$. By 8.4 (iii), $|(Q_a \cap Q_c)Z_c/Z_c| = 2^4$. By 8.10 with the rôles of b and d interchanged, $(Q_d \cap M_b)Q_b^*/Q_b^* = O_2(M_{bc}/Q_b^*)$ and so by 4.7 $Q_d \cap M_b$ acts transitively on the 32 elements x in $\Gamma_1(b)$ with $\angle\, xbc = 96$. Thus $M_{bc} = M_{abc}(Q_d \cap M_b)$. Suppose $(Q_a \cap Q_c)Z_c = Q_b \cap Q_d$. Then $M_{bc} = M_{abc}(Q_d \cap M_b)$ normalizes $(Q_a \cap Q_c)Z_c/Z_c$. A contradiction, since by 8.1 M_{bc} does not normalize a subgroup of order 2^4 in Q_c/Z_c. So $(Q_a \cap Q_c)Z_c \neq Q_b \cap Q_d$. Since 5 divides $|M_{abcd}|$, $Q_a \cap Q_c \cap Q_d \leq Z_c$.

Since $O_2(M_{cd}/Q_d^*) \cap M_{de}/Q_d^* = 1$, $Q_a \cap M_{cde} \leq Q_d$. Similarly $Q_a \cap M_c \leq Q_b$. By 8.10 $Q_b \cap Q_d \leq Q_c$. Hence $Q_a \cap M_{cde} \leq Q_a \cap Q_c \cap Q_d \leq Z_c$. By 8.4 (iii), $Q_a \cap Z_c \leq Z_b$ and thus $Q_a \cap M_{cde} = Z_b$. By symmetry $Q_e \cap M_{abc} = Z_d$. By 8.11 $M_{abcd} = M_{ad}$ and so $Q_e \cap M_a = Q_e \cap M_{abc} = Z_d$. Thus $M_{ae} \leq C_{M_e}(Z_d) = M_{ed}$ and $M_{ae} = M_{abcde}$. By symmetry, $Q_a \cap M_e = Z_b$ and so $Q_a \cap Q_e = 1$. Note that $M_{ae}/O_2(M_{ae}) \cong Sym(5)$ and $O_2(M_{ae}) \leq Q_b \cap Q_d$ and so $M_{ae}/(Q_b \cap Q_d) \cong Sym(5)$. □

Lemma 8.13 Γ *has five classes of nd-paths* $\overset{a}{1} - \overset{b}{3} - \overset{c}{1} - \overset{d}{3} - \overset{e}{1}$. *The classes can be described as follows:*

Class 1: $\angle\, bcd = 42$, $e \in X_1(a)$ *and* $|\Gamma_1(d) \cap X_1(a)| = 3$.

Class 2: $\angle\, bcd = 42$, $e \notin X_1(a)$ *and there exists a unique* $f \in \Gamma_4(a) \cap \Gamma_4(e)$. *For* f *we have* $\angle\, afe = 16$, $M_{ae}/Q_f^* \cong Sym(6)$ *and* $M_{ae}Q_a = M_{af}$. *n*

Class 3: $\angle\, bcd = 56$ *and* $\Gamma_1(d) \subset X_1(a)$.

Class 4: $\angle\, bcd = 1008$ *and there exists an nd-path* $\overset{a}{1} - \overset{l}{3} - \overset{j}{2} - \overset{m}{3} - \overset{e}{1}$ *with* $V_l(j) \cap V_m(j) = 0$.

Class 5: $\angle\, bcd = 2688$ *and there exists an nd-path* $\overset{a}{1} - \overset{f}{4} - \overset{g}{3} - \overset{h}{4} - \overset{e}{1}$ *with* $f, g, h \in \Gamma(c)$ *and* $\angle\, afg = 96 = \angle\, ehg$.

Proof. By 8.6 (i), $M_{ac}Q_c = M_{bc}$ and so by $D_b(c)$, $\angle\, bcd$ determines the nd-path (a, b, c, d) up to conjugacy.

Since Q_cQ_d/Q_d acts regularly on the four elements of $\Gamma_1(d) \setminus \{c\}$ we conclude that $Q_b \cap Q_c$ acts transitively on $\Gamma_1(d) \setminus \{c\}$ provided that $(Q_b \cap Q_c)(Q_d \cap Q_c) = Q_c$ and has two orbits if $(Q_b \cap Q_c)(Q_d \cap Q_c)$ is a hyperplane in Q_c. Thus 8.4 (i) implies:

(*) For $r = 56, 1008$ and 2688 there exists exactly one class of nd-paths $\overset{a}{1} - \overset{b}{3} - \overset{c}{1} - \overset{d}{3} - \overset{e}{1}$ with $\angle\, bcd = r$, and for $r = 42$ there exist at most two classes of such paths.

Assume now that $\angle\, bcd = 42$. Then by $D_b(c)$ there exist $f \in \Gamma_4(bcd)$ and $g \in \Gamma_2(bcd)$. Note that f and g are adjacent. By 8.5, f is adjacent to a and e. Replacing g by $R_c(g)$ if necessary, we may assume that g is adjacent to a (see 2.6 applied to $\Gamma(b)$). Then by $D_b(c)$, f and g are uniquely determined by (a, b, c, d) and so $M_{abcd} \leq M_{fg}$. Consider $D_a(f)$. Exactly three of the five elements in $\Gamma_1(d)$ are adjacent to g. If e is adjacent to g then $\angle\, afe = 60$ and $e \in X_1(a)$. If e is not adjacent to g, then $\angle\, afe = 16$, $M_{afe}/Q_f^* \cong Sym(6)$, $Q_a \cap M_e = Q_a \cap Q_f$, $Q_a M_{afe} = M_{af}$ and, since M_{af} is maximal in M_a, $M_{ae} \leq N_{M_a}(Q_a \cap M_e) = N_{M_a}(Q_a \cap Q_f) = M_{af}$. Thus (a, b, c, d, e) is in Class 1 if e is adjacent to g and in Class 2 if e is not adjacent to g.

Assume next that $\angle\, bcd = 56$. Then by $D_b(c)$ there exists $f \in \Gamma_4(bcd)$ and a and e are adjacent to f. Since $\Gamma_2(bcd) = \emptyset$, $D_a(f)$ shows that $\angle\, afd = 96$ and $\Gamma_1(d) \subset X_1(a)$. So (a, b, c, d, e) is in Class 3.

Assume now that $\angle\, bcd = 1008$. Then by $D_b(c)$ there exists an nd-path $\overset{b}{3} - \overset{f}{2} - \overset{g}{3} - \overset{h}{2} - \overset{d}{3}$ in $\Gamma(c)$ with $R_c(f) \neq h$. Using 2.6 and replacing f by $R_c(f)$ and h by $R_c(h)$, if necessary, we assume that a is adjacent to f and e is adjacent to h. Note $c \in \Gamma_1(fgh)$ and $f \neq R_c(h)$, which means that $\Omega_f(g) \cap \Omega_h(g) \neq \emptyset$. In $\Gamma(g)$ we find a unique nd-path $\overset{f}{2} - \overset{i}{1} - \overset{j}{2} - \overset{k}{1} - \overset{h}{2}$ with $R_i(f) = j = R_k(h)$ (indeed if $\Omega_f(g) = \{1, 2\}$ and $\Omega_h(g) = \{1, 3\}$, then $\Omega_j(g) = \{4, 5\}$, $\Omega_i(g) = 3$ and $\Omega_k(g) = 2$). In $\Gamma(f)$ there exists a unique vertex l of type 3 adjacent to a and i (indeed, l is defined by $V_l(f) = V_a(f) + V_i(f)$) and similarly there exists a unique $m \in \Gamma_3(ehk)$. Since $R_i(f) = j = R_k(h)$, l and m are both adjacent to j. Furthermore, a is adjacent to f and $j = R_i(f)$. Thus by 2.6 applied to $\Gamma(l)$, a and j are not adjacent and by symmetry e and j are not adjacent. Suppose that there exists $x \in \Gamma_4(ljm)$. By 8.5 x is adjacent to i and k and we see in $\Gamma(j)$ that x is adjacent to g and so also to f and c. Moreover, a and x are adjacent to l and thus a and x are adjacent. So x is adjacent to a and c, $V_b(f) = V_a(f) + V_c(f) \leq V_x(f)$ and b is adjacent to x. By symmetry x is adjacent to d and so $x \in \Gamma_4(bcd)$, a contradiction to $\angle\, bcd = 1008$ and $D_b(c)$. Thus no such x exists and we found an nd-path $\overset{a}{1} - \overset{l}{3} - \overset{j}{2} - \overset{m}{3} - \overset{e}{1}$ with $V_l(j) \cap V_m(j) = 0$. Hence (a, b, c, d, e) is in Class 4.

Assume finally that $\angle\, bcd = 2688$. Then by $D_b(c)$ there exists an nd-path $\overset{b}{3} - \overset{f}{4} - \overset{g}{3} - \overset{h}{4} - \overset{d}{3}$ in $\Gamma(c)$ with $\angle\, b \overset{f}{c} g = 8 = \angle\, d \overset{h}{c} g$. By 8.5, a is adjacent to f and e is adjacent to h. Since $\angle\, b \overset{f}{c} g = 8$ we conclude from $D_a(f)$ that $\angle\, afg = 96$ and by symmetry, $\angle\, ehg = 96$. Hence (a, b, c, d, e) is in Class 5. □

Lemma 8.14 *M_a has exactly three orbits $X_2(a)$, $X_3(a)$ and $X_4(a)$ on $\{e \in \Gamma_1 \mid d(a, e) = 2\}$. Moreover we can choose notation so that*

(i) $|X_2(a)| = 2^4 \cdot 7 \cdot 11 \cdot 23 = 28,336$ *and* $M_{ae} \sim 2^{1+6+6}3 \cdot Sym(6)$ *if* $e \in X_2(a)$,

(ii) $|X_3(a)| = 2^7 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 3,400,320$ *and* $M_{ae} \sim 2^{12}(Sym(3) \times Sym(3))$ *if* $e \in X_3(a)$,

(iii) $|X_4(a)| = 2^{11} \cdot 3^2 \cdot 7 \cdot 11 \cdot 23 = 32,643,072$ *and* $M_{ae} \sim 2^{3+4}Sym(5)$ *if* $e \in X_4(a)$.

Proof. 8.13, 8.9 and 8.12. □

Lemma 8.15 *Given a path* $\overset{a}{1} - \overset{b}{4} - \overset{c}{1} - \overset{d}{3} - \overset{e}{1}$ *with* $d(a, e) = 3$. *Then* $\angle\, abc = 16$ *and* $\angle\, bcd = 2880$.

Proof. Clearly $d(a, c) = 2$ and so $\angle\, abc = 16$. If d is adjacent to b, b is adjacent to e and by $D_a(b)$, $d(a, e) \leq 2$, a contradiction. Thus $\angle\, bcd \neq 15$.

Let $x \in \Gamma_3(bc)$ and suppose that $\angle\, xcd = 56$. Then by 8.13, $\Gamma_1(x) \leq X_1(e)$. On the other hand by $D_a(b)$, $X_1(a) \cap \Gamma_1(x) \neq \emptyset$. Thus $X_1(a) \cap X_1(e) \neq \emptyset$ and $d(a, e) \leq 2$, a contradiction.

Thus $\angle\, xcd \neq 56$ for all $x \in \Gamma_3(bc)$. Suppose that $\angle\, bcd = 720$. Then by $D_b(c)$ there exists an nd-path $\overset{b}{4} - \overset{x}{3} - \overset{y}{4} - \overset{d}{3}$ in $\Gamma(c)$ with $\angle\, x \overset{c}{y} d = 8$. Thus by $D_x(c)$, $\angle\, xcd = 56$, a contradiction.

Suppose that $\angle\, bcd = 180$. Then there exists an nd-path $\overset{b}{4} - \overset{x}{2} - \overset{d}{3}$ in $\Gamma(c)$. Since $\angle\, abc = 16$, replacing x by $R_c(x)$, if necessary, we may assume that $\angle\, abx = 60$ (compare $D_a(b)$), in which case there exists $y \in \Gamma_3(abx)$. So we found an nd-path $\overset{a}{1} - \overset{y}{3} - \overset{x}{2} - \overset{d}{3} - \overset{e}{1}$. Since $d(a,e) = 3$, $\Gamma_1(yxd) = \emptyset$, and so $V_y(x) \cap V_d(x) = 0$. By 8.9 and 8.13 Class 4, $d(a,e)$=2, a contradiction.

Thus $\angle\, bcd = 2880$ and the lemma is proved. □

Lemma 8.16 *There exists a unique class of nd-paths* $\overset{a}{1} - \overset{b}{4} - \overset{c}{1} - \overset{d}{3} - \overset{e}{1}$ *with* $\angle\, abc = 16$ *and* $\angle\, bcd = 2880$. *Moreover, for any such nd-path there exists an nd-path* $\overset{a}{1} - \overset{h}{3} - \overset{f}{2} - \overset{g}{4} - \overset{e}{1}$ *with* $V_h(f) \cap V_g(f) = 0$ *and* $\angle\, fge = 56$.

Proof. By 8.13 Class 2, $M_{ac}Q_c = M_{bc}$ and so there exists a unique class of nd-paths $\overset{a}{1} - \overset{b}{4} - \overset{c}{1} - \overset{d}{3}$ with $\angle\, abc = 16$ and $\angle\, bcd = 2880$. Moreover, $\Gamma_2(bcd) = \emptyset$ and so by 8.4 (ii), $(Q_b \cap Q_c)(Q_d \cap Q_c) = Q_c$. Since Q_c acts transitively on $\Gamma_1(d) \setminus \{c\}$, the uniqueness part of the lemma is proved.

By $D_b(c)$ there exists an nd-path $\overset{b}{4} - \overset{f}{2} - \overset{g}{4} - \overset{d}{3}$ in $\Gamma(c)$. Replacing f by $R_c(f)$, if necessary, we may assume that $\angle\, abf = 60$ (compare $D_a(b)$), in which case there exists $h \in \Gamma_3(abf)$. Note that d is adjacent to e and g and so e and g are adjacent by 8.5. Since $\angle\, bcd = 2880$ we see from $D_b(c)$ that $\Gamma_3(bcg) = \emptyset$ and so $V_b(f) \cap V_g(f) = V_c(f)$. Moreover, since $\angle\, abc = 16$, c and h are not adjacent. So $V_c(f) \not\leq V_h(f) \leq V_b(f)$ and $V_h(f) \cap V_g(f) = 0$.

Since $\angle\, bcd = 2880$, d is not adjacent to f. On the other hand both d and f are adjacent to c and we see from the $D_f(g)$ that $\angle\, fgd = 84$ and hence $\angle\, fge = 56$. □

Lemma 8.17 *Given a path* $\overset{a}{2} - \overset{b}{1} - \overset{c}{1}$ *with* $c = R_b(a)$. *Then*

(i) $Q_a \cap Q_c = Q_b \cap Q_c$ *and* $Q_a \cap Q_c$ *is maped to* $\bigwedge^2 V_b^*(c)$ *under the isomorphism* $Q_c \to \bigwedge^2 V^*(c)$.

(ii) $[Q_c, V(a)] = V_b(a)$.

Proof. Since $Q_a \cap Q_b$ is centralized by Q_b and $c = R_b(a)$ we have $Q_a \cap Q_b = Q_c \cap Q_b = Q_a \cap Q_b \cap Q_c$. Note that $\bigwedge^2 V_b^*(c)$ is the unique proper M_{bc}-submodule in $\bigwedge^2 V^*(c)$. Thus $Q_b \cap Q_c$ is maped to $\bigwedge^2 V_b^*(c)$. Moreover, since $M_{bc} = M_{abc}$ we also conclude that $Q_a \cap Q_c = Q_b \cap Q_c$. Thus (i) holds.

(ii) holds since $V_b(a)$ is the unique proper M_{abc}-submodule in $V(a)$. □

Lemma 8.18 (i) *There exists a unique class of nd-paths* $\overset{a}{1} - \overset{b}{3} - \overset{c}{2} - \overset{d}{4} - \overset{e}{1}$ *with* $V_b(c) \cap V_d(c) = 0$ *and* $\angle\, cde = 56$. *Moreover, for any such path* $d(a,e) = 3$, $|M_{ae}| = 2^{10} \cdot 3^2$, $M_{ae} = M_{abcde}$, $Q_a \cap M_e = 1$ *and there exists an nd-path* $\overset{e}{1} - \overset{h}{3} - \overset{g}{2} - \overset{i}{4} - \overset{a}{1}$ *with* $V_h(g) \cap V_i(g) = 0$ *and* $\angle\, gia = 56$.

(ii) *Put* $X_5(a) = e^{M_a}$. *Then* $|X_5(a)| = 2^{11} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 54,405120$.

(iii) *Given a path* $\overset{a}{1} - \overset{j}{4} - \overset{k}{1} - \overset{l}{3} - \overset{e}{1}$. *Then* $e \in \bigcup_{i=0}^{5} X_i(a)$.

Proof. (i) and (ii) By 8.7, $(Q_a \cap M_c)Q_c = O_2(M_{bc})$ and so $Q_a \cap M_c$ acts transitively on the 64 elements of $\{x \in \Gamma_4(c) \mid V_b(c) \cap V_x(c) = 0\}$. Hence there exists a unique class of nd-paths $\overset{a}{1} - \overset{b}{3} - \overset{c}{2} - \overset{d}{4}$ with $V_b(c) \cap V_d(c) = 0$. Moreover, we see in M_c/Q_c that M_{bcd}/Q_c is a complement to $O_2(M_{cd}/Q_c) = Q_dQ_c/Q_c$ in M_{cd}/Q_c. Thus $M_{cd} = M_{bcd}Q_d$. By 8.7 $M_{bc} = M_{abc}Q_c$ and so $M_{cd} = M_{bcd}Q_d = ((M_{abc}Q_c) \cap M_d)Q_d = M_{abcd}Q_cQ_d$. We claim that $Z_b(Q_c \cap Q_d) = Q_c$. Indeed, identifying Q_c with $\bigwedge^2 V(c)^*$ we have $Z_b = \bigwedge^2 V_b(c)^*$ and $Q_c \cap Q_d = V(c)^* \bigwedge V_d(c)^*$ and the claim follows from $V(c)^* = V_b(c)^* \oplus V_d(c)^*$. Since $Z_b \leq M_{abcd}$, $M_{cd} = M_{abcd}Q_cQ_d = M_{abcd}Q_d$. In particular, the uniqueness statement is proved.

Put $K = M_{abcde}$. Let x be the number of paths as in (i) starting with a. Then $x = |\Gamma_3(a)| \cdot \angle\, abc \cdot \angle\, bcd \cdot \angle\, cde = (3 \cdot 5 \cdot 11 \cdot 23) \cdot 4 \cdot 64 \cdot 56 = 2^{11} \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and so $|K| = |M_a|/x = 2^{10} \cdot 3^2$.

Since $\angle\, cde = 56$, $\angle\, edc = 240$ and so by $D_e(d)$ there exists a unique $f \in \Gamma_1(cd)$ with $\angle\, edf = 16$. Moreover, if we define $g = R_f(c)$ then there exists $h \in \Gamma_3(gde)$. Since $V_f(c) \leq V_d(c)$, $V_f(c) \not\leq V_b(c)$ and so there exists a unique element $i \in \Gamma_4(bcf)$, namely i is determined by $V_i(c) = V_f(c) + V_b(c)$. Then i is adjacent to a by 8.5 and, since $g = R_f(c)$, to g as well. Since $V_b(c) \cap V_d(c) = 0$ and $V_b(c) \leq V_i(c)$ we have $V_i(c) \cap V_d(c) = V_f(c)$. Conjugation under Q_f yields $V_i(g) \cap V_d(g) = V_f(g)$. Since $\angle\, fde = 16$, f is not adjacent to h and so since $V_h(g) \leq V_d(g)$, $V_i(g) \cap V_h(g) = 0$. Consider the nd-path $\overset{g}{2} - \overset{f}{1} - \overset{c}{2} - \overset{b}{3} - \overset{a}{1}$ in $\Gamma(i)$. By $D_g(i)$ since $R_f(g) = c$, we have $\angle\, gic = 7$ and $\angle\, gib = 28$. Now one can see from $D_c(i)$ that $\Gamma_1(ic)$ and $\Gamma_3(ic)$ are points and lines of the projective plane of order 2 with the natural incidence relation. In view of this observation and indexes in $D_g(i)$, every element from $\{x \mid x \in \Gamma_1(ib), \angle\, gix = 14\}$ is adjacent to c. Hence $\angle\, gia = 56$. In particular, the path (e, h, g, i, a) has all the properties stated in the lemma.

Since $[Q_i \cap M_h, V(g)] = [Q_i \cap M_h, V_i(g) + V_h(g)] \leq V_i(g) \cap V_h(g) = 0$ we have

$$Q_i \cap M_h \leq Q_g$$

and so $Z_b \cap M_h \leq Q_c \cap Q_g$. Identify Q_c with $\bigwedge^2 V(c)^*$. By 8.17 (i), $Q_c \cap Q_g = \bigwedge^2 V_f^*(c)$ and so

$$Z_b \cap Q_c \cap Q_g = \bigwedge^2 V_b(c)^* \cap \bigwedge^2 V_f(c)^* = \bigwedge^2 (V_b(c)^* \cap V_f(c)^*) = \bigwedge^2 V_i(c)^* = Z_i.$$

Hence $Z_b \cap M_h = Z_i$. Since $V_i(g) \cap V_h(g) = 0$, $Z_i = \bigwedge^2 V_i(g)^* \not\leq V(g)^* \bigwedge V_h(g)^* = Q_g \cap Q_h$. Moreover, $Q_g \cap M_e \leq Q_h$ and thus $Z_i \not\leq M_e$. Since f, g, h and i are uniquely determined in terms of (a, b, c, d, e), $K \leq M_{fghi}$ and, in particular, $Z_b \cap M_e = Z_b \cap K \leq Z_i \cap M_e = 1$. From $Q_a \cap M_c \leq Q_b$, $Q_b \cap M_d \leq Q_c$ and (see 8.7) $Q_a \cap Q_c = Z_b$ we conclude that $Q_a \cap K = 1$. Recall that $|K| = 2^{10} \cdot 3^2$. Since $K \leq M_{abi}$ and $|M_{abi}/Q_a| = |M_{ab}/Q_a|/7 = 2^{10} \cdot 3^2$ we have $KQ_a = M_{abi}$.

Suppose, that $Q_a \cap M_e \neq 1$ and pick a Sylow 2-subgroup R of K. Then $C_{Q_a \cap M_e}(R) \neq 1$ and since $C_{Q_a}(R) = C_{Q_a}(RQ_a) = Z_i$ we get $Z_i \leq M_e$, a contradiction.

Hence $Q_a \cap M_e = 1$. Since $Q_a \cap M_y \neq 1$ for all $y \in \Gamma_1$ with $d(a, y) \leq 2$, $d(a, e) = 3$.

Suppose $K \neq M_{ae}$. Then by 3.4, $M_{ae}Q_a$ is equal to one of M_a, M_{ab} or M_{ai}. Suppose $M_{abe} \neq K$. Then $M_{abe}Q_a = M_{ab}$. In particular, $|M_{abe}/K| = 7$ and so $M_{abe} = O^{2,3}(M_{abe})K$. From $\angle\, abc = 4$ we conclude $O^{2,3}(M_b) \leq M_c$. Now also $K \leq M_c$ and so $M_{abe} \leq M_c$. Note that $Q_c \cap M_e = Z_b(Q_c \cap Q_d) \cap M_e = Q_c \cap Q_d$ and so $M_{ce} \leq N_{M_c}(Q_c \cap Q_d) = M_{cd}$. Thus $M_{ce} = M_{cde}$ and $M_{abe} = M_{abce} = M_{abcde} = K$, a contradiction.

Thus $M_{abe} = K$ and $M_{ae}Q_a = M_{ai}$. On the other hand, by 8.7 $M_{ge} \leq M_{ghe}$ and as seen above $Q_i \cap M_h \leq Q_g$. In particular, $Q_i \cap M_e \leq Q_i \cap M_h = Q_i \cap Q_g$. Since Z_i acts transitively on the two elements in $\Gamma_1(h) \setminus \Gamma_1(gh)$, $Q_i \cap Q_g = Z_i(Q_i \cap M_e)$. Thus $M_{ae} = M_{aie} \leq N_{M_i}(Q_i \cap Q_g) = M_{ig}$, a contradiction, since 3^3 divides $|M_{ae}| = |M_{ai}/Q_a|$ but not $|M_{ig}|$.

Thus $M_{ae} = M_{abcde} = M_{abcdefghi}$ and (i) and (ii) hold.

(iii) follows from 8.16, 8.15, 8.14 and (i). □

Lemma 8.19 *Given an nd-path* $\overset{a}{1} - 4 - 1 - 4 - \overset{e}{1}$. *Then* $e \in X_l(a)$ *for some* $0 \leq l \leq 5$.

Proof. Consider first a path $\overset{a}{1} - \overset{b}{2} - \overset{c}{4} - \overset{d}{2} - \overset{e}{1}$. Then by $D_b(c)$ there exists a path $\overset{b}{2} - \overset{f}{3} - \overset{g}{1} - \overset{d}{2}$ in $\Gamma(c)$. Pick $i \in \Gamma_4(abf)$ and $h \in \Gamma_3(gde)$. Then by 8.5 i is adjacent to g and we found a path $\overset{a}{1} - \overset{i}{4} - \overset{g}{1} - \overset{h}{3} - \overset{e}{1}$. So by 8.18 (iii), $e \in X_l(a)$, for some $0 \leq l \leq 5$.

Consider now an nd-path $\overset{a}{1} - \overset{b}{4} - \overset{c}{1} - \overset{d}{4} - \overset{e}{1}$. By $D_b(c)$ there exists a path $\overset{b}{4} - \overset{f}{3} - \overset{g}{4} - \overset{h}{3} - \overset{d}{4}$ in $\Gamma(c)$. If $d(a,c) \leq 1$ or $d(c,e) \leq 1$ we are done by 8.18 (iii). So suppose that $\angle\, abc = 16 = \angle\, cde$. Then by $D_a(b)$ and $D_e(d)$ we have $\angle\, abf \neq 96 \neq \angle\, edh$ and there exist $i \in \Gamma_2(abf)$ and $j \in \Gamma_2(hde)$. Then by 8.5 both i and j are adjacent to g and we are done by the first paragraph of the proof. □

Lemma 8.20 *Let $\overset{b}{4} - \overset{c}{3} - \overset{d}{4} - \overset{e}{3} - \overset{f}{4}$ be an nd-path with $[Z_b, Z_f] \neq 1$. Then $\angle\, cde = 160$, $Q_b \cap Q_f = Z_d$, $Q_b \cap M_e$ acts transitively on the four elements in $\{\alpha \in \Gamma_4(e) \mid [Z_b, Z_\alpha] \neq 1\}$, $M_{bdf}Q_d/Q_d \cong Sym(3) \times C_2$ and $(Q_b \cap Q_d \cap M_f)Q_f^*/Q_f^* = (Q_d \cap M_f)Q_f^*/Q_f^* = O_2(M_{ef}/Q_f^*)$.*

Proof. By $D_c(d)$, if $\angle\, cde \neq 160$, then there exists $\alpha \in \Gamma_1(cde)$. Hence $Z_cZ_e \leq Q_\alpha$ and $[Z_c, Z_e] = 1$, a contradiction.

Thus $\angle\, cde = 160$ and by $D_c(d)$ there exists a unique nd-path $\overset{c}{3} - \overset{g}{2} - \overset{h}{1} - \overset{i}{2} - \overset{e}{3}$ in $\Gamma(d)$, such that $R_h(g) = i$, in which case h is not adjacent to c. Moreover, by 8.5 b is adjacent to g and f is adjacent to i. Notice that $V_b(g) \cap V_d(g) = V_c(g)$, $V_h(g) \not\leq V_c(g)$ and $V_h(g) \leq V_d(g)$. Thus $V_h(g) \not\leq V_b(g)$ and so h is not adjacent to b. By symmetry, h is not adjacent to f. Hence $Z_f \not\leq Q_h \cap Q_i = Q_h \cap Q_g$. Put $R = Q_g \cap Q_h$. We compute in Q_g:

$$R \cap Q_b = (Q_b \cap Q_g) \cap (Q_h \cap Q_g) = (V_b(g)^* \wedge V(g)^*) \cap (V_h(g)^* \wedge V_h(g)^*) = (V_b(g)^* \cap V_h(g)^*) \wedge V_h(g)^*.$$

Since $V_b(g)^* \cap V_h(g)^*$ has order two we conclude that the subgroups of order two in $R \cap Q_b$ are all of the form Z_δ for some $\delta \in \Gamma_4(gh) = \Gamma_4(ghi)$ with $V_b(g)^* \cap V_h(g)^* \leq V_\delta(g)^* \leq V_h(g)^*$. Notice that for any such δ there exists $\gamma \in \Gamma_3(bg\delta)$ and so $Z_b \leq Z_\gamma \leq Q_\delta$.

Suppose that $R \cap Q_b \cap Q_f \neq Z_d$ and pick $\delta \in \Gamma_4(ghi) \setminus \{d\}$ with $Z_\delta \leq R \cap Q_b \cap Q_f$. Then $Z_b \leq Q_\delta$ and similarly, $Z_f \leq Q_\delta$. Thus Z_b and Z_f are both contained in the elementary abelian group $Q_d \cap Q_\delta$, a contradiction.

Thus $R \cap Q_b \cap Q_f = Z_d$. Since $|R| = 2^6$ and $|R \cap Q_b| = 2^3 = |R \cap Q_f|$ we get $|R/(R \cap Q_b)(R \cap Q_f)| = 2$. Since $[R \cap Q_b, Q_b \cap Q_f] \leq Z_b \cap R = 1$ we have $[(R \cap Q_b)(R \cap Q_f), Q_b \cap Q_f] = 1$. Now R is a natural $\Omega_6^+(2)$-module for $M_{ig}/Q_iQ_g \cong L_4(2)$ and so no element of M_{ig} acts as a transvection on R. Thus $Q_b \cap Q_f \leq C_{M_{ig}}(R) = Q_iQ_g$. Now by 8.17 (ii) $[Q_iQ_g, V(i)] \leq V_h(i), [Q_f, V(i)] \leq V_f(i)$, $[Q_iQ_g \cap Q_f, V(i)] \leq V_f(i) \cap V_h(i) = 1$ and so $Q_gQ_i \cap Q_f \leq Q_i$. By symmetry, $Q_gQ_i \cap Q_b \leq Q_g$ and thus

$$Z_d \leq Q_b \cap Q_f \leq (Q_iQ_g \cap Q_b) \cap (Q_iQ_g \cap Q_f) = Q_b \cap Q_g \cap Q_i \cap Q_f = Q_b \cap R \cap Q_f = Z_d.$$

Since $|Q_b \cap Q_d| = 2^7 = |Q_f \cap Q_d|$ and $|Q_d| = 2^{13}$ we conclude

$$Q_d = (Q_b \cap Q_d)(Q_f \cap Q_d) \leq (Q_b \cap Q_d)Q_e. \qquad (*)$$

By 8.10 $(Q_b \cap M_d)Q_d^*/Q_d^* = O_2(M_{cd}/Q_d^*)$ and $M_{bcd}Q_d^* = M_{cd}$. Thus $M_{bcde}Q_d^* = M_{cde}$. Also $((Q_b \cap M_d)Q_d^* \cap M_{bcde}Q_d^*) = ((Q_b \cap M_d) \cap (M_{bcde}Q_d^*))Q_d^* = (Q_b \cap M_e)Q_d^*$ and and $(Q_b \cap M_e)Q_d^*/Q_d^* = O_2(M_{cd}/Q_d^*) \cap M_{cde}/Q_d^*$. Thus by 4.8, $M_{bde}Q_d/(Q_b \cap M_e)Q_d \cong Sym(3) \times C_2$ and $(Q_b \cap M_e)Q_d/Q_d$ has order two and inverts Q_d^*/Q_d. Since $Q_d^*Q_e/Q_e \cong Alt(4)$ and since by (*), $Q_dQ_e/Q_e = (Q_b \cap Q_d)Q_e/Q_e$ we conclude that $Q_b \cap M_e$ acts as a dihedral group of order eight on Z_e with Z_b mapping onto the centre of the D_8. Hence $Q_b \cap M_e$ acts transitively on $\{\alpha \in \Gamma_4(e) \mid [Z_b, Z_\alpha] \neq 1\}$, $[Q_b \cap M_f, Z_e] = [Q_b \cap M_f, C_{Z_e}(Z_b)Z_f] \leq Z_d$, $Q_b \cap M_f \leq Q_d$ and $M_{bdf}Q_d/Q_d \cong Sym(3) \times C_2$.

Since by (*) $(Q_b \cap M_f)Q_f^* = (Q_b \cap Q_d \cap M_f)Q_f^* = (Q_d \cap M_f)Q_f^*$, the last statement follows from 8.10. □

Lemma 8.21 *There exists a unique class of nd-paths* $\overset{a}{1} - \overset{b}{4} - \overset{c}{3} - \overset{d}{4} - \overset{e}{3} - \overset{f}{4} - \overset{g}{1}$ *such that* $\angle\, abc = 96 = \angle\, gfe$ *and* $[Z_b, Z_f] \neq 1$. *Moreover, for any such path,* $C_{M_{ag}}(Z_d) = M_{adg} = M_{abcdefg}$, $|M_{adg}| = 24$, $M_{ag} \cap Q_a = 1$, $Q_d \cap M_{adg} = Z_d$, $M_{adg}Q_d/Q_d \cong Sym(3) \times C_2$, $g \notin \bigcup_{i=0}^{5} X_i(a)$ *and there exists an nd-path* $p = \overset{a}{1} - 3 - 2 - 3 - \overset{\alpha}{1} - \overset{\beta}{3} - \overset{g}{1}$ *with* $|M_{adg}/M_{dp}| = 3$, $\alpha \in \Gamma_1(d)$ *and* $Z_d \leq Q_\alpha \cap M_{dp}$.

Proof. By 8.11 there exists a unique class of nd-paths $\overset{a}{1} - \overset{b}{4} - \overset{c}{3} - \overset{d}{4}$ with $\angle\, abc = 96$. Moreover, by the same lemma $M_{abcd}Q_d^* = M_{cd}$, $M_{abcd}Q_a/Q_a = C_{M_a/Q_a}(Z_d)$ and $M_{abcd} = C_{M_a}(Z_d) = M_{ad}$. Thus $C_{M_{ag}}(Z_d) = M_{adg} = M_{abcdefg}$.

By 8.20 $\angle\, cde = 160$, $Q_b \cap Q_f = Z_d$ and $Q_b \cap M_e$ acts transitively on $Z_e \backslash M_b$. Thus our path from b to f is unique up to conjugation and $M_{bdf}Q_d/Q_d \cong Sym(3) \times C_2$. By 4.7 $O_2(M_{ef}/Q_f^*)$ acts regularly on $\{\alpha \in \Gamma_1(f) \mid \angle\, ef\alpha = 32\}$. By 8.20 $O_2(M_{ef}/Q_f^*) = (Q_b \cap Q_d \cap M_f)Q_f^*/Q_f^* = (Q_d \cap M_f)Q_f^*/Q_f^*$, so $Q_b \cap Q_d \cap M_f$ is transitive on the same set and $Q_d \cap M_{fg} \leq Q_f$. Similarly $Q_d \cap Q_f \cap M_b$ is transitive on the set $\{\alpha \in \Gamma_1(b) \mid \angle\, cb\alpha = 32\}$ and $Q_d \cap M_{ba} \leq Q_b$. Since $(Q_b \cap Q_d) \cap (Q_f \cap Q_d) = Z_d$ the uniqueness follows and $M_{adg}Q_d/Q_d \cong Sym(3) \times C_2$. Notice that $C_{Q_b \cap M_g}(Z_d) \leq Q_b \cap M_{gf} \leq Q_b \cap Q_f = Z_d$ and so $Q_b \cap M_g = Z_d$. Thus $Q_d \cap M_{adg} = Q_b \cap Q_d \cap Q_f = Z_d$. Furthermore, $|M_{adg}| = |M_{ab}|/(96 \cdot 6 \cdot 160 \cdot 4 \cdot 32) = 24$, $Q_a \cap M_c \leq Q_b$ and so $Q_a \cap M_{dg} = Q_a \cap Q_b \cap M_g = Q_a \cap Z_d = 1$. Hence $C_{Q_a \cap M_g}(Z_d) = 1$ and $Q_a \cap M_g = 1$.

Suppose that $g \in X_i(a)$ for some $0 \leq i \leq 5$. Since $|Q_a \cap M_g| = 1$, $i = 5$. It follows from 8.18 that M_{ag} has an elementary abelian normal subgroup A of order 2^6. If Z_d is in A, then 2^6 divides $|C_{M_{ag}}(Z_d)|$ and if $Z_d \not\leq A$, 2^3 divides $|C_A(Z_d)|$ and so 2^4 divides $|C_A(Z_d)Z_d|$, and in any case we get contradiction to $|M_{adg}| = 24$. Thus $g \notin X_i(a)$ for all $0 \leq i \leq 5$.

By $D_c(d)$ there exist three nd-paths of type $3 - 2 - 1 - 3$ from c to e in $\Gamma(d)$. Moreover they are transitively permuted by M_{cde}. Since the elements of order three in Q_d^* act fixed-point freely on Q_d/Z_d and since $Z_b \leq Q_d$, $Q_d^* \cap M_b \leq Q_d$. Thus $O^3(M_{adg}) \not\leq Q_d^*$ and $|M_{cde}/Q_d^*|_3 = 3$ implies that M_{adg} acts transitively on those three nd-paths. Let (c, h, i, e) be one of them. Then h is adjacent to b and since $\angle\, abc = 96$, $D_a(b)$ yields a unique nd-path $\overset{a}{1} - \overset{l}{3} - \overset{k}{2} - \overset{j}{1} - \overset{h}{2}$ in $\Gamma(b)$ with $R_j(h) = k$. Let m be the unique vertex of type 3, adjacent to j, h and i. Since $k = R_j(h)$, m is adjacent to k. Since $\angle\, gfe = 96$ and i is adjacent to e and f, $D_g(f)$ shows that there exists a unique $n \in \Gamma_3(gfi)$. Put $p = (a, l, k, m, i, n, g)$. Then $M_{pd} = M_{adghi}$ and so $|M_{adg}/M_{pd}| = 3$. Since h and i are adjacent to d, $Z_d \leq M_{adghi} = M_{pd}$ and $Z_d \leq Q_i$. Thus the lemma holds (with $\alpha = i$ and $\beta = n$). □

Lemma 8.22 *There exists a unique class of nd-paths* $q = \overset{a}{1} - \overset{b}{3} - \overset{c}{2} - \overset{d}{3} - \overset{e}{1} - \overset{f}{3} - \overset{g}{1}$ *with* $g \notin \bigcup_{i=0}^{5} X_i(a)$. *Moreover, for any such path* $|M_q| = 24$ *and* $M_q/O_2(M_q) \cong Sym(3)$.

Proof. The existence of such a path has been established in 8.21.

Suppose there exists $x \in \Gamma_4(bcd)$. Then by 8.5 x is adjacent to a and e, a contradiction to 8.19. So $V_b(c) \cap V_d(c) = 0$ and the path $\overset{a}{1} - \overset{b}{3} - \overset{c}{2} - \overset{d}{3} - \overset{e}{1}$ is as in 8.9.

Suppose that $Z_d \cap M_a \cap Q_f \neq 1$ and pick $x \in \Gamma_4(de)$ with $Z_x \leq Z_d \cap M_a \cap Q_f$. Then x is adjacent to c and $Z_x \leq Q_c$. Since $Q_c \cap M_a \leq Q_b$ we get $Z_x \leq Q_b$. Thus by 8.8, $V_x(c) \cap V_b(c) \neq 0$. In particular, there exists $y \in \Gamma_1(bcx)$. Since $Z_x \leq Q_f$, 8.3 (iii) implies also that $\angle\, fex \neq 1344$ and so by $D_f(e)$ there exists a path $\overset{x}{4} - \overset{z}{3} - \overset{u}{4} - \overset{f}{3}$ in $\Gamma(e)$. Then g is adjacent to u. Put $v = R_y(c)$. Since

c is not adjacent to a, 2.6 applied to $\Gamma(b)$ implies that v is adjacent to a. Clearly v is also adjacent to x. By $D_z(x)$ there exists a path $\overset{v}{2} - \overset{w}{3} - \overset{t}{1} - \overset{z}{3}$ in $\Gamma(x)$. Then u is adjacent to t and g. Pick $s \in \Gamma_4(avw)$. Then s is adjacent to t and we found a path $\overset{a}{1} - \overset{s}{4} - \overset{t}{1} - \overset{u}{4} - \overset{g}{1}$. Since $g \notin \bigcup_{i=0}^{5} X_i(a)$ this must be an nd-path, a contradiction to 8.19.

Hence $Z_d \cap M_a \cap Q_f = 1$. If $\angle\, def \neq 2688$, then $D_f(e)$ and 8.3 (iii) imply $|Z_d \cap Q_f| \geq 4$. By 8.9, $|Z_d \cap M_a| = 4$. Since $|Z_d| = 8$, the latter means that $Z_d \cap Q_f \cap M_a \neq 1$. So $\angle\, def = 2688$, $|Z_d \cap Q_f| = 2$ and by 8.9, $M_{ae}Q_e = N_{M_{de}}(Z_d \cap M_a)$. Put $X = C_{M_{de}}(Z_d)$. Then by 3.5 $M_{def}X = N_{M_{de}}(Z_d \cap Q_f)$ and $M_{def}/Q_e \cong Sym(4)$. It follows that M_{def} acts transitively on $\{A \leq Z_d \mid |A| = 4, A \cap Q_f = 1\}$. Moreover, $N_{M_{def}}(A)/Q_e \cong Sym(3)$ for any such A. Thus both $N_{M_{de}}(Z_d \cap M_a)$ and M_{ae} act transitively on $\{f \in \Gamma_3(e) \mid \angle\, def = 2688, Z_d \cap M_a \cap Q_f = 1\}$ and $M_{aef}Q_e/Q_e \cong Sym(3) \cong M_{aef}/O_2(M_{aef})$. Moreover, $Q_e = (Z_d \cap M_a)(Q_e \cap Q_f)$ and so $Z_d \cap M_a$ acts transitively on $\Gamma_1(f) \setminus \{e\}$. Thus our nd-path from a to g is unique up to conjugacy and $M_q Q_e/Q_e \cong Sym(3) \cong M_q/O_2(M_q)$. Finally, $|M_{abcdefg}| = |M_{ae}|/(\frac{4}{7} \cdot 2688 \cdot 4) = 24$ and the lemma is proved. □

Let $X_6(a) = g^{M_a}$, where g is as in 8.21 or equally well as in 8.22.

Lemma 8.23 (i) *Let $g \in X_6(a)$. Then $|M_{ag}| = 2^3 \cdot 3 \cdot 11 \cdot 23$, M_{ag} has two orbits on $\Gamma_2(g)$ and acts transitively on $\{\{\rho, \rho^{Q_g}\} \mid \rho \in \Gamma_2(g)\}$.*

(ii) $|X_6(a)| = 2^{18} \cdot 3^2 \cdot 5 \cdot 7 = 82,575,360.$

Proof. Let (a, b, c, d, e, f, g) and p be as in 8.21. Then by 8.21 and 8.22, $|M_{adg}| = 24 = |M_p|$, $M_{adg}/Z_d \cong Sym(3) \times C_2$, $|M_{pd}| = 8$, $C_{M_{ag}}(Z_d) = M_{adg}$, $M_p/O_2(M_p) \cong Sym(3)$ and $M_{ag} \cap Q_g = 1$. Put $A = M_p \cap Q_\alpha$, then by 8.21, $Z_d \leq A$. Hence A is a nontrivial normal 2-subgroup of M_p and $C_{M_p}(A) \leq M_{pd}$ is a 2-group. Since $|O_2(M_p)| = 4$ we get that A is elementary abelian of order 4 and that $M_p \cong Sym(4)$. Thus M_{dp} is a dihedral group of order 8 and so $N_{M_{ag}}(M_{dp}) \leq C_{M_{ag}}(Z_d)$. In particular, M_{pd} is a Sylow 2-subgroup of M_{ag}. Moreover, there exists t in M with $(a, b, c, d, e, f, g)^t = (g, f, e, d, c, b, a)$. Notice that $t \in M_d$ and so t normalizes M_{adg}. Thus we may assume that $M_{dp}^t = M_{dp}$.

We claim that $A \cap A^t = Z_d$. Clearly $Z_d \leq A \cap A^t$. By 8.21 α is adjacent to d and since $t \in M_d$, also α^t is adjacent to d. Since $d(a, g) > 2$, $\alpha \neq \alpha^t$. We claim that $Q_\alpha \cap Q_\alpha^t \leq Q_d$. Indeed if $\Gamma_3(\alpha d \alpha^t) = \emptyset$, i.e. if $\angle\, \alpha d \alpha^t = 16$, then $Q_\alpha \cap Q_\alpha^t \leq O_2(M_{\alpha d \alpha^t}) \leq Q_d$, and if $\delta \in \Gamma_3(\alpha d \alpha^t)$ 8.6 implies $Q_\alpha \cap Q_\alpha^t = Z_\delta \leq Q_d$. By 8.21, $Q_d \cap M_{ag} = Z_d$ and so $A \cap A^t \leq M_{ag} \cap Q_\alpha \cap Q_\alpha^t \leq M_{ag} \cap Q_d \leq Z_d$.

In particular, $A \neq A^t$. Put $E = O_2(M_{adg})$. Since $M_{adg}/Z_d \cong Sym(3) \times C_2$, $|E| = 4$. Moreover, t normalizes E and so $A \neq E \neq A^t$, $E \cong C_4$ and M_{adg} is a dihedral group of order 24. Let $D = O_3(M_{adg})$ and note that $ED = C_{M_{ag}}(Z_d D)$. Since D centralizes Z_d, 3.1 implies $C_{M_g/Q_g}(D) \cong C_3 \times L_3(2)$. Now a subgroup of $L_3(2)$ with a centralizer of an involution a cyclic group of order four clearly is a cyclic group of order four and so $C_{M_{ag}}(D) = DE$. In particular, D is a Sylow 3-subgroup of M_{ag}. Note that all involutions in M_{pd} are contained in $A \cup A^t$ and so conjugate into Z_d under M_p and M_p^t, respectively. Thus M_{ag} has a unique class of involutions. Let z be any involution in M_{ag} and put $C(z) = M_{adg} \cap C_{M_{ag}}(z)$. If 3 divides $|C(z)|$, $z \in C_{M_{ag}}(D) = DE$ and $z \in Z_d$. Hence exactly one of the following holds: $z \in M_{adg}$, $|C(z)| = 2$ or $C(z) = 1$. Moreover, if $C(z) = \langle y \rangle$ for one of the twelve involutions $y \in M_{adg} \setminus Z_d$, then z is one of the ten involutions in $C_{M_{ag}}(y) \setminus M_{adg}$. Thus, if r is the number of involutions in M_{ag}, i.e. $r = |M_{ag}/M_{adg}|$, then $r = 13 + 12 \cdot 10 + 24s = 133 + 24s$ for some non-negative integer s. On the other hand, since $|M_g/Q_g| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, r divides $5 \cdot 7 \cdot 11 \cdot 23$ and we conclude that $r = 11 \cdot 23$ or $r = 5 \cdot 7 \cdot 23$. The latter case is impossible by Burnside's p-complement theorem for $p = 23$ and so $r = 11 \cdot 23$.

Hence $|M_{ag}| = 2^3 \cdot 3 \cdot 11 \cdot 23$. In particular M_{adg} and M_p are maximal $\{2,3,5,7\}$-subgroups of M_{ag}. Since both M_{fg} and $M_{\beta g}$ are $\{2,3,5,7\}$-groups we conclude that $M_{afg} = M_{adg}$ and $M_{a\beta g} = M_p$.

Since $|\Gamma_2(\alpha\beta g)| = 3$ we can choose $x \in \Gamma_2(\alpha\beta g)$ with $M_{dp} \leq M_x$. Since the non-trivial elements of odd order in M_{ag} act fixed-point freely on $\Gamma_2(g)$ we conclude that $M_{agx} = M_{dp} = M_{agy} = M_{ag\{x,y\}}$, where $y = R_g(x)$. In particular, $|M_{ag}/M_{agx}| = 759$ and the lemma is proved. □

We remark that using the list of maximal subgroups of Mat_{24} or the classification of groups with dihedral Sylow 2-subgroups it is not difficult to see that M_{ag} for $g \in X_6(a)$ is isomorphic to $L_2(23)$. But we will not need this fact.

Lemma 8.24 *Let $g \in \Gamma_1$ with $d(g,a) = 3$. Then $g \in X_5(a) \cup X_6(a)$.*

Proof. Pick $e \in \Gamma_1$ with $d(e,a) = 2$ and $e \sim g$. If $e \in X_2(a)$, then $g \in X_5(a)$ by 8.18 (iii). If $e \in X_3(a)$ then $g \in X_5(a) \cup X_6(a)$ by 8.22.

So we may assume that $e \in X_4(a)$. In particular, by 8.13 there exists an nd-path $\overset{a}{1} - \overset{b}{4} - \overset{c}{3} - \overset{d}{4} - \overset{e}{1} - \overset{f}{3} - \overset{g}{1}$ with $\angle\, abc = 96 = \angle\, edc$. By $D_d(e)$ there exists a path $\overset{d}{4} - \overset{h}{2} - \overset{i}{3} - \overset{j}{2} - \overset{f}{3}$ in $\Gamma(e)$. Note that $\angle\, cde = 32$ and so by 4.7 there exists $k \in \Gamma_3(deh)$ with $\angle\, cdk = 40$. Thus by $D_c(d)$ there exists $l \in \Gamma_1(cdk)$. Then l is adjacent to b by 8.5. By $D_a(b)$ and $\angle\, abc = 96$ there exists $m \in \Gamma_3(abl)$ and so $a \sim l$. Considering the path $\overset{k}{3} - \overset{h}{2} - \overset{i}{3} - \overset{j}{2} - \overset{f}{3}$ in $\Gamma(e)$ we see in $D_k(e)$ that $\angle\, kef \neq 2688$. Thus by 8.13 applied to (g,f,e,k,l), $l \in X_r(g)$ for some $0 \leq r \leq 3$. Thus, by the first paragraph of the proof (applied to (g,l,a)) in place of (a,e,g), $a \in X_5(g) \cup X_6(g)$ and the lemma is established. □

Lemma 8.25 *Let $z \in \Gamma_1$. Then $d(a,z) \leq 3$. In particular, $\Gamma_1 = \bigcup_{i=0}^{6} X_i(a)$.*

Proof. Suppose not. Since $(\Gamma_1, \sim)$ is connected, there exists $z \in \Gamma_1$ with $d(a,z) = 4$.

We claim that there does not exist an nd-path $\overset{a}{1} - \overset{b}{4} - \overset{c}{1} - \overset{d}{3} - \overset{e}{1} - \overset{f}{3} - \overset{z}{1}$. If such a path exists then $d(a,e) = 3$ and $d(c,z) = 2$. By 8.15 and 8.13 this means that $\angle\, abc = 16, \angle\, bcd = 2880$ and $c \in X_2(z) \cup X_3(z) \cup X_4(z)$.

Suppose first that $c \in X_2(z)$. Then there exists $\rho \in \Gamma_4(cz)$ and we found an nd-path $\overset{a}{1} - \overset{b}{4} - \overset{c}{1} - \overset{\rho}{4} - \overset{z}{1}$. Hence by 8.19, $d(a,z) \leq 3$, a contradiction.

Suppose next that $c \in X_3(z)$ and choose an nd-path $\overset{c}{1} - \overset{g}{3} - \overset{h}{2} - \overset{i}{3} - \overset{z}{1}$ with $V_g(h) \cap V_i(h) = 0$. By $D_b(c)$ there exists $j \in \Gamma_2(cg)$ with $\angle\, bcj \neq 384$. Replacing j by $R_c(j)$ if necessary, we may assume that $j = R_k(h)$ for some $k \in \Gamma_1(gh)$. (Indeed, we may assume $\Omega_c(g) = 1, \Omega_h(g) = \{1,2\}$ and $\Omega_j(g) = \{2,3\}$ or $\{4,5\}$. Replacing j by $R_c(j)$ in the first case we may assume that the second case holds and so $k \in \Gamma_1(g)$ with $\Omega_k(g) = 3$ does the trick.) Pick $l \in \Gamma_4(khi)$. Then l is adjacent to j and z. Since $c \in X_3(z)$, $\Gamma_4(cz) = \emptyset$ thus l is not adjacent to c. Let $\Lambda = \{V_x(j)/V_c(j) \mid x \in \Gamma_3(cj), V_x(j) \leq V_c(j) + V_l(j)\}$ and $\Theta = \{V_x(j)/V_c(j) \mid x \in \Gamma_3(cj), \angle\, bcx \neq 2880\}$. Then Λ is the set 1-spaces in a 3-subspace of $V(j)/V_c(j)$. Moreover, since $\angle\, bcj \neq 384$ we get from $D_b(c)$, that Θ is the set of 1-spaces in a 3- or 4-subspace of $V(j)/V_c(j)$. Thus $|\Theta \cap \Lambda| \geq 3$ and there exists $m \in \Gamma_3(cj)$ with $m \neq g$, $\angle\, bcm \neq 2880$ and $V_m(j) \leq V_c(j) + V_l(j)$. In particular, $V_m(j) \cap V_l(j) = V_n(j)$ for some $n \in \Gamma_1(mjl)$. If $n = k$, $V_g(j) = V_k(j) + V_c(j) = V_m(j)$, a contradiction to $m \neq g$. Thus $n \neq k$ and there exists a unique $o \in \Gamma_3(kjn)$. Since $V_o(j) = V_k(j) + V_n(j) \leq V_l(j)$, o is adjacent to l. Since $h = R_k(j)$, o and i are both adjacent to l and h. Hence there exists $p \in \Gamma_1(ihol)$. Put $q = R_p(h)$. Since n is adjacent to j and o, n is not adjacent to $h = R_k(j)$ by 2.6 applied to $\Gamma(o)$. Since also $p \in \Gamma(o)$, n is adjacent to $q = R_p(h)$. Similarly, since z is not adjacent to h, we see in $\Gamma(i)$ that z is adjacent to q. Hence there exists $r \in \Gamma_3(nqz)$ and we found an nd-path $\overset{a}{1} - \overset{b}{4} - \overset{c}{1} - \overset{m}{3} - \overset{n}{1} - \overset{r}{3} - \overset{z}{1}$ with $\angle\, bcm \neq 2880$, a contradiction to the second paragraph of the proof.

Suppose finally that $c \in X_4(z)$ and choose an nd-path $\overset{c}{1} - \overset{g}{4} - \overset{h}{3} - \overset{i}{4} - \overset{z}{1}$ with $\angle\, cgh = 96 = \angle\, zih$. Regard $\Gamma_3(cg)$ as 1-spaces and $\Gamma_2(cg)$ as the isotropic 2-spaces of a four dimensional symplectic space S over $GF(2)$. With the help of $D_b(c)$ we will show that there exists $y \in \Gamma_2(cg)$ such that $\angle\, bcm \neq 2880$ for all $m \in \Gamma_3(cgy)$. Indeed, if $\angle\, bcg \neq 1440$ choose y such that y is adjacent to b. If $\angle\, bcg = 1440$, there exists $u \in \Gamma_2(cg)$ such that $v \in \Gamma_3(cg)$ is perpendicular to u in S if and only if $\angle\, bcv \neq 2880$. Choose $y = u$ in this case. By 4.7 there exists $m \in \Gamma_3(cgy)$ with $\angle\, ghm = 40$. Hence by $D_h(g)$ there exists $n \in \Gamma_1(mgh)$. Then n is adjacent to h and i and since $\angle\, zih = 96$, there exists $r \in \Gamma_3(hnz)$ and again we found an nd-path $\overset{a}{1} - \overset{b}{4} - \overset{c}{1} - \overset{m}{3} - \overset{n}{1} - \overset{r}{3} - \overset{z}{1}$ with $\angle\, bcm \neq 2880$, a contradiction.

This completes the proof of the claim. Pick $g \in \Gamma_1$ with $g \sim z$ and $d(g,a) = 3$. By 8.24, $g \in X_5(a) \cup X_6(a)$. By the claim $g \notin X_5(a)$ and so $g \in X_6(a)$. Pick an nd-path $\overset{e}{1} - \overset{f}{3} - \overset{g}{1} - \overset{h}{3} - \overset{z}{1}$ with $d(e,a) = 2$. Let $j \in \Gamma_2(ghz)$. Then by 8.23 there exists $t \in M_{ag}$ such that j^t is adjacent to f and so $f^{t^{-1}}$ is adjacent to j. Replacing (e,f) by $(e^{t^{-1}}, f^{t^{-1}})$ we may assume that f is adjacent to j. Since $V_g(j) \leq V_f(j) \cap V_h(j)$, there exists $k \in \Gamma_4(fjh)$. Then k is adjacent to e and z and we get a contradiction to the claim applied with the rôles of a and z interchanged. □

Theorem 8.26 *Let M be a faithful completion of the J_4-triangle of groups (M_1, M_2, M_3) and let M_4 be as above. Then*

(i) *M_1 has seven orbits on M/M_1. The lengths of these orbits are* 1; $2^2{\cdot}3{\cdot}5{\cdot}11{\cdot}23 = 15,180$; $2^4{\cdot}7{\cdot}11{\cdot}23 = 28,336$; $2^7{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}23 = 3,400,320$; $2^{11}{\cdot}3^2{\cdot}7{\cdot}11{\cdot}23 = 32,643,072$; $2^{11}{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}23 = 54,405,120$ *and* $2^{18} \cdot 3^2 \cdot 5 \cdot 7 = 82,575,360$.

(ii) $|M| = 2^{21} \cdot 3^3 \cdot 5 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43 = 86,775,571,046,077,562,880$.

(iii) *M is simple.*

(iv) *Let $1 \neq z \in Z(M_4)$. Then $C_M(z) = M_4$.*

Proof. (i) follows from 8.25, 8.6, 8.14, 8.18 and 8.23 and (ii) follows from (i).

(iii) Let N be a normal subgroup of M. If $N \cap M_i \neq 1$ for some $1 \leq i \leq 3$ we conclude that $Z_4 \leq Z_i \leq N$ (notice that $Z_1 = Q_1$ and $Z_2 = Q_2$). Hence $Q_2 = \langle Z_4^{M_2} \rangle \leq N$, $M_1 = \langle Q_2^{M_1} \rangle \leq N$ and for $j = 2,3$, $M_j = \langle M_{1j}^{M_j} \rangle \leq N$. Thus $M = N$. So suppose that $M_i \cap N = 1$ for all i. Let $1 \leq i < j \leq 3$. Then M_{ij} is a maximal subgroup of M_j and M_i is not isomorphic to an overgroup of M_{ij} in M_j. Thus $M_iN \cap M_jN = M_{ij}N$ and so M/N is a faithful completion of a J_4-triangle. By (ii) $|M| = |M/N|$ and so $N = 1$.

(iv) Let $t \in C_M(z)$ and put $a = M_1$, $b = M_1t = a^t$ and $c = M_4$. Then $z \in Q_a \cap Q_b$ and so by 8.21, 8.18, 8.12 and 8.9, $b \in X_i(a)$ for some $0 \leq i \leq 2$. It is now easy to check that $\{d \in \Gamma_4(b) \mid Z_d \leq Q_a \cap Q_b\} = \Gamma_4(a,b)$. Thus $c, c^t \in \Gamma_4(a)$ and $Z_c = Z_c^t$ implies $c = c^t$ and $t \in M_4$.□

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