Supplemental Material for Section 3.6: The Formula for $\frac{d}{dx}x^r$ When r Is a Rational Number

One of the major applications of the Chain Rule is to show that for any rational number r (Recall that a rational number is one that can be written as the quotient of two integers.) and for any $x > 0$ the formula for the derivative of the function x^r is $\dfrac{d}{dx}x^r = rx^{r-1}$. For example $\dfrac{d}{dx}x^{\frac{7}{3}} = \frac{7}{3}x^{\frac{4}{3}}$. We begin with a special case $r = \frac{1}{n}$ where $n \in \mathbb{N}$. To employ the Chair Rule in this special case, let $g(x) = x^{\frac{1}{n}}$ and let $f(x) = x^n$. Then clearly $f \circ g(x) = f(g(x)) = \left(x^{\frac{1}{n}}\right)^n = x$. The derivative of the right hand side is clearly 1. We would like to use the Chain Rule to compute the derivative of the right hand side, but first, according to the hypothesis of the Chair Rule, we must show that g is differentiable. From what we've learned about derivatives, we need to show that the graph of $y = g(x)$ has a tangent line at each of its points. This fact will be established using geometry. Begin with a point P on the graph of $y = g(x) = x^{\frac{1}{n}}$ other that $(0, 0)$. Reflect the graph in the line $y = x$. The resulting graph is that of $y = x^n$, which we know is differentiable. Consequently there is a line tangent to $y = x^n$ and the reflected version of P. Now reflect that tangent line in the line $y = x$ and clearly it becomes a line tangent to $y = g(x)$ at P. Consequently g is differentiable and we may apply the chain Rule. This entire geometric process is demonstrated with the square root function and the point $(2, \sqrt{2})$ in figure 1 below. This geometric argument establishes Proposition 1 .

Proposition 1. *The function $f(x) = x^{\frac{1}{n}}$ is differentiable at each x in its domain except for $x = 0$.*

Now that the differentiability of g has been established, we may apply the Chain Rule to compute the derivative of $f \circ g$. For any x in the domain of g by the Chain Rule we have,

$$\frac{d}{dx}g^n(x) = ng^{n-1}(x)g'(x).$$

But because $g^n(x) = x$ as was pointed out above, the derivative of the composition can also be computed without the Chain Rule. We know $\dfrac{d}{dx}g^n(x) =$

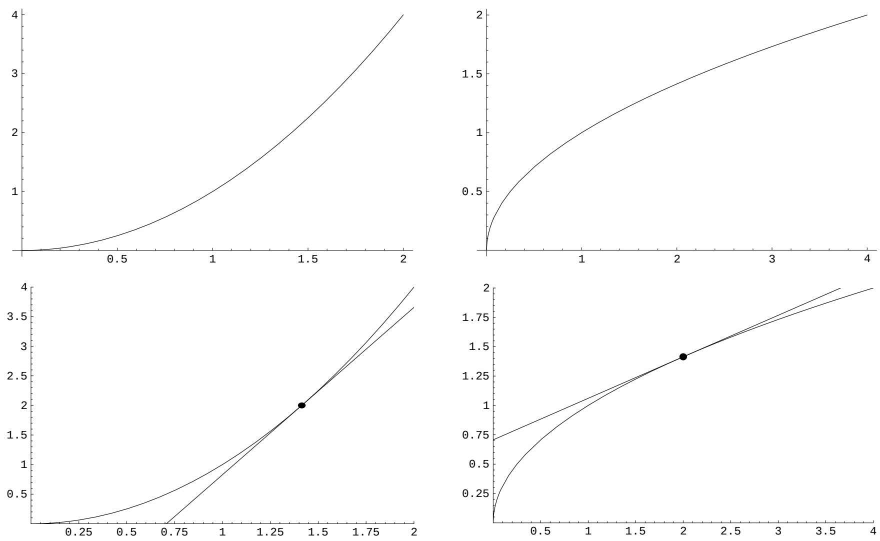

Figure 1:

$\dfrac{d}{dx}x = 1$. Equating the two expressions for $\dfrac{d}{dx}g^n(x)$ we get that

$$1 = ng^{n-1}(x)g'(x) = n(x^{\frac{1}{n}})^{n-1}g'(x) = nx^{1-\frac{1}{n}}g'(x).$$

Solving this equation for $g'(x)$ yields

$$g'(x) = \frac{1}{n}x^{\frac{1}{n}-1} \text{ or } \frac{d}{dx}x^{\frac{1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

Note that the exponent $\frac{1}{n} - 1$, is negative (except for $n = 1$) indicating why x can't be 0. Also Note that this formula is the same as $\dfrac{d}{dx}x^n = nx^{n-1}$ with n replaced by $\frac{1}{n}$. We now proceed to show that the same formula holds if n is replaced by any rational number. For any rational number r there are two integers, m and n with n positive such that $r = \frac{m}{n}$. We will now use the Chain Rule to prove the following assertion.

Theorem 1. *Let r be a rational number and let $x > 0$. Then* $\dfrac{d}{dx}x^r = rx^{r-1}$.

Proof. First note that $x^r = x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$, a composition with outer function $f(x) = x^m$ and inner function, $g(x) = x^{\frac{1}{n}}$. We know the derivative of the outer function, $f'(x) = mx^{m-1}$, and by Proposition 1, we also know the derivative of the inner function, $g'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$. Thus by the Chain Rule

$$\frac{d}{dx}x^r = \frac{d}{dx}\left(x^{\frac{1}{n}}\right)^m = m\left(x^{\frac{1}{n}}\right)^{m-1}\frac{1}{n}x^{\frac{1}{n}-1} = \frac{m}{n}x^{\frac{m}{n}-\frac{1}{n}+\frac{1}{n}-1} = rx^{r-1}.$$