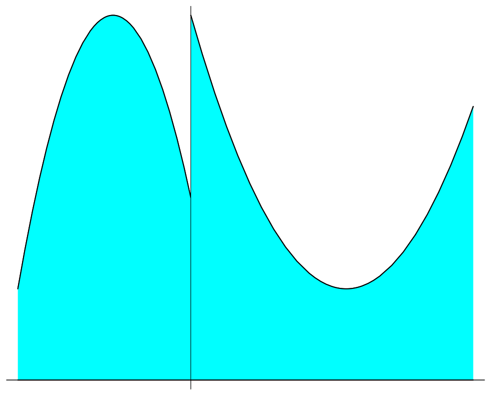
Supplemental Material for Section 5.3:
The Definite Integral

We now turn our attention to the second aspect of Integration Theory; the Definite Integral. The motivation for this part of the theory is the area problem. However the applications of the definite integral are numerous and go well beyond finding areas as will be seen from subsequent study. The area problem can be stated as follows.



Let f be a function defined on a closed interval $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. Let R be the region indicated in the picture to the left. Formally R is defined by

$$R = \{(x, y) : x \in [a, b] \text{ and } 0 \leq y \leq f(x)\}$$

and is called the region under the graph of f. The area problem is to define the area of R and to discover an easy method to compute it in a large number of cases. We begin with several examples.

Example 1. Let p be any positive real number and let $[a, b]$ be any interval. Set $f(x) = p$ for all $x \in [a, b]$. Then the region under the graph of f is a rectangle of length $b - a$ and of height p. Consequently the area of the region R is $p(b - a)$.

As simple as this example is, it is of fundamental importance in dealing with general area.

Example 2. Let $f(x) = 2x$ for $x \in [0, 2]$. Then the region under the graph of f is a triangle with base equal to the length of the interval $[0, 2]$ and height $f(2) = 4$. So the area of the region R in this case is $\frac{1}{2}(2)(4) = 4$.

Example 3. Let $f(x) = -(\frac{1}{2})x + 4$ for $x \in [1, 4]$. In this case the region under the graph of f can be seen to be composed of a rectangle surmounted by a triangle. So the area can be computed as $\frac{33}{4}$. The details are left to the student.

Example 4. Let $f(x) = |x|$ for $x \in [-2, 4]$. Here the the region under the graph of f is composed of two triangles. It is easy to see that the area of the region is $2 + 8 = 10$.

Example 5. Let $f(x) = 1 + |x|$ for $x \in [-3, 2]$. Here the the region under the graph of f is composed of two triangles and a rectangle. Clearly the area of the region is $\frac{23}{2}$.

Example 6. Let $f(x) = \sqrt{9 - x^2}$ for $x \in [-3, 3]$. Then the graph of f is a semicircle and the region under the graph of f is the region inside of this semicircle. Thus the area of this region is $\frac{1}{2}3^2\pi = \frac{9}{2}\pi$.

For all of these examples the area under the graph of the given function could by found using known geometric results. However for the region under the graph of $f(x) = x^2$ for $x \in [0, 2]$ no such simple solution is possible. Instead we approximate the region by a sequence of regions whose areas we can compute by geometric methods and investigate what happens as the sequence of approximations gets better and better. Specifically we will approximate the region by rectangles. We will first describe the process for an arbitrary function f defined on an arbitrary interval $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. We begin by dividing the given interval into two subintervals of equal length. In the first select any number, c_1, at random and in the second select a second number, c_2, at random. Then the area of the region R under the graph is approximately

$$f(c_1)\frac{b-a}{2} + f(c_2)\frac{b-a}{2} = \frac{b-a}{2}\big(f(c_1) + f(c_2)\big).$$

This approximation isn't particularly good, but it can be improved by partitioning the interval $[a,b]$ into three subintervals of equal length and selecting a number at random in each of the subintervals. Label these three numbers as c_1, c_2 and c_3 respectively. Then the area of R is approximated by

$$f(c_1)\frac{b-a}{3} + f(c_2)\frac{b-a}{3} + f(c_3)\frac{b-a}{3} = \frac{b-a}{3}\big(f(c_1)+f(c_2)+f(c_3)\big).$$

This approximation is a slight improvement over the previous one, but still needs to be refined. So next the interval is divided into four subintervals of equal length and a number selected at random from each of the four intervals. Then the corresponding sum $\frac{b-a}{4}\big(f(c_1)+f(c_2)+f(c_3)+f(c_4)\big)$ is a better approximation of the area of R. Each time the number of subintervals is increased the resulting approximation to the area of R improves. So to investigate the general situation, let n be any positive integer and imagine partitioning the interval $[a,b]$ into n subintervals of equal length. The common length of each of these is then $\frac{b-a}{n}$. The left endpoint of the first interval is, of course, a while its right endpoint is $a+\frac{b-a}{n}$. The left endpoint of the second subinterval is $a+\frac{b-a}{n}$ and its right endpoint is $a+\frac{b-a}{n}+\frac{b-a}{n} = a+2\frac{b-a}{n}$. Continuing in this fashion we see that the kth subinterval has left endpoint $a+(k-1)\frac{b-a}{n}$ and right endpoint $a+k\frac{b-a}{n}$. An arbitrary element is then selected from each of these subintervals. The one selected from the kth interval is denoted by c_k. Thus $c_k \in \left[a+(k-1)\frac{b-a}{n}, a+k\frac{b-a}{n}\right]$. The area of R can be approximated very well, if n is very large, by

$$f(c_1)\frac{b-a}{n} + f(c_2)\frac{b-a}{n} + \cdots + f(c_n)\frac{b-a}{n} = \frac{b-a}{n}\left(f(c_1)+f(c_2)+\cdots+f(c_n)\right).$$

To shorten the amount of writing we introduce the following notation.

$$\sum_{k=1}^{n} f(c_k) = f(c_1)+f(c_2)+\cdots+f(c_n).$$

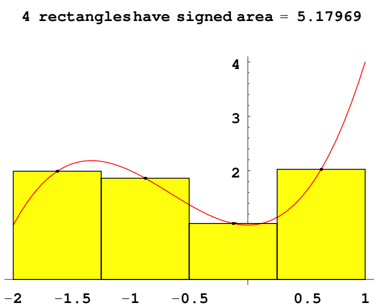
As n gets larger and larger, this approximation will get closer and closer to the area of the region under the graph. Consequently the formal definition of this area is as follows.

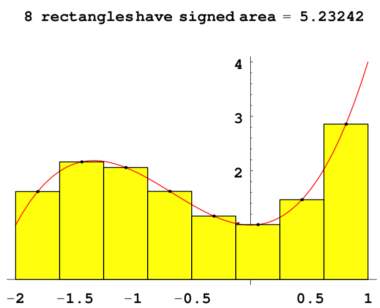
Definition 7. Let f be a function defined on a closed interval $[a,b]$ with $f(x) \geq 0$ for all $x \in [a,b]$. Then the region under the graph of f has area means there is a number A such that

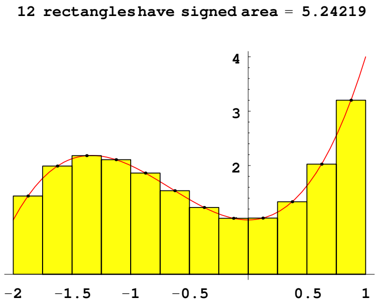
$$\lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f(c_k) = A$$

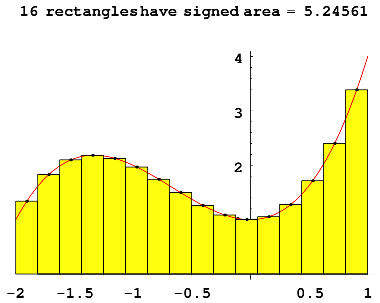
where A doesn't depend on the choice of $c_k \in \left[a+(k-1)\frac{b-a}{n}, a+k\frac{b-a}{n}\right]$.

The following figures demonstrate how the approximation to the area improves as n increases.









To illustrate the definition, we will show that for the case of a triangle it gives us the expected answer.

Example 8. Let $f(x) = mx$ for $x \in [0, b]$ where $0 < b$ and $m > 0$. Let n be any positive integer. In this case for each $k = 1,\ 2, \cdots, n$ we have that $c_k \in \left[\frac{k-1}{n}b,\ \frac{k}{n}b\right]$. Because f is increasing on $[0, b]$, for each $k = 1,\ 2, \cdots, n$ we have that $m\frac{k-1}{n}b \le f(c_k) \le m\frac{k}{n}b$. Hence

$$\frac{b}{n}\sum_{k=1}^{n} m\frac{k-1}{n}b \le \frac{b}{n}\sum_{k=1}^{n} f(c_k) \le \frac{b}{n}\sum_{k=1}^{n} m\frac{k}{n}b \text{ or } \frac{b^2}{n}\frac{m}{n}\sum_{k=1}^{n}(k-1) \le \frac{b^2}{n}\sum_{k=1}^{n} f(c_k) \le \frac{b^2}{n}\frac{m}{n}\sum_{k=1}^{n} k. \tag{1}$$

To compute the two sums $\sum_{k=1}^{n}(k-1)$ and $\sum_{k=1}^{n} k$ we use formula $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$. (See page 311 of the text for this formula and others of a similar nature.) By direct substitution the right hand side of equation (1) is $b^2 m \dfrac{1}{n^2}\dfrac{n(n+1)}{2}$. To compute the left hand side, note that $\sum_{k=1}^{n}(k-1) = \sum_{k=1}^{n-1} k$. Thus the left hand side of equation (1) is $b^2 m \dfrac{1}{n^2}\dfrac{(n-1)n}{2}$. Consequently, (1) can be rewritten as

$$b^2 m \frac{1}{2}\frac{(n-1)n}{n^2} \le \frac{b}{n}\sum_{k=1}^{n} f(c_k) \le b^2 m \frac{1}{2}\frac{n(n+1)}{n^2}.$$

Computing the limits as n tends to ∞ in the same way that we did for functions and $x \to \infty$ we see that $\lim_{n\to\infty} \dfrac{(n-1)n}{n^2} = 1$ and $\lim_{n\to\infty} \dfrac{n(n+1)}{n^2} = 1$. So by the Sandwich Theorem $\lim_{n\to\infty} \dfrac{b}{n}\sum_{k=1}^{n} f(c_k) = \dfrac{b^2 m}{2}$. Hence the area under the graph is exactly what we expected; namely $\dfrac{b^2 m}{2}$.

We will now use essentially the same technique to find an area that can't be computed by simple geometry.

Example 9. Let $f(x) = x^2$ for $x \in [a, b]$ where for ease of calculation, let $0 \le a$. Let n be any positive integer. For each $k = 1, 2, \ldots, n$ let $c_k \in \left[a + (k-1)\dfrac{b-a}{n},\ a + k\dfrac{b-a}{n}\right]$. Because f is increasing on $[a, b]$, as in the previous example we have

$$\frac{b-a}{n}\sum_{k=1}^{n}\left(a + (k-1)\frac{b-a}{n}\right)^2 \le \frac{b-a}{n}\sum_{k=1}^{n} f(c_k) \le \frac{b-a}{n}\sum_{k=1}^{n}\left(a + k\frac{b-a}{n}\right)^2. \tag{2}$$

We will compute the sum on the right hand side of equation (2) and leave the computation of the left hand side as an exercise for the reader.

$$\sum_{k=1}^{n}\left(a + k\frac{b-a}{n}\right)^2 = \sum_{k=1}^{n}\left(a^2 + 2ak\frac{b-a}{n} + k^2\left(\frac{b-a}{n}\right)^2\right) = na^2 + \frac{2a(b-a)}{n}\sum_{k=1}^{n} k + \left(\frac{b-a}{n}\right)^2\sum_{k=1}^{n} k^2.$$

To complete this calculation we will once again use the formula used in Example 8 but we also clearly need a formula for $\sum_{k=1}^{n} k^2$. It is $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$. Thus

$$\sum_{k=1}^{n}\left(a + k\frac{b-a}{n}\right)^2 = na^2 + \frac{2a(b-a)}{n}\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2\frac{n(n+1)(2n+1)}{6}. \tag{3}$$

To finish this part of the calculation, we multiply equation (3) by $\dfrac{b-a}{n}$ and evaluate the limit of each term on the right hand side. First we have $\lim_{n\to\infty} \dfrac{b-a}{n} na^2 = (b-a)a^2$. Next $\lim_{n\to\infty} \dfrac{b-a}{n}\dfrac{2a(b-a)}{n}\dfrac{n(n+1)}{2} = a(b-a)^2$ and lastly $\lim_{n\to\infty} \dfrac{b-a}{n}\Big(\dfrac{b-a}{n}\Big)^2 \dfrac{n(n+1)(2n+1)}{6} = \dfrac{(b-a)^3}{3}$. Now adding these three and doing the algebra we get that $\lim_{n\to\infty} \dfrac{b-a}{n}\sum_{k=1}^{n}\Big(a+k\dfrac{b-a}{n}\Big)^2 = \dfrac{1}{3}(b^3-a^3)$. Doing a similar calculation you will get that $\lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n}\Big(a+(k-1)\dfrac{b-a}{n}\Big)^2 = \dfrac{1}{3}(b^3-a^3)$. Thus again by the Sandwich Theorem the area under the graph of x^2 on the interval $[a,b]$ is $\dfrac{1}{3}(b^3-a^3)$.

A moment's thought will convince you that the only place in the above presentation where the assumption of $f(x) \geq 0$ was used was in the motivation. If we ignore that we started out to define area and drop the assumption $f(x) \geq 0$, we see that it is still possible to write down the same definition. The resulting concept is called the Definite Integral (or the Riemann integral) of the function. The number that results from the definition is called the Riemann integral of f from a to b and is denoted by $\int_a^b f(x)\,dx$. The sign $\int$ is an elongated letter "S" and is to remind us that the number is obtained by computing a sum and then taking a limit. So to be specific here is the definition.

Definition 10. Let f be a function defined on a closed interval $[a,b]$. Then f is (Riemann) integrable on $[a,b]$ means there is a number, denoted by $\int_a^b f(x)\,dx$, such that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n} f(c_k)$$

regardless of the choice of $c_k \in \big[a+(k-1)\dfrac{b-a}{n}, a+k\dfrac{b-a}{n}\big]$.

The number, $\int_a^b f(x)\,dx$, is called the Definite (or Riemann) integral of f from a to b. It has a geometric interpretation in terms of the region **between** the x-axis and the graph of f; namely, $\int_a^b f(x)\,dx$, is the area of that part of the region lying above the x-axis minus the area of that part of the region lying below the x-axis. To compute the approximations to $\int_a^b f(x)\,dx$ you use exactly the same method as you did above. In particular for $0 \leq a < b$ we have $\int_a^b x^2\,dx = \frac{1}{3}(b^3-a^3)$. To interpret $\int_a^b f(x)\,dx$ geometrically first visualize the region <u>between</u> the graph and the x axis. Then $\int_a^b f(x)\,dx$ is the area of that part of the region above the x axis minus the area of that part of the region below the x axis.

The following properties of definite integrals follow directly from the definition.

1. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. $\int_a^a f(x)\,dx = 0$
3. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ for any number k
4. $\int_a^b f(x) - g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
5. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. If $m \leq f(x) \leq M$ for all x in $[a,b]$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$
7. If $g(x) \leq f(x)$ for all x in $[a,b]$, then $\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx$

Exercises

1. Let $f(x) = \frac{1}{x}$. Compute $\frac{b-a}{n}\sum_{k=1}^{n} f(c_k)$ for each of the following choices.

(a) $[a,b] = [1,5]$, $n = 4$ and c_k is the left endpoint of each subinterval.

(b) $[a,b] = [1,5]$, $n = 4$ and c_k is the right endpoint of each subinterval.

(c) $[a,b] = [1,5]$, $n = 4$ and c_k is the midpoint of each subinterval.

(d) $[a,b]=[1,5]$, $n=6$ and c_k is the left endpoint of each subinterval.

(e) $[a,b]=[1,5]$, $n=6$ and c_k is the right endpoint of each subinterval.

(f) $[a,b]=[1,5]$, $n=6$ and c_k is the midpoint of each subinterval.

(g) $[a,b]=[1,5]$, $n=8$ and c_k is the left endpoint of each subinterval.

(h) $[a,b]=[1,5]$, $n=8$ and c_k is the right endpoint of each subinterval.

(i) $[a,b]=[1,5]$, $n=8$ and c_k is the midpoint of each subinterval.

2. Repeat the above with $f(x)=\frac{x}{x+1}$. 3. Let $b<0$, let $m>0$ and let $f(x)=mx$. Proceeding as in Example 8 compute $\int_b^0 f(x)\,dx$. 4. Express each of the following limits as a definite integral. Each has more than one correct answer.

(a) $\lim_{n\to\infty}\dfrac{1}{n}\sum_{k=1}^n k\dfrac{1}{n}$.

(b) $\lim_{n\to\infty}\dfrac{3}{n}\sum_{k=1}^n \left(k\dfrac{3}{n}\right)^{\frac{1}{2}}$.

(c) $\lim_{n\to\infty}\dfrac{2}{n}\sum_{k=1}^n \left(1+k\dfrac{2}{n}\right)^{\frac{1}{3}}$.

(d) $\lim_{n\to\infty}\dfrac{1}{n}\sum_{k=1}^n \left(1+k\dfrac{2}{n}\right)^{\frac{1}{3}}$.

(e) $\lim_{n\to\infty}\dfrac{2}{n}\sum_{k=1}^n \left(1+k\dfrac{2}{n}\right)^{\frac{2}{3}}$.

(f) $\lim_{n\to\infty}\dfrac{1}{n}\sum_{k=1}^n \left(1+k\dfrac{1}{n}\right)^{-2}$.

(g) $\lim_{n\to\infty}\dfrac{2}{n}\sum_{k=1}^n \left(-1+k\dfrac{1}{n}\right)^{2}$.

(h) $\lim_{n\to\infty}\dfrac{2}{n}\sum_{k=1}^n \left(1+(-1+k\dfrac{2}{n})^2\right)^{3}$.

(i) $\lim_{n\to\infty}\dfrac{2}{n}\sum_{k=1}^n \sin\left(k\pi\dfrac{2}{n}\right)$.

Selected Answers

1(d) $\frac{4}{6}\sum_{k=1}^6 \frac{1}{1+\frac{2(k-1)}{3}} = \frac{2}{3}\left(1+\frac{3}{5}+\frac{3}{7}+\frac{3}{9}+\frac{3}{11}+\frac{3}{13}\right)$

1(i) $\frac{4}{8}\sum_{k=1}^8 \frac{1}{1+\frac{2k-1}{4}} = \frac{1}{2}\left(\frac{4}{5}+\frac{4}{7}+\frac{4}{9}+\frac{4}{11}+\frac{4}{13}+\frac{4}{15}+\frac{4}{17}+\frac{4}{19}\right)$

4(b) $\int_0^3 x^{\frac{1}{2}}\,dx$

4(c) $\int_0^2 (1+x)^{\frac{1}{3}}\,dx$ or $\int_1^3 x^{\frac{1}{3}}\,dx$

4(f) $\int_0^1 (1+x)^{-2}\,dx$ or $\int_1^2 x^{-2}\,dx$