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Stability of Mean Flows over an Infinite Flat Plate

MILAN MIKLAVČIČ & MICHAEL WILLIAMS

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1. Introduction

The main concern of this paper is a stability analysis for the often encountered linearized Navier-Stokes equations for parallel and nonparallel mean flows over an infinite flat plate [1, 2, 5, 6, 7, 12, 13, 14, 18, 19]. The system of equations for parallel flows (see Section 5 for the formulation and treatment of nonparallel flows) can be written as

$$\frac{\partial u_1}{\partial t} - \frac{1}{R}\frac{\partial^2 u_1}{\partial y^2} + h(y)\,u_1 + v_1'(y)\,u_2 + i\alpha p = 0, \tag{1}$$

$$\frac{\partial u_2}{\partial t} - \frac{1}{R}\frac{\partial^2 u_2}{\partial y^2} + h(y)\,u_2 + \frac{\partial p}{\partial y} = 0, \tag{2}$$

$$\frac{\partial u_3}{\partial t} - \frac{1}{R}\frac{\partial^2 u_3}{\partial y^2} + h(y)\,u_3 + v_3'(y)\,u_2 + i\beta p = 0, \tag{3}$$

$$i\alpha u_1 + i\beta u_3 + \frac{\partial u_2}{\partial y} = 0, \tag{4}$$

$$u_j(0, t) = 0 \quad \text{for} \quad j = 1, 2, 3 \tag{5}$$

where $h(y) = (\alpha^2 + \beta^2)/R + i\alpha v_1(y) + i\beta v_3(y)$ and the primes denote derivatives. $u = u(y, t) = (u_1, u_2, u_3)$ and $p = p(y, t)$ denote the velocity and pressure of the fluid at a point $y \geqq 0$ and time $t \geqq 0$ respectively; R as usual is the Reynolds number. v_1 and v_3 are the x (streamwise) and z (spanwise) components of the mean flow while α and β are the wave numbers in the x and z directions for perturbations of the mean flow. This system of equations is viewed as a generalized Orr-Sommerfeld equation.

In bounded regions the Navier-Stokes equations have been successfully treated for many different mean flows [*e.g.* 3, 4, 8, 9, 16, 20]. In these cases much is known about the stability of solutions, completeness of eigenfunctions, spectrum and bifurcation. Efforts have been made [1, 2, 5, 6, 7, 12, 13, 14, 18, 19] to obtain similar results for generalized Orr-Sommerfeld equations and the modified versions for nonparallel flow. These works consist of numerical studies of eigenvalues and of basically formal manipulations.

An explicit criterion characterizing the case when the stability of all physically reasonable solutions is determined by the eigenvalues is presented here. From some of the works mentioned above, which treat only special forms of the generalized Orr-Sommerfeld equations, one can guess such a criterion. However, the proof given here is applicable to both the generalized Orr-Sommerfeld equations and the modified equations for nonparallel flow. This and the fact that the criterion is independent of the completeness or incompleteness of eigenfunctions is contrary to some expectations [*e.g.* 6, 13, 18].

MACK's numerical results [12] led to a belief that the generalized Orr-Sommerfeld equations may have only finitely many eigenvalues [*e.g.* 6, 13, 18]. An example is given for which this is not true; in addition, a nontrivial condition implying that there are no eigenvalues is proven.

Effects of small perturbations of the mean flow are presented. Some bounds and other facts on eigenvalues which can be used, for example, to estimate critical Reynolds numbers are given also.

In Section 2 the setting and the statement of the main theorem are given. The generalized Orr-Sommerfeld equations are transformed into a more convenient equivalent form in Section 3. The proof of the main theorem is presented in Section 4. Section 5 contains the extensions of the results to nonparallel flows and perturbations of the mean flow are treated in Section 6.

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2. Preliminaries and Statement of Main Theorem

Throughout $\mathscr{H}(\mathscr{H}^j)$ denotes the Hilbert space $L^2(0,\infty)$ (j-fold product of $L^2(0,\infty)$) and $\|\cdot\|$ represents the norm in $\mathscr{H}$ or $\mathscr{H}^j$ without confusion; the usual L^∞ norm is given by $\|\cdot\|_\infty$. The set of all complex-valued functions which are absolutely continuous on $[0, a]$ for every $a > 0$ is denoted by $\mathscr{AC}$.

Several operators on $\mathscr{H}$ appear frequently and are represented as follows: for $\operatorname{Re}(z) > 0$ and $g \in \mathscr{H}$, define $F_z, G_z \in B(\mathscr{H})$ by

$$(F_z g)(x) = \int_0^x e^{z(s-x)} g(s)\, ds$$

and

$$(G_z g)(x) = \int_x^\infty e^{z(x-s)} g(s)\, ds.$$

The operator T is defined by $Tf = -f''$ for $f \in \mathscr{D}(T) = \{f \mid f, f' \in \mathscr{H} \cap \mathscr{AC}, f'' \in \mathscr{H}, f(0) = 0\}$.

A map (u, p) from an interval $(0, t_0)$, $0 < t_0 \leqq \infty$, into $\mathscr{H}^4$ is said to be a solution of equations (1–5) if for each $t \in (0, t_0)$ the following conditions are satisfied:

1. $u_j, p, \dfrac{\partial u_j}{\partial y} \in \mathscr{H} \cap \mathscr{AC}, \dfrac{\partial^2 u_j}{\partial y^2} \in \mathscr{H}$ for $j = 1, 2, 3$ and $u = (u_1, u_2, u_3)$,

2. u is continuously differentiable in t,

3. $\dfrac{\partial u_2}{\partial t} \in \mathscr{AC}$ and $\dfrac{\partial^2 u_2}{\partial t\, \partial y} = \dfrac{\partial^2 u_2}{\partial y\, \partial t}$,

4. u, p satisfy equations (1–5),

5. $\lim\limits_{t \to 0+} u(t)$ exists.

$\mathscr{S}_0$ is the set of all such maps. $(u, p) \in \mathscr{S}_0$ is a stable solution if $\sup\limits_t \|u(t)\| < \infty$; otherwise it is unstable. $(u, p) \in \mathscr{S}_0$ is an eigenvector if $u(t) = e^{-zt} u_0$ and $p(t) = e^{-zt} p_0$ for some $z \in \mathbb{C}$, $u_0 \in \mathscr{H}^3$ and $p_0 \in \mathscr{H}$. The set of all such z is denoted by σ_{os} and its members are called eigenvalues.

The main theorem may now be stated.

Theorem 1. *Suppose*

(i) $v_1, v_3 \in \mathscr{AC}$;

(ii) *the limits* $\lim\limits_{y \to \infty} v_1(y) = \bar{v}_1$ *and* $\lim\limits_{y \to \infty} v_3(y) = \bar{v}_3$ *exist and are finite*;

(iii) $v_1 - \bar{v}_1, v_3 - \bar{v}_3, v_1', v_3' \in \mathscr{H}$;

(iv) $\alpha^2 + \beta^2 \in \mathbb{C} \setminus (-\infty, 0]$, $R > 0$.

Let $\mu = (\alpha^2 + \beta^2)/R + i\alpha\bar{v}_1 + i\beta\bar{v}_3$ *and* $\sigma_c = \{z \in \mathbb{C} \mid \mathrm{Re}\,(z) \geqq \mathrm{Re}\,(\mu)$ *and* $\mathrm{Im}\,(z) = \mathrm{Im}\,(\mu)\}$. *Then*

(a) *If* $u_0 = (u_{01}, u_{02}, u_{03}) \in \mathscr{H}^3$, $u_{02} \in \mathscr{AC}$, $u_{02}' \in \mathscr{H}$, $u_{02}(0) = 0$ *and equation* (4) *is satisfied, then there is a unique* $(u, p) \in \mathscr{S}_0$ *defined on* $(0, \infty)$ *satisfying* $\lim\limits_{t \to 0+} u(t) = u_0$. *If* $(u, p) \in \mathscr{S}_0$, *then* $u = u(y, t)$ *is infinitely differentiable in* y *and* t *and there is a unique* $(\tilde{u}, \tilde{p}) \in \mathscr{S}_0$ *defined on* $(0, \infty)$ *which restricts to* (u, p).

(b) $\sigma_{os} \subset \{\mu + z \mid z \in \mathbb{C}$ *with* $|\mathrm{Im}\,(z)| \leqq M_1(\alpha, \beta)$ *for* $\mathrm{Re}\,(z) \geqq 0$ *and* $|z| \leqq M_1(\alpha, \beta)$ *for* $\mathrm{Re}\,(z) < 0\}$.
If $K \subset \mathbb{C} \setminus \sigma_c$ *is compact, then* $K \cap \sigma_{os}$ *is finite. The constant* M_1 *is given by equation* (13). σ_{os} *itself may be an infinite set.*

(c) *If* $R < M_2(\alpha, \beta)$, *then* σ_{os} *is empty. The constant* M_2 *is given by equation* (14) *and Theorem* 3.

(d) *If* $\mathrm{Re}\,(\mu) > 0$, *then* $\mathscr{S}_0$ *contains an unstable solution if and only if* $\mathscr{S}_0$ *contains an unstable eigenvector. If* $\mathrm{Re}\,(\mu) < 0$, *then* $\mathscr{S}_0$ *contains an unstable solution.*

3. Reformulation of Basic Problem

In this section the generalized Orr-Sommerfeld problem given by equations (1–5) is transformed into a more convenient equivalent form.

For $(u, p) \in \mathscr{S}_0$, $u = (u_1, u_2, u_3)$ note that

$$\frac{\partial p}{\partial y} + \lambda p = 2G_\lambda g_1' u_2,$$

where $\lambda = \sqrt{\alpha^2 + \beta^2}$, $\operatorname{Re}(\lambda) > 0$ and $g_1 = i\alpha(v_1 - \bar{v}_1) + i\beta(v_3 - \bar{v}_3)$. Define

$$w_1 \equiv \beta u_1 - \alpha u_3, \tag{6}$$

$$w_2 \equiv \lambda u_2 - i\alpha u_1 - i\beta u_3 = \lambda u_2 + \frac{\partial u_2}{\partial y} \tag{7}$$

so that the new dependent variables w_1, w_2 satisfy

$$\frac{\partial w_1}{\partial t} + \mu w_1 + A w_1 + C w_2 = 0 \tag{8}$$

and

$$\frac{\partial w_2}{\partial t} + \mu w_2 + B w_2 = 0 \tag{9}$$

where the operators A, B, C are defined by

$$A = \frac{1}{R} T + g_1, \tag{10}$$

$$B = \frac{1}{R} T + g_1 + 2\lambda G_\lambda g_1' F_\lambda - g_1' F_\lambda, \tag{11}$$

$$C = g_2 F_\lambda, \quad g_2 = \beta v_1' - \alpha v_3'. \tag{12}$$

Next let $\mathscr{S}_m$ be the set of all maps $w = (w_1, w_2)$ from the interval $(0, t_0)$, $0 < t_0 \leqq \infty$ into $\mathscr{H}^2$ which satisfy the following conditions for $t \in (0, t_0)$:

(1) $w_1, w_2 \in \mathscr{D}(T)$,

(2) w is continuously differentiable in t,

(3) w satisfies equations (8), (9),

(4) $\lim_{t \to 0+} w(t)$ exists.

$w \in \mathscr{S}_m$ is stable if $\sup_t \|w(t)\| < \infty$ and unstable otherwise.

Clearly $(u, p) \in \mathscr{S}_0$ implies that w defined by equations (6), (7) belongs to $\mathscr{S}_m$ and if (u, p) is stable, so is w.

Conversely, if $w \in \mathscr{S}_m$ then equations (6), (7) define a unique $(u, p) \in \mathscr{S}_0$ with the same domain. $u_2 = F_\lambda w_2$, u_1 and u_3 follow from equations (6), (7) by linear algebra and p is then determined by equation (1) or (3). Moreover, if w is stable so is (u, p). These results are summarized by the following theorem:

Theorem 2. *There is a one-to-one correspondence between $\mathscr{S}_0$ and $\mathscr{S}_m$ given by equations* (6), (7). *Moreover,* $(u, p) \in \mathscr{S}_0$ *is stable if and only if the corresponding* $w \in \mathscr{S}_m$ *is stable.*

4. Proof of the Main Theorem

Part (a). Let $\mathscr{A}$ denote the set of generators of analytic semigroups, and note that $\frac{1}{R} T \in \mathscr{A}$. Since $g_1 \in B(\mathscr{H})$, it follows that $A \in \mathscr{A}$ [11] and in the same way $B, D \in \mathscr{A}$ where D is defined on $\mathscr{H}^2$ by

$$D = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \; \mathscr{D}\,(D) = \mathscr{D}(T) \times \mathscr{D}(T).$$

Thus, since $D \in \mathscr{A}$, $e^{-Dt} w_0 \in \mathscr{S}_m$ for each $w_0 \in \mathscr{H}^2$ and for each $w \in \mathscr{S}_m$, there is a $w_0 \in \mathscr{H}^2$ such that w is a restriction of $e^{-Dt} w_0$. Moreover, each $w \in \mathscr{S}_m$ is infinitely differentiable in y and t. The transformation equations (6), (7) together with the properties of the integral operator F_λ then give part (a) of the main theorem.

Part (b). Because g_1 is in $\mathscr{H}$, g_1 as a multiplication operator is $\frac{1}{R}$ T-compact. Therefore $\sigma(A) \supset [0, \infty)$ and $\sigma_p(A) \supset \sigma(A) \setminus [0, \infty)$, where $\sigma_p(A)$ is the point spectrum of A [11]. Furthermore, for compact K with $K \subset \mathbb{C} \setminus [0, \infty)$, $K \cap \sigma(A)$ is finite [11]. Similarly, these conclusions hold for the operators B and D using the same relatively compact perturbation with relative-bound-0 argument.

From the form of D and the transformation itself, it is easily seen that $\sigma_{os} - \mu = \sigma_p(D) \subset \sigma_p(A) \cap \sigma_p(B)$. In order to analyze the resolvent of B, for $z \in \mathbb{C} \setminus [0, \infty)$ consider the representation

$$B - z = \left(1 + (g_1 + 2\lambda G_\lambda g_1' F_\lambda - g_1' F_\lambda) \left(\frac{1}{R} T - z\right)^{-1}\right) \left(\frac{1}{R} T - z\right).$$

With the norm values, $\|F_\lambda\| = \|G_\lambda\| = 1/\mathrm{Re}\,(\lambda)$ and $\left\| \left(\frac{1}{R} T - z\right)^{-1} \right\| = \frac{1}{|\mathrm{Im}\,(z)|}$ for $\mathrm{Re}\,(z) \geqq 0$ and $= \frac{1}{|z|}$ for $\mathrm{Re}\,(z) < 0$ the estimates of part (b) of Theorem 1 follow immediately when M_1 is taken to satisfy

$$M_1(\alpha, \beta) = \|g_1\|_\infty + \left(\frac{2\,|\lambda|}{\mathrm{Re}\,(\lambda)} + 1\right) \min \left\{ \frac{\|g_1'\|_\infty}{\mathrm{Re}\,(\lambda)}, \frac{\|g_1'\|}{\sqrt{2\,\mathrm{Re}\,(\lambda)}} \right\}. \tag{13}$$

To show that σ_{os} may be infinite, it is enough to exhibit an example for which $\sigma_p(A)$ is infinite since $\sigma_p(A) \subset \sigma_{os} - \mu$. Take g_1 to be a real-valued function such that for some positive numbers a, ε and ℓ_0

$$g_1(x) \leqq -ax^{-2+\varepsilon} \quad \text{for} \quad x > \ell_0.$$

A is then self-adjoint and its essential spectrum is $\sigma_{\text{ess}}(A) = [0, \infty)$ [15]. This example is motivated from quantum mechanics and the following proof is adapted from the proof of Theorem XIII.6 in [15].

Let ϕ_0 be a real-valued, infinitely differentiable function supported on the interval [1, 2] and so normalized that $\|\phi_0\| = 1$. Let $\phi_\ell = \ell^{-\frac{1}{2}} \phi_0(x/\ell)$ so that the support of ϕ_ℓ is contained in $[\ell, 2\ell]$ and again $\|\phi_\ell\| = 1$. If $\ell > \ell_0$, then

$$(A\phi_\ell, \phi_\ell) = -\int_0^\infty \phi_\ell''(x)\, \phi_\ell(x)\, dx + \int_0^\infty g_1(x)\, \phi_\ell^2(x)\, dx$$

$$\leqq -\ell^{-2} \int_0^\infty \phi_0''(x)\, \phi_0(x)\, dx - a\ell^{-2+\varepsilon} \int_0^\infty x^{-2+\varepsilon}\, \phi_0^2(x)\, dx.$$

Since $\varepsilon > 0$, this last expression is negative for large ℓ. Thus $q > \ell_0$ may be chosen such that $(A\phi_\ell, \phi_\ell) < 0$ for all $\ell > q$. Define $\phi^{(n)} = \phi_{q2^n}$, $n = 1, 2, \ldots$; $\{\phi^{(n)}\}$ is an orthonormal sequence and $(A\phi^{(n)}, \phi^{(m)}) = 0$ if $n \neq m$. For given N, set $V_N = \text{span}\,\{\phi^{(1)}, \ldots, \phi^{(N)}\}$ and note that A restricted to V_N has eigenvalues $\{(A\phi^{(n)}, \phi^{(n)})\}_{n=1}^N$. Since all the eigenvalues are strictly negative, the Rayleigh-Ritz principle implies that there exist at least N eigenvalues (counting multiplicities) of A below 0. Since N is arbitrary and $\sigma_{\text{ess}}(A) = [0, \infty)$, A has infinitely many eigenvalues.

Part (c). The techniques used here are similar to those used by KATO in [10]; therefore, not all details will be presented.

The following technical lemma is needed.

Lemma 1. (i) *For $f \in \mathscr{H}$ and* $\operatorname{Im}(\xi) \neq 0$, *if h_1 and h_2 are measurable functions from* $[0, \infty)$ *to* $\mathbb{C}$ *such that* $\sqrt{|h_2|}\, f \in \mathscr{H}$, *then*

$$\left\|\sqrt{|h_1|}\, (T - \xi)^{-1} \sqrt{|h_2|}\, f\right\|^2 \leqq \left(\int_0^\infty x\, |h_1(x)|\, dx\right) \left(\int_0^\infty x\, |h_2(x)|\, dx\right) \|f\|^2.$$

(ii) *If* $h \in L^1(0, \infty)$, $\operatorname{Re}(z) > 0$ *and* $g \in \mathscr{H}$, *then for* $f(x) = \int_x^\infty |h(t)|\, dt$ *the following inequality holds*:

$$2 \operatorname{Re}(z) \left\|\sqrt{|h|}\, F_z g\right\|^2 \leqq \left\|\sqrt{f}\, g\right\|^2.$$

Proof. To establish (i) it is sufficient to note that $(T - \xi)^{-1}$ is an integral operator with kernel which is pointwise dominated by $k(x, y) = \min\{x, y\}$

From the observation that

$$|(\sqrt{h}\ F_z g)(x)|^2 \leqq |h(x)| \left(\int_0^x e^{\mathrm{Re}(z)(s-x)} |g(s)|\ ds\right)^2$$
$$\leqq \frac{1}{2\ \mathrm{Re}\ (z)} |h(x)| \int_0^x |g(s)|^2\ ds,$$

a simple integration by parts proves (ii).

The following additional assumptions on g_1 are now temporarily made: $\int_0^\infty x\,|g_1(x)|\,dx < \infty$, $\int_0^\infty x\,|g_1'(x)|\,dx < \infty$ and $\int_0^\infty x\tilde{g}(x)\,dx < \infty$ where $\tilde{g}(x) = \int_x^\infty |g_1'(t)|\,dt$.

Next, an auxiliary set of operators are defined by

$$A_1 = \sqrt{|g_1|}, \qquad B_1 = \overline{(\mathrm{sgn}\ g_1)}\sqrt{|g_1|},$$
$$A_2 = \sqrt{|g_1'|}\,F_\lambda, \qquad B_2 = 2\bar{\lambda}\,\overline{(\mathrm{sgn}\ g_1')}\sqrt{|g_1'|}\,F_{\bar{\lambda}},$$
$$A_3 = A_2, \qquad B_3 = -\overline{(\mathrm{sgn}\ g_1')}\sqrt{|g_1'|},$$

where $(\mathrm{sgn}\ g)(x) = g(x)/|g(x)|$ if $g(x) \neq 0$ and 1 otherwise. Note that except for possibly B_3 all of the above operators are bounded. Let the domain of B_3 be its maximal domain so that $\mathscr{D}(T) \subset \mathscr{D}(B_3)$ and B_3 is closed.

As in [10], A_1, B_1 and B_3 are T-smooth and by Lemma 1, for $g \in \mathscr{H}$ and $\mathrm{Im}\,(\xi) \neq 0$, the inequality

$$\|A_2(T-\xi)^{-1}g\|^2 \leqq \frac{1}{2\ \mathrm{Re}\ (\lambda)} \left\|\sqrt{\tilde{g}}\ (T-\xi)^{-1}g\right\|^2$$

holds. Since $\sqrt{\tilde{g}}$ is T-smooth, so are A_2, B_2 and A_3. Let A_0 and B_0 be linear operators from $\mathscr{H}$ to $\mathscr{H}^3$ defined by

$$A_0 \equiv A_1 \oplus A_2 \oplus A_3 \quad \text{and} \quad B_0 \equiv B_1 \oplus B_2 \oplus B_3;$$

A_0 and B_0 are T-smooth and

$$B_0^* A_0 = B_1^* A_1 + B_2^* A_2 + B_3^* A_3 = g_1 + 2\lambda G_\lambda g_1' F_\lambda - g_1' F_\lambda.$$

Therefore $B = \frac{1}{R} T + B_0^* A_0$. If for all $f \in \mathscr{D}(B_0^*)$ with $\|f\| = 1$, all $\xi \in \mathbb{C} \setminus \mathbb{R}$ and some $N < \frac{1}{R}$, the inequality

$$\|A_0(T-\xi)^{-1}B_0^* f\| \leqq N$$

holds, then B is similar to $\frac{1}{R} T$ [10]. Thus, since $\sigma_p(T)$ is empty, $\sigma_p(B)$ is empty.

Now suppose that constants $N_{ij} \in [0, \infty]$ are such that $\|A_i(T-\xi)^{-1}B_j^* f_j\|^2 \leqq N_{ij}\|f_j\|^2$ for $f_j \in \mathscr{D}(B_j^*)$, $\operatorname{Im}(\xi) \neq 0$ and $i, j = 1, 2, 3$. Then for $f \in \mathscr{D}(B_0^*)$, $f = (f_1, f_2, f_3)$, $\|f\| = 1$ and $\operatorname{Im}(\xi) \neq 0$,

$$\|A_0(T-\xi)^{-1}B_0^* f\|^2 = \sum_{i=1}^{3}\left\|\sum_{j=1}^{3} A_i(T-\xi)^{-1}B_j^* f_j\right\|^2 \leqq \sum_{i=1}^{3}\sum_{j=1}^{3} N_{ij}$$

obtains, so that with the choice

$$M_2(\alpha, \beta) \equiv \left(\sum_{i=1}^{3}\sum_{j=1}^{3} N_{ij}\right)^{-\frac{1}{2}}, \tag{14}$$

the condition $R < M_2(\alpha, \beta)$ implies that $\sigma_p(B)$ is empty. For the same reasons, $\sigma_p(A)$ is empty when $R < N_{11}^{-\frac{1}{2}}$ so that the conclusion of Part (c) holds with the choice of M_2 given in equation (14). The following theorem gives suitable choices for the N_{ij}.

Theorem 3. *The following assignment of the values of the N_{ij} satisfies the requirements of the preceding argument*:

$$N_{11} = \left(\int_0^\infty x\,|g_1(x)|\,dx\right)^2,$$

$$N_{12} = \frac{2\,|\lambda|^2}{\operatorname{Re}(\lambda)}\left(\int_0^\infty x\,|g_1(x)|\,dx\right)\left(\int_0^\infty x\tilde{g}(x)\,dx\right),$$

$$N_{13} = \left(\int_0^\infty x\,|g_1(x)|\,dx\right)\left(\int_0^\infty x\,|g_1'(x)|\,dx\right),$$

$$N_{21} = N_{31} = \frac{1}{2\operatorname{Re}(\lambda)}\left(\int_0^\infty x\tilde{g}(x)\,dx\right)\left(\int_0^\infty x\,|g_1(x)|\,dx\right),$$

$$N_{22} = N_{32} = \left|\frac{\lambda}{\operatorname{Re}(\lambda)}\right|^2\left(\int_0^\infty x\tilde{g}(x)\,dx\right)^2$$

$$N_{23} = N_{33} = \frac{1}{2\operatorname{Re}(\lambda)}\left(\int_0^\infty x\tilde{g}(x)\,dx\right)\left(\int_0^\infty x\,|g_1'(x)|\,dx\right).$$

Remark. The assumptions on g_1 made at the beginning of this subsection may now be dropped by allowing $+\infty$ for the values of the above integrals.

Proof. As a representative example, the estimate for N_{12} is considered. Set $f(x) = (A_1(T-\xi)^{-1}\,B_2^* f_2)(x)$ for $f_2 \in \mathscr{H}$ and $\operatorname{Im}(\xi) \neq 0$. Let $k(x, y)$ be the integral kernel of $(T-\xi)^{-1}$ so that

$$f(x) = 2\lambda\sqrt{|g_1(x)|}\int_0^\infty dy\int_y^\infty ds\,k(x, y)\,e^{\lambda(y-s)}\operatorname{sgn}(g_1')\sqrt{|g_1'(s)|}\,f_2(s).$$

Interchanging the order of integration and applying absolute values gives

$$|f(x)| \leqq \int_0^\infty k_1(x, y)\, |f_2(y)|\, dy$$

where

$$k_1(x, y) = |2\lambda| \sqrt{|g_1(x)\, g_1'(y)|} \int_0^y |k(x, s)|\, e^{\mathrm{Re}(\lambda)(s-y)}\, ds.$$

Since

$$|k_1(x, y)|^2 \leqq \frac{2\,|\lambda|^2}{\mathrm{Re}\,(\lambda)} |g_1(x)\, g_1'(y)| \int_0^y |k(x, s)|^2\, ds,$$

integration by parts together with the bound $|k(x, y)|^2 \leqq xy$ give the defined value of N_{12}. This procedure and Lemma 1 give the results for the other N_{ij}.

Part (d). The desired result is a direct consequence of the following theorem which, despite its being an expected result, does not seem to appear in the literature.

Theorem 4. *Let A be the generator of a strongly continuous semigroup, e^{-At}, on a Banach space X and suppose that $\sigma(A)$ is not empty. Let $a = \inf\{\mathrm{Re}\,(z) \mid z \in \sigma(A)\}$ and $b > a$. Then there exists an $x \in \mathscr{D}(A)$ such that for any $t_0 \geqq 0$ there is a $t > t_0$ for which $\|e^{-At}x\| > e^{-bt}$. When A is the generator of an analytic semigroup, then x may be found in $\bigcap_{n=1}^{\infty} \mathscr{D}(A^n)$.*

Proof. Suppose that A is the generator of a strongly continuous semigroup and that for all $x \in \mathscr{D}(A)$ there exists a $t_0 \geqq 0$ such that for all $t > t_0$ the inequality $\|e^{-At}x\| \leqq e^{-bt}$ holds.

Let $z \in \sigma(A)$ with $\mathrm{Re}\,(z) \in [a, b)$. Define

$$Rx = \lim_{t\to\infty} \int_0^t e^{zs} e^{-As} x\, ds$$

(Bochner integral) for those $x \in X$ for which this limit exists. Obviously R is a linear operator and $\mathscr{D}(R) \supset \mathscr{D}(A)$.

For $x \in \mathscr{D}(A)$, $e^{zt}e^{-At}x$ is continuously differentiable on $[0, \infty)$ and

$$\frac{d}{dt} e^{zt} e^{-At} x = e^{zt} e^{-At} (z - A)\, x.$$

An integration then gives

$$e^{zt}e^{-At}x - x = \int_0^t e^{zs}e^{-As}(\xi - A)\, x\, ds + (z - \xi) \int_0^t e^{zs}e^{-As}\, x\, ds \tag{15}$$

for each $\xi \in \mathbb{C}$ and all $t \in [0, \infty)$. Since $x \in \mathscr{D}(R)$ and $\lim_{t\to\infty} e^{zt}e^{-At}x = 0$ by hypothesis, then from (15) it is seen that $(\xi - A)\, x \in \mathscr{D}(R)$ and $x = R(A - z)\, x$. Therefore $\mathscr{D}(R) = X$ and $A - z$ is one-to-one.

Next let $A_r = (1 - e^{-Ar})/r$ for $r > 0$. For $x \in X$,

$$A_r Rx = \lim_{t\to\infty} \frac{1}{r} \int_0^t [e^{zs} e^{-As} - e^{-zr} e^{z(r+s)} e^{-A(r+s)}]\, x\, ds$$

$$= \frac{1 - e^{-rz}}{r} Rx + \frac{e^{-rz}}{r} \int_0^r e^{zt} e^{-At} x\, dt.$$

Hence $Rx \in \mathscr{D}(A)$, $(A - z)\, Rx = x$ and $\operatorname{Ran}(A - z) = X$, Ran denoting "range". Because A is closed, this is a contradiction.

When A is a generator of an analytic semigroup, $e^{-Ar}x \in \bigcap_{n=1}^{\infty} \mathscr{D}(A^n)$ for $r > 0$ and $x \in X$ so that assuming the theorem to be false leads to the following statement: $\forall x \in X$ there exist constants M, $t_0 \in [0, \infty)$ such that $\|e^{-At}x\| \leq Me^{-bt}$ for all $t > t_0$. The foregoing argument then gives a contradiction once again.

5. Nonparallel Model of Flow

When it is not possible to assume that the mean flow is parallel, for example, during the initial stages of turbulence, the linearized Navier-Stokes equations become

$$\frac{\partial u_1}{\partial t} - \frac{1}{R} \frac{\partial^2 u_1}{\partial y^2} + hu_1 + v_2 \frac{\partial u_1}{\partial y} + v_1' u_2 + i\alpha p = 0,$$

$$\frac{\partial u_2}{\partial t} - \frac{1}{R} \frac{\partial^2 u_2}{\partial y^2} + hu_2 + v_2 \frac{\partial u_2}{\partial y} + v_2' u_2 + \frac{\partial p}{\partial y} = 0,$$

$$\frac{\partial u_3}{\partial t} - \frac{1}{R} \frac{\partial^2 u_3}{\partial y^2} + hu_3 + v_2 \frac{\partial u_3}{\partial y} + v_3' u_2 + i\beta p = 0.$$

Equations (4), (5) remain unchanged and h is as given in Equations (1)–(3). The definition of $\mathscr{S}_0$ is changed accordingly.

If, in addition to the hypotheses of the Main Theorem, it is assumed that $v_2 \in \mathscr{AC} \cap \mathscr{H}$ and $v_2' \in \mathscr{H}$, then the conclusions of the Main Theorem again hold, except for Part (c) and the bounds on σ_{os} in Part (b). The proof is almost the same but more cluttered; hence only the important differences will be presented.

First, note that $(u, p) \in \mathscr{S}_0$ implies that $p + v_2 u_2, \frac{\partial}{\partial y}(p + v_2 u_2) \in \mathscr{AC} \cap \mathscr{H}$ and $\frac{\partial^2}{\partial y^2}(p + v_2 u_2) \in \mathscr{H}$, and that

$$\frac{\partial^2}{\partial y^2}(p + v_2 u_2) = \lambda^2(p + v_2 u_2) - 2g_1' u_2 - \lambda^2 v_2 u_2 + v_2 \frac{\partial^2 u_2}{\partial y^2}.$$

This equation leads to

$$\frac{\partial p}{\partial y} + \lambda p = G_\lambda \left(2g_1' u_2 + v_2' \left(\lambda u_2 + \frac{\partial u_2}{\partial y}\right)\right) - u_2 v_2'.$$

Therefore the transformation given by equations (6), (7) still applies, giving an equivalent modified system. The modified operators, A_m, B_m, C_m and D_m are

$$A_m = A + v_2 T_0, \qquad \mathscr{D}(A_m) = \mathscr{D}(T),$$

$$B_m = B + v_2 T_0 + \lambda G_\lambda v_2', \qquad \mathscr{D}(B_m) = \mathscr{D}(T),$$

$$C_m = C,$$

$$D_m = \begin{pmatrix} A_m & C_m \\ 0 & B_m \end{pmatrix}, \qquad \mathscr{D}(D_m) = \mathscr{D}(T) \times \mathscr{D}(T)$$

where $T_0 f = f'$ for $f \in \mathscr{D}(T_0) = \{f \mid f \in \mathscr{AC} \cap \mathscr{H}, f' \in \mathscr{H}\}$.

Since $\mathscr{D}(T_0) \supset \mathscr{D}(T)$, T_0 is T-bounded with relative bound zero because

$$\lim_{\varkappa \to \infty} \|T_0(T + i\varkappa)^{-1}\| = 0.$$

This gives Part (a). Noting that $v_2 T_0$ is also T-compact, Part (b) is also established except for different bounds on σ_{os} which are easily computable. Part (d) follows unchanged.

6. Perturbations of the Mean Flow

Call $v = (v_1, v_2, v_3)$ a mean flow if the components, v_i, satisfy the conditions of Theorem 1 and Section 5. Fix the complex numbers α, β with $\alpha^2 + \beta^2 \in \mathbb{C} \setminus (-\infty, 0]$. For a mean flow, v, define $D_m = D_m(v)$ and $\sigma_{os} = \sigma_{os}(v)$ as in Sections 5 and 2.

When mean flows differ by a constant, the corresponding σ_{os} are translates of each other. Hence, attention will be restricted to those mean flows for which $\lim_{y \to \infty} v(y) = 0$. Let $\mathscr{M}$ denote the set of all such mean flows. For $v, \tilde{v} \in \mathscr{M}$ define

$$d(v, \tilde{v}) = \sum_{i=1}^{3} \|v_i - \tilde{v}_i\|_\infty + \|v' - \tilde{v}'\|.$$

Let Γ be a positively-oriented circle in $\sigma(D_m(v))^c$ for some $v \in \mathscr{M}$. Inside of the circle there are only finitely many eigenvalues with finite (algebraic) multiplicity [11]. Let n be the total multiplicity ($n = 0$ means that there are no eigenvalues inside Γ). The dimension of Ran P for the projection

$$P(v) = \frac{1}{2\pi i} \int_\Gamma (\lambda - D_m(v))^{-1} \, d\lambda$$

is n [11]. Suppose now that $\tilde{v} \in \mathscr{M}$ and that

$$\sup_{\lambda \in \Gamma} \|(D_m(v) - D_m(\tilde{v}))\,(D_m(v) - \lambda)^{-1}\|$$

is sufficiently small that

$$\|P(\tilde{v}) - P(v)\| = \left\| \frac{1}{2\pi} \int_\Gamma (\lambda - D_m(v))^{-1} (D_m(v) - D_m(\tilde{v})) (D_m(v) - \lambda)^{-1} \right.$$
$$\left. \times (1 - (D_m(v) - D_m(\tilde{v})) (D_m(v) - \lambda)^{-1})^{-1} \, d\lambda \right\| < 1.$$

Then $\dim \operatorname{Ran}(P(v)) = \dim \operatorname{Ran}(P(\tilde{v})) = n$ [11], meaning that the total multiplicity of eigenvalues of D_m inside Γ remains unchanged. These statements along with the usual compactness argument give the following theorem.

Theorem 5. *If K is a compact set in $\mathbb{C} \setminus [0, \infty)$ and $v \in \mathcal{M}$, then for any $\varepsilon > 0$ there exist an open set, $\mathcal{O} \supset K$, such that $\operatorname{dist}(K, \mathcal{O}^c) < \varepsilon$ and a $\delta > 0$ such that for each $\tilde{v} \in \mathcal{M}$ with $d(v, \tilde{v}) < \delta$, $\sigma_{os}(v) \cap K$ and $\sigma_{os}(\tilde{v}) \cap \mathcal{O}$ contain eigenvalues of the same total multiplicity.*

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Virginia Polytechnic Institute and State University
Blacksburg, Virginia

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