Eigenvalues of the Orr-Sommerfeld Equation in an Unbounded Domain

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Communicated by D. D. JOSEPH

1. Introduction

The main purpose of this paper is to prove that in the space $(L^2(0, \infty))^4$ the generalized Orr-Sommerfeld equation [10] has only finitely many eigenvalues when the mean flow exponentially approaches a constant. This surprising fact was discovered by numerical studies of eigenvalues for Blasius mean flow [4, 9]. It has been proven [10] that when the mean flow approaches a constant slowly enough, the generalized Orr-Sommerfeld equation can have infinitely many eigenvalues. There is also a nontrivial condition [10] (involving the Reynolds number) which implies that the Orr-Sommerfeld equation has no eigenvalues. The proof given here is based on Lemma 2, which can be considered a generalization of some standard results [*e.g.* 3, 11].

Given are several properties of, and bounds for eigenvalues which can be used to estimate the critical Reynolds number and to help in the numerical search for eigenvalues.

An expectation [12] that eigenvalues should not be imbeded in the continuous spectrum [10] is also proven. These facts may suggest a way [3, 10] to obtain a spectral resolution. One can show, however, that there can exist (finitely many) spectral singularities not corresponding to eigenvalues, *i.e.* -1 can be an eigenvalue of $RQ_0(\sqrt{-Rz})$ (see Section 3) even if z is not an eigenvalue of the generalized Orr-Sommerfeld equation. In such a case it is still possible to define a spectral resolution in a suitable subspace [3, 10]. If the Reynolds number is sufficiently small, then the corresponding operator is spectral. This can easily be seen from [3, 7, 10]. Since these spectral results are rather far from what one would want [1, 2, 12] and since the proofs are very cluttered, details will not be presented.

In Section 2 the main theorem is given. The idea of the proof is worked out in Section 3 and the proof of the main theorem is presented in Section 4.

I wish to thank MICHAEL WILLIAMS for suggesting the problem and I am grateful to DAVID ISAACSON for a critical reading of the manuscript.

2. The Main Theorem

The generalized Orr-Sommerfeld equation is given by [6, 10]

$$\frac{\partial u_1}{\partial t} - \frac{1}{R}\frac{\partial^2 u_1}{\partial y^2} + h(y)\, u_1 + v_1'(y)\, u_2 + i\alpha p = 0, \tag{1}$$

$$\frac{\partial u_2}{\partial t} - \frac{1}{R}\frac{\partial^2 u_2}{\partial y^2} + h(y)\, u_2 + \frac{\partial p}{\partial y} = 0, \tag{2}$$

$$\frac{\partial u_3}{\partial t} - \frac{1}{R}\frac{\partial^2 u_3}{\partial y^2} + h(y)\, u_3 + v_3'(y)\, u_2 + i\beta p = 0, \tag{3}$$

$$i\alpha u_1 + i\beta u_3 + \frac{\partial u_2}{\partial y} = 0, \tag{4}$$

$$u_j(0, t) = 0 \quad \text{for} \quad j = 1, 2, 3 \tag{5}$$

where $h(y) = (\alpha^2 + \beta^2)/R + i\alpha v_1(y) + i\beta v_3(y)$ and the primes denote derivatives. $u = u(y, t) = (u_1, u_2, u_3)$ and $p = p(y, t)$ denote the velocity and pressure of the fluid at a point $y \geqq 0$ and time $t \geqq 0$ respectively; R as usual is the Reynolds number. v_1 and v_3 are the x and z components of the mean flow while α and β are the wave numbers in the x and z directions of the mean flow.

Throughout $\mathscr{H}(\mathscr{H}^j)$ denotes the Hilbert space $L^2(0, \infty)$ (j-fold product of $L^2(0, \infty)$). The set of all complex-valued functions which are absolutely continuous on $[0, a]$ for every $a > 0$ is denoted by $\mathscr{AC}$.

A map (u, p) from the interval $(0, \infty)$ into $\mathscr{H}^4$ is said to be a solution of equations (1–5) if for each $t \in (0, \infty)$ the following conditions are satisfied [10]:

(i) $u_j, p, \dfrac{\partial u_j}{\partial y} \in \mathscr{H} \cap \mathscr{AC}$, $\dfrac{\partial^2 u_j}{\partial y^2} \in \mathscr{H}$ for $j = 1, 2, 3$ and $u = (u_1, u_2, u_3)$

(ii) u is continuously differentiable in t

(iii) $\dfrac{\partial u_2}{\partial t} \in \mathscr{AC}$ and $\dfrac{\partial^2 u_2}{\partial t\, \partial y} = \dfrac{\partial^2 u_2}{\partial y\, \partial t}$

(iv) (u, p) satisfy equations (1–5)

(v) $\lim_{t \to 0+} u(t)$ exists.

$\mathscr{S}_0$ is the set of all such maps. $(u, p) \in \mathscr{S}_0$ is an eigenvector if $u(t) = e^{-zt} u_0$ and $p(t) = e^{-zt} p_0$ for some $z \in \mathbb{C}$, $u_0 \in \mathscr{H}^3 \setminus \{0\}$ and $p_0 \in \mathscr{H}$. The set of all such z is denoted by σ_{0s}.

The main theorem may now be stated.

Theorem 1. *Suppose:*

(i) $v_1, v_3 \in \mathscr{AC}$,

(ii) *the limits* $\lim_{y \to \infty} v_1(y) = \bar{v}_1$ *and* $\lim_{y \to \infty} v_3(y) = \bar{v}_3$ *exist and are finite,*

(iii) $v_1 - \bar{v}_1, v_3 - \bar{v}_3, v_1', v_3' \in \mathscr{H}$,

(iv) $\alpha^2 + \beta^2 \in \mathbb{C} \setminus (-\infty, 0]$, $\quad R > 0$.

Let $\lambda = \sqrt{\alpha^2 + \beta^2}$, $\operatorname{Re}(\lambda) > 0$, $\mu = \lambda^2/R + i\alpha\bar{v}_1 + i\beta\bar{v}_3$ *and* $g_1 = \alpha(\bar{v}_1 - v_1) + \beta(\bar{v}_3 - v_3)$. *Then*

a) *If* $g_1, g_1' \in L^1(0, \infty)$, *then for every* $z \in \sigma_{0s}$

$$|z - \mu| \leqq R\left(\|g_1\|_1 + \frac{2|\lambda| + 1}{(\operatorname{Re}(\lambda))^2}\|g_1'\|_1\right)\left(\|g_1\|_1 + \left(\frac{2|\lambda|}{(\operatorname{Re}(\lambda))^2} + 1\right)\|g_1'\|_1\right).$$

b) *If for some* $\varepsilon > 0$ $\int_0^\infty e^{\varepsilon x}|g_1'(x)|\,dx < \infty$, *then* σ_{0s} *is finite.*

c) *If* g_1 *and* λ *are real valued, then for every* $z \in \sigma_{0s}$

$$\operatorname{Re}(z - \mu) > -\frac{1}{\lambda}\inf_{p \in [2,\infty]}\left(\frac{\lambda}{2}\right)^{\frac{1}{p}}\|g_1'\|_p.$$

d) *If* $g_1' \in \mathscr{AC}$, $g_1'' \in \mathscr{H}$, $\lambda > 0$ *and if* $g_1(x) > 0$, $2\lambda^2 g_1(x) + g_1''(x) \geqq 0$ *for all* $x \in (0, \infty)$, *then* $\operatorname{Im}(z - \mu) < 0$ *for every* $z \in \sigma_{0s}$.

Remark: If $v_3 = 0$ and v_1 is the usual Blasius mean flow, then the assumptions in parts a and b are satisfied. If, in addition, $\alpha > 0$ and $\beta \in \mathbb{R}$ then the assumptions in parts c and d are also satisfied.

3. Preliminaries

In this section the stage is set for the proof of parts a and b of the Main Theorem. The main idea is represented in the following lemmas. The notation used is standard [8]; for $a \in \mathbb{R}$ let $\mathscr{V}(a) = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > a\}$.

Lemma 1. *Suppose:*

(i) $T, A_1, \ldots, A_n, B_1, \ldots, B_n$ *are operators on a Banach space* X, $R \in (0, \infty)$. *Set* $S = \frac{1}{R}T + B_1A_1 + \ldots + B_nA_n$.

(ii) *There exists a family of operators* $K(z)$ *on* X *for* $z \in \overline{\mathscr{V}}(0)$ *such that* $K(z)(T + z^2)f = f$ *for all* $f \in \mathscr{D}(T)$ *and all* $z \in \overline{\mathscr{V}}(0)$.

(iii) *There exists a family of operators* $C_i(z)$ *on* X *for* $z \in \overline{\mathscr{V}}(0) \setminus \{0\}$, $i = 1, \ldots, n$ *such that* $C_i(z) \supset A_iK(z)$ *and* $\operatorname{Range}(B_j) \subset \mathscr{D}(C_i(z))$ *for all* $z \in \overline{\mathscr{V}}(0) \setminus \{0\}$ *and all* $i, j \in \{1, \ldots, n\}$.

(iv) *There exists a family of operators* $Q_{ij}(z)$ *on* X *for* $z \in \overline{\mathscr{V}}(0) \setminus \{0\}$, $i, j \in \{1, \ldots, n\}$ *such that* $Q_{ij}(z) \supset C_i(z)B_j$ *and* $\|Q_{ij}(z)\| \leqq q_{ij}/|z| < \infty$ *for all* $z \in \overline{\mathscr{V}}(0) \setminus \{0\}$ *and all* $i, j \in \{1, \ldots, n\}$.

Then for every $z \in \sigma_p(S)$

$$|z| \leqq R\sum_{ij} q_{ij}^2.$$

Lemma 2. *If assumptions* (i) *through* (iv) *of Lemma 1 are satisfied and if there is an* $\varepsilon > 0$ *such that* $Q_{ij}(z)$ *are holomorphic families of compact operators on* X *for* $z \in \mathscr{V}(-\varepsilon)$ *and* $i, j \in \{1, \ldots, n\}$, *then* $\sigma_p(S)$ *is a finite set.*

Proof. Suppose that $z \in \sigma_p(S) \setminus \{0\}$. Let $f \neq 0$ be such that

$$\sum_{i=1}^{n} B_i(A_i f) = \left(z - \frac{1}{R} T\right) f.$$

Then

$$RK\left(\sqrt{-Rz}\right) \sum_{i=1}^{n} B_i(A_i f) = -f, \quad \operatorname{Re}\left(\sqrt{-Rz}\right) \geqq 0,$$

$$RA_j K\left(\sqrt{-Rz}\right) \sum_{i=1}^{n} B_i(A_i f) = -A_j f, \text{for } j = 1, \ldots, n,$$

$$RC_j\left(\sqrt{-Rz}\right) \sum_{i=1}^{n} B_i(A_i f) = -A_j f,$$

$$R \sum_{i=1}^{n} Q_{ji}\left(\sqrt{-Rz}\right)(A_i f) = -A_j f.$$

Let $x = (A_1 f, \ldots, A_n f) \in X^n$ and let $Q_0(\xi)$ be the matrix $\{Q_{ij}(\xi)\}_{ij}$. Clearly, $x \neq 0$ and

$$1 \leqq \left\| RQ_0\left(\sqrt{-Rz}\right) \right\|^2 \leqq \frac{R}{|z|} \sum_{ij} q_{ij}^2,$$

which proves Lemma 1. Lemma 2 is now obvious [8].

Now several operators on $\mathscr{H}$ will be introduced. For $z \in \mathscr{V}(0)$ and $g \in \mathscr{H}$, define $F_z, G_z \in \mathscr{B}(\mathscr{H})$ by

$$(F_z g)(x) = \int_0^x e^{z(s-x)} g(s)\, ds$$

and

$$(G_z g)(x) = \int_x^\infty e^{z(x-s)} g(s)\, ds.$$

The operator T is defined by $Tf = -f''$ for $f \in \mathscr{D}(T) = \{f \mid f, f' \in \mathscr{H} \cap \mathscr{AC}, f'' \in \mathscr{H}, f(0) = 0\}$.

For $z \in \mathbb{C}$ and $x, y \in [0, \infty)$ define

$$k(z, x, y) = \int_0^{\min\{x,y\}} e^{z(2s-x-y)}\, ds.$$

Observe that $|k(z, x, y)| \leqq \dfrac{1}{|z|}$ for $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$. If $\varepsilon \in (0, \infty)$, $\delta \in [0, \varepsilon)$ and $z \in \mathscr{V}(-\delta)$ then

$$|k(z, x, y)| \leqq \frac{1}{\varepsilon - \delta} e^{\varepsilon(x+y)}. \tag{6}$$

If $\xi \in \mathbb{C} \setminus \{0\}$, $\varepsilon \in (0, \infty)$, $\delta \in [0, \varepsilon)$ and $z \in \mathscr{V}(|\xi| - \delta)$, then

$$\left| \frac{k(z+\xi, x, y) - k(z, x, y)}{\xi} - \frac{\partial k(z, x, y)}{\partial z} \right| \leqq |\xi| \left(\frac{3}{\varepsilon - \delta} \right)^3 e^{\varepsilon(x+y)}. \qquad (7)$$

Define the family of operators $K(z)$ for $z \in \bar{\mathscr{V}}(0)$ by $\mathscr{D}(K(z)) = \{f \in \mathscr{H} \mid$ for all $x \in [0, \infty)$ $\lim_{s\to\infty} \int_0^s k(z, x, y) f(y)\, dy \equiv g(x)$, and $g \in \mathscr{D}(T)\}$,

$$(K(z)f)(x) = \lim_{s\to\infty} \int_0^s k(z, x, y) f(y)\, dy \text{ for } f \in \mathscr{D}(K(z)).$$

Integration by parts gives $K(z)(T + z^2) f = f$ for every $f \in \mathscr{D}(T)$ and every $z \in \bar{\mathscr{V}}(0)$. Note that if $z \in \mathbb{C}$ and $e^{-z(\cdot)} f(\cdot) \in L^1(0, \infty)$, then $k(z, x, \cdot) f(\cdot) \in L^1(0, \infty)$ for all $x \in [0, \infty)$.

Suppose that $h_1, h_2 \in \mathscr{H}$ and that $\lambda, \lambda_1 \in \mathscr{V}(0)$. In $\mathscr{H}$ define operators A and B in the following way:

Case I: $A = h_1$, $B = h_2$,

Case II: $A = h_1$, $B = G_{\lambda_1} h_2$,

Case III: $A = h_1 F_\lambda$, $B = h_2$,

Case IV: $A = h_1 F_\lambda$, $B = G_{\lambda_1} h_2$.

$G_{\lambda_1} h_2$ is a product of operators G_{λ_1} and the multiplication operator h_2.

Case I. Define the family $C(z)$ for $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$ by $\mathscr{D}(C(z)) = \{f \in \mathscr{H} \mid$ for all $x \in [0, \infty)$ $\lim_{s\to\infty} \int_0^s k(z, x, y) f(y)\, dy \equiv g(x)$, and $g h_1 \in \mathscr{H}\}$,

$$(C(z)f)(x) = h_1(x) \lim_{s\to\infty} \int_0^s k(z, x, y) f(y)\, dy \text{ for } f \in \mathscr{D}(C(z)).$$

Clearly, $C(z) \supset AK(z)$ for all $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$.

For $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$ define the family $Q(z)$ by

$$(Q(z)f)(x) = h_1(x) \int_0^\infty k(z, x, y)\, h_2(y) f(y)\, dy, \quad f \in \mathscr{H}.$$

Clearly, $\|Q(z)\| \leqq \|h_1\|_2 \|h_2\|_2 / |z|$, Range $(B) \subset \mathscr{D}(C(z))$ and $C(z) B \subset Q(z)$ *for* all $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$.

Case II. Define the family $C(z)$ as in Case I. For $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$ define the family $Q(z)$ by

$$(Q(z)f)(x) = h_1(x) \int_0^\infty k(z, x, y) \left(\int_y^\infty e^{\lambda_1(y-s)}\, h_2(s) f(s)\, ds \right) dy$$

$$= h_1(x) \int_0^\infty \left(\int_0^y k(z, x, s)\, e^{\lambda_1(s-y)}\, ds \right) h_2(y) f(y)\, dy, \; f \in \mathscr{H}.$$

Thus $\|Q(z)\| \leq \frac{\|h_1\|_2 \, \|h_2\|_2}{|z| \operatorname{Re}(\lambda_1)}$, Range $(B) \subset \mathscr{D}(C(z))$ and $C(z)\, B \subset Q(z)$ for all $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$.

Case III. Now, define the family $C(z)$ for $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$ by $\mathscr{D}(C(z)) = \{f \in \mathscr{H} |$ for all $x \in [0, \infty)$, $\lim_{s \to \infty} \int_0^s k(z, x, y) f(y)\, dy \equiv g(x)$ and if $h(x) = \int_0^x e^{\lambda(s-x)} g(s)\, ds$ then $h h_1 \in \mathscr{H}\}$,

$$(C(z) f)(x) = h_1(x) \int_0^x e^{\lambda(t-x)} \left(\lim_{s \to \infty} \int_0^s k(z, t, y) f(y)\, dy \right) dt, \quad f \in \mathscr{D}(C(z)).$$

Clearly, $C(z) \supset AK(z)$ for all $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$.

For $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$ and $f \in \mathscr{H}$ let

$$(Q(z) f)(x) = \int_0^x h_1(x)\, e^{\lambda(s-x)} \left(\int_0^\infty k(z, s, y)\, h_2(y) f(y)\, dy \right) ds$$

$$= \int_0^\infty h_1(x) \left(\int_0^x k(z, s, y)\, e^{\lambda(s-x)}\, ds \right) h_2(y) f(y)\, dy.$$

Again $\|Q(z)\| \leq \frac{\|h_1\|_2 \, \|h_2\|_2}{|z| \operatorname{Re}(\lambda)}$, Range $(B) \subset \mathscr{D}(C(z))$ and $C(z)\, B \subset Q(z)$ for all $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$.

Case IV. Let the family $C(z)$ be as in Case III. For $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$ and $f \in \mathscr{H}$ define

$$(Q(z) f)(x) = h_1(x) \int_0^x e^{\lambda(t-x)} \left(\int_0^\infty k(z, t, y) \left(\int_y^\infty e^{\lambda_1(y-s)}\, h_2(s) f(s)\, ds \right) dy \right) dt$$

$$= \int_0^\infty h_1(x) \left(\int_0^x dt \int_0^y ds\, k(z, t, s)\, e^{\lambda(t-x) + \lambda_1(s-y)} \right) h_2(y) f(y)\, dy.$$

Thus $\|Q(z)\| \leq \frac{\|h_1\|_2 \, \|h_2\|_2}{|z| \operatorname{Re}(\lambda) \operatorname{Re}(\lambda_1)}$, Range $(B) \subset \mathscr{D}(C(z))$ and $C(z)\, B \subset Q(z)$ for all $z \in \bar{\mathscr{V}}(0) \setminus \{0\}$.

If, in addition, there is an $\varepsilon > 0$ such that $\int_0^\infty |h_i(x)\, e^{\varepsilon x}|^2\, dx < \infty$, then inequalities (6) and (7) imply that in all of the above cases $Q(z)$ can be extended to a holomorphic family of compact operators for $z \in \mathscr{V}(-\varepsilon)$.

3. Proof of the Main Theorem

Parts a and b. It has been shown [10] that

$$\sigma_{0s} - \mu \subset \sigma_p(D_{11}) \cup \sigma_p(D_{22})$$

where

$$D_{11} = \frac{1}{R} T - ig_1,$$

$$D_{22} = \frac{1}{R} T - ig_1 - 2i\lambda G_\lambda g_1' F_\lambda + ig_1' F_\lambda.$$

Therefore, it is enough to prove the following theorem.

Theorem 2. *Suppose that* $\phi_1, \phi_2, \phi_3 \in L^2(0, \infty) \cap L^1(0, \infty)$, $R \in (0, \infty)$ *and* $\lambda, \lambda_1 \in \mathscr{V}(0)$. *Set* $S = \frac{1}{R} T + \phi_1 + G_{\lambda_1} \phi_2 F_\lambda + \phi_3 F_\lambda$. *Then*

a) *For every* $z \in \sigma_p(S)$

$$|z| \leqq R \left(\|\phi_1\|_1 + \frac{\|\phi_2\|_1}{(\mathrm{Re}\,(\lambda))^2} + \frac{\|\phi_3\|_1}{(\mathrm{Re}\,(\lambda))^2} \right) \left(\|\phi_1\|_1 + \frac{\|\phi_2\|_1}{(\mathrm{Re}\,(\lambda_1))^2} + \|\phi_3\|_1 \right).$$

b) *If, in addition, there is an* $\varepsilon > 0$ *such that* $\int_0^\infty |\phi_i(x)|\, e^{\varepsilon x}\, dx < \infty$, $i = 1, 2, 3$ *then* $\sigma_p(S)$ *is finite.*

Proof. Define

$$A_1 = |\phi_1|^{\frac{1}{2}}, \quad B_1 = \mathrm{sgn}\,(\phi_1)\, |\phi_1|^{\frac{1}{2}},$$

$$A_2 = |\phi_2|^{\frac{1}{2}} F_\lambda, \quad B_2 = G_{\lambda_1}\, \mathrm{sgn}\,(\phi_2)\, |\phi_2|^{\frac{1}{2}},$$

$$A_3 = |\phi_3|^{\frac{1}{2}} F_\lambda, \quad B_3 = \mathrm{sgn}\,(\phi_3)\, |\phi_3|^{\frac{1}{2}},$$

where $\mathrm{sgn}\,(\phi)\,(x) = \phi(x)/|\phi(x)|$ if $\phi(x) \neq 0$ and 1 otherwise. B_2 is considered as a product of operators. Hence

$$S = \frac{1}{R} T + B_1 A_1 + B_2 A_2 + B_3 A_3.$$

Define the families $C_i(z)$, $Q_{ij}(z)$ as in the above cases. An application of Lemma 1 and Lemma 2 completes the proof.

Parts c and d. Suppose that $\lambda > 0$ and that g_1 is a real valued function. If $z \in \sigma_p(D_{11})$, then

$$\frac{1}{R} Tf - ig_1 f = zf, \quad f \in \mathscr{D}(T) \setminus \{0\},$$

so that

$$\|f\|_2^2\, \mathrm{Im}\,(z) = -(g_1 f, f),$$

$$\|f\|_2^2\, \mathrm{Re}\,(z) = \frac{1}{R} \|f'\|_2^2.$$

If $z \in \sigma_p(D_{22})$, then

$$\frac{1}{R} Tf - ig_1 f - 2i\lambda G_\lambda g_1' F_\lambda f + ig_1' F_\lambda f = zf, \quad f \in \mathscr{D}(T) \setminus \{0\}.$$

Hence

$$\|f\|_2^2 \operatorname{Re}(z) = \frac{1}{R}\|f'\|_2^2 - \operatorname{Im}(g_1' F_\lambda f, f)$$
$$> -\|f\|_2^2 \frac{1}{\lambda} \inf_{p\in[2,\infty]} \left(\frac{\lambda}{2}\right)^{\frac{1}{p}} \|g_1'\|_p$$

which proves part c. This bound is somewhat weaker than those obtained in the bounded domain [5]; however, it does not require that $g_1' \in L^\infty(0, \infty)$. Assuming, in addition, that $g_1' \in \mathscr{AC}$ and $g_1'' \in \mathscr{H}$ gives

$$-\|f\|_2^2 \operatorname{Im}(z) = (g_1 f, f) + 2\lambda(g_1' F_\lambda f, F_\lambda f) - \operatorname{Re}(g_1' F_\lambda f, f)$$
$$= (g_1 f, f) + 2\lambda(g_1' F_\lambda f, F_\lambda f) - \left(\left(\lambda g_1' - \frac{1}{2} g_1''\right) F_\lambda f, F_\lambda f\right)$$
$$= (g_1 (F_\lambda f)', (F_\lambda f)') + \left(\left(\lambda^2 g_1 + \frac{1}{2} g_1''\right) F_\lambda f, F_\lambda f\right)$$

which proves part d. Note that this equality can also give bounds on Im (z), which are similar to those in [5].

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(Received May 18, 1982)