

MidMichigan Mathematical Olympiad
April 25, 2009
Solutions grades 7-9

1. Arrange the whole numbers 1 through 15 in a row so that the sum of any two adjacent numbers is a perfect square. In how many ways this can be done?

Solution. There are two solutions

8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7, 9,

or, in the reverse order

9, 7, 2, 14, 11, 5, 4, 12, 13, 3, 6, 10, 15, 1, 8.

To show that these are only two possible sequences note that there are only following complete squares that can be obtained as sums: 4, 9, 16, 25. Number 1 is less than any sum while $36 > 14 + 15$. Hence for the neighbor of number 8 there is only one possibility 1. The only neighbor of 9 is 7. Therefore the sequence, if exists, must have 8 and 9 at the ends. Similarly we see that number 7 can have only two neighbors 9 and 2, number 2 can have only two neighbors 7 and 14 etc. Therefore we can restore the whole sequence.

2. Prove that if p and q are prime numbers which are greater than 3 then $p^2 - q^2$ is divisible by 24.

Solution. To show that $p^2 - q^2$ is divisible by 24, it is enough to show that it is divisible by 8 and simultaneously is divisible by 3. Let us show that $p^2 - q^2$ is divisible by 8. Note that $p^2 - q^2 = (p - q)(p + q)$. Since both p and q are odd, we notice that both $p - q$ and $p + q$ are even, and moreover, one of them is divisible by 4. Therefore the product is divisible by 8. To show that they are divisible by 3 let us consider again $(p - q)(p + q)$. Since both p and q are primes their remainders modulo 3 are 1 or 2. If their remainder are the same then $p - q$ is divisible by 3. If they have different remainders then $p + q$ has remainder 0 modulo 3 and , hence, is divisible by 3.

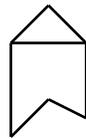
3. If a polyleg has even number of legs he always tells truth. If he has an odd number of legs he always lies. Once a green polyleg told a dark-blue polyleg "- I have 8 legs. And you have only 6 legs!" The offended dark-blue polyleg replied "-It is me who has 8 legs, and you have only 7 legs!" A violet polyleg added "-The dark-blue polyleg indeed has 8 legs. But I have 9 legs!" Then a stripped polyleg started "None of you has 8 legs. Only I have 8 legs!"

Which polyleg has exactly 8 legs?

Solution. Note that if a polyleg claims that it has odd number of legs it lies. So,

- The violet polyleg lies, and the dark-blue does not have 8 legs.
- The dark-blue polyleg lies and it has odd number of legs.
- The green polyleg lies and it can not have 8 legs.
- The stripped polyleg tells the truth and it indeed has 8 legs.

4. There is a small puncture (a point) in the wall (as shown in the figure below to the right). The housekeeper has a small flag of the following form (see the figure left). Show on the figure all the points of the wall where you can hammer in a nail such that if you hang the flag it will close up the puncture.



flag



puncture

Solution. Let us denote the position of of puncture by P and the position of the nail by N . To shade the area of N let us note that if we turn the wall bottom up, put a nail in the puncture and a flag then it will exactly cover the whole area of N . Hence the following picture explains what part of the wall we have to shadow.



5. Assume a, b, c are odd integers. Show that the quadratic equation $ax^2 + bx + c = 0$ has no rational solutions. (A number is said to be rational, if it can be written as a fraction of two integers).

Solution. We have to show that the discriminant of the quadratic equation $D = b^2 - 4ac$ is not a perfect square. Note that D is odd number. Assume that it is a perfect square, $D = Q^2$. Then Q is also odd. Its remainder modulo 8 can be 1, 3, 5, or 7. Then the remainder of D modulo 8 is 1. On the other hand, the remainder of b^2 modulo 8 is 1 while the remainder of $4ac$ modulo 8 is 4 since both a and b are odd. Hence the remainder of $b^2 - 4ac$ modulo 8 must be 5. The contradiction shows that D is not a perfect square.