MATH OLYMPIAD 2010
10–12 grades

Solutions

1. Find all solutions a, b, c, d, e, f, g if the letters represent distinct digits and satisfy the following:

$$\begin{array}{cccccc} & & a & b & c & b \\ \times & & & & a & b \\ \hline & c & d & b & d & b \\ + \; c & e & b & f & b & \\ \hline c & g & a & e & g & b \end{array}$$

Solution: When we multiply b by itself, we obtain a number that ends with b. Thus b is 0, 1, 5, or 6. b cannot be 0: otherwise the result of multiplication by b would be 0. b cannot be equal to 1. Otherwise $\overline{abcb}$ multipled by b cannot be equal to $\overline{cdbdb}$.

Suppose that $b = 6$. Then b times a ands with b. Since $a \neq 6$, a must be equal to 1. Then $\overline{abcb}$ would be equal to $\overline{cgaegb}$, which is impossible. Thus $b = 5$.

When we multiply $\overline{a5c5}$ by 5, we get $\overline{cd5d5}$, whose third digit is 5. Therefore $\overline{c5}$ times 5 is less than 100. The only possibility is that $c = 1$.

b times a ends with 5. Thus a is odd. On the other hand $\overline{abcb}$ times a starts with 1. Thus $a \leq 3$. Hence, $a = 3$.

We can multiply now 3515 by 35 to find the remaining digits. We obtain $d = 7$, $e = 0$, $f = 4$, $g = 2$.

2. 5 numbers are placed on the circle. One finds that the sum of any two neighboring numbers is not divisible by 3, and the sum of any three consecutive numbers is not divisible by 3. How many numbers on the circle are divisible by 3?

Solution: Clearly, among the numbers there are those that are not divisible by 3. Let us take one such number. Suppose that it has remainder 1 when divided by 3.

Consider two neighbors. None of the neighbors can have remainder 2. Otherwise the sum of the first number and that neighbor is divisible by 3. At least one of the neighbors must be divisible by 3. Otherwise, all the three consecutive numbers have remainder 1 and their sum is divisible by 3.

Assume that the left neighbor is divisible by 3. Then the left neighbor of this number divisible by 3 has remainder 1. Otherwise, we have two consecutive numbers divisible by 3 or 3 consecutive numbers divisible by 3.

The two remaining numbers must have remainders 1 and 0. Otherwise we can find 2 consecutive or 3 consecutive numbers whose sum is divisible by 3.

Thus in the case when there is a number with remainder 1, there are precisely two numbers divisible by 3.

Now it remains to consider the case when the first chosen number not divisible by 3 has remainder 2. We can argue in the same way as above and show that in this case 3 numbers have remainders 2 and two numbers are divisible by 3. Or, we can multiply all numbers by -1 and obtain a number with remainder one, in which case there precisely two numbers divisible by 3.

Answer: there are two numbers divisible by 3.

3. n teams played in a volleyball tournament. Each team played precisely one game with each of the other teams. If x_j is the number of victories and y_j is the number of losses of the jth team, show that

$$\sum_{j=1}^{n} x_j^2 = \sum_{j=1}^{n} y_j^2.$$

Solution: Each team played $n-1$ games.

Thus $y_j = n - 1 - x_j$, and so

$$y_j^2 = (n-1-x_j)^2 = x_j^2 + (n-1)^2 - 2(n-1)x_j.$$

Hence,

$$\sum_{j=1}^{n} y_j^2 = \sum_{j=1}^{n} x_j^2 + n(n-1)^2 - 2(n-1)\sum_{j=1}^{n} x_j.$$

It remains to observe that

$$\sum_{j=1}^{n} x_j = \frac{n(n-1)}{2}.$$

The last equality is true because the total number of games is $\frac{n(n-1)}{2}$ and each game is won by exactly one team.

4. Three cars participated in the car race: a Ford [F], a Toyota [T], and a Honda [H]. They began the race with F first, then T, and H last. During the race, F was passed a total of 3 times, T was passed 5 times, and H was passed 8 times. In what order did the cars finish?

Solution: Let $\mathrm{T}_p\mathrm{H}$ be the number of times T passed H. We use similar notation for the other cars. We have

$$\mathrm{T}_p\mathrm{F} + \mathrm{H}_p\mathrm{F} = 3,$$

$$\mathrm{H}_p\mathrm{T} + \mathrm{F}_p\mathrm{T} = 5,$$

$$\mathrm{T}_p\mathrm{H} + \mathrm{F}_p\mathrm{H} = 8.$$

On the other hand since at the beginning T was ahead of H, we have

$$\mathrm{T}_p\mathrm{H} \le \mathrm{H}_p\mathrm{T}.$$

Similarly,

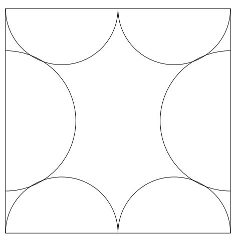
$$\mathrm{F}_p\mathrm{H} \le \mathrm{H}_p\mathrm{F}.$$

Thus

$$8 = \mathrm{T}_p\mathrm{H}+\mathrm{F}_p\mathrm{H} \le \mathrm{H}_p\mathrm{T}+\mathrm{H}_p\mathrm{F} \le \mathrm{H}_p\mathrm{T}+\mathrm{F}_p\mathrm{T}+\mathrm{T}_p\mathrm{F}+\mathrm{H}_p\mathrm{F} = 5+3 = 8.$$

Thus $\mathrm{T}_p\mathrm{H} = \mathrm{H}_p\mathrm{T} = 5$ and $\mathrm{F}_p\mathrm{H} = \mathrm{H}_p\mathrm{F} = 3$. Thus H passed T the same number of times as T passed H, T passed F the same number of times as F passed T, while H never passes F and F never passed H. It follows that the cars finished in the same order they started.

5. The side of the square is 4 cm. Find the sum of the areas of the six half-disks shown on the picture.



Solution: Obviously, the radius of each half-disk on the bottom and on the top side is 1. Let us now connect the center of the left half-disk on the top side with the center of the half-disk on the left side. By the Pythagoras theorem, the sum of their radii is $\sqrt{5}$. Hence, the radius of the disks on the left side and on the right side is $\sqrt{5}-1$. Thus the sum of the areas is equal to

$$4 \cdot \frac{\pi}{2} + 2\frac{\pi(\sqrt{5}-1)^2}{2} = \pi(2 + (\sqrt{5}-1)^2) = (8-2\sqrt{5})\pi.$$