

# MATH OLYMPIAD 2010

7–9 grades

## Solutions

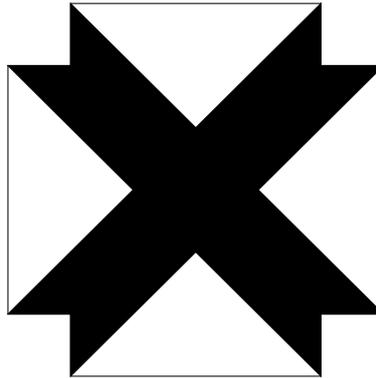
1. Find the smallest whole number  $n \geq 2$  such that the product

$$(2^2 - 1)(3^2 - 1) \cdots (n^2 - 1)$$

is the square of a whole number.

*Solution:*  $(2^2 - 1) \cdots (n^2 - 1) = 1 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot 5 \cdots (n - 1) \cdot (n + 1) = p^2 \cdot 2 \cdot n \cdot (n + 1)$ , where  $p = 3 \cdot 4 \cdots (n - 1)$ . Hence we need to find a minimal  $n$  such that  $2 \cdot n(n + 1)$  is a complete square. The answer is  $n = 8$ .

2. The figure below shows a  $10 \times 10$  square with small  $2 \times 2$  squares removed from the corners. What is the area of the shaded region?



*Solution:* Move the four remaining parts together. They form a  $6 \times 6$  square. Then the area of the shaded region is  $10 \times 10 - 4 \times (2 \times 2) - 6 \times 6 = 100 - 16 - 36 = 48$ .

3. Three cars are racing: a Ford [F], a Toyota [T], and a Honda [H]. They began the race with F first, then T, and H last. During the race, F was passed a total of 3 times, T was passed 5 times, and H was passed 8 times. In what order did the cars finish?

*Solution:* Let  $T_p H$  be the number of times  $T$  passed  $H$ . We use similar notation for the other cars. We have

$$T_p F + H_p F = 3,$$

$$H_p T + F_p T = 5,$$

$$T_p H + F_p H = 8.$$

On the other hand since at the beginning  $T$  was ahead of  $H$ , we have

$$T_p H \leq H_p T.$$

Similarly,

$$F_p H \leq H_p F.$$

Thus

$$8 = T_p H + F_p H \leq H_p T + H_p F \leq H_p T + F_p T + T_p F + H_p F = 5 + 3 = 8.$$

Thus  $T_p H = H_p T = 5$  and  $F_p H = H_p F = 3$ . Thus  $H$  passed  $T$  an odd number of times,  $T$  passed  $F$  an odd number of times, while  $H$  never passes  $F$ . It follows that the cars finished in the same order they started.

4. There are 11 big boxes. Each one is either empty or contains 8 medium-sized boxes inside. Each medium box is either empty or contains 8 small boxes inside. All small boxes are empty. Among all the boxes, there are a total of 102 empty boxes. How many boxes are there altogether?

*Solutions:* Let  $B_e$  denote the number of empty large boxes,  $M_e$  the number of empty middle sized boxes, and  $S$  the number of empty small boxes. Then  $B_e + M_e + S = 102$ . The total number of medium-sized boxes is  $8(11 - B_e) = 88 - 8B_e$ . The total number of small boxes is  $S = 8(88 - 8B_e - M_e) = 704 - 64B_e - 8M_e$ . Then the total number of empty boxes is  $102 = 704 - 64B_e - 8M_e + B_e + M_e = 704 - 63B_e - 7M_e = 704 - 7(9B_e + M_e)$ . Hence,  $9B_e + M_e = \frac{1}{7}(704 - 102) = 86$ . Now, the total number of boxes is  $11 + 88 - 8B_e + 704 - 64B_e - 8M_e = 803 - 72B_e - 8M_e = 803 - 8(9B_e + M_e) = 803 - 8 \cdot 86 = 115$ .

5. Ann, Mary, Pete, and finally Vlad eat ice cream from a tube, in order, one after another. Each eats at a constant rate, each at his or her own rate. Each eats for exactly the period of time that it would take the three remaining people, eating together, to consume half of the tube. After Vlad eats his portion there is no more ice cream in the tube. How many times faster would it take them to consume the tube if they all ate together?

*Solutions:* Denote the times of eating of the corresponding persons by  $a, m, p,$  and  $v$ . By the  $A_a$  we denote the part of tube consumed by Ann for the time  $a$ , by  $M_p$  we denote the part of tube that Mary may eat up for the time  $p$ . Finally,  $C$  denote the whole tube. Let us assume that all people eat together for the total time  $a + m + p + v$ . Let us figure out how many tubes of ice cream they can consume. We have to compute  $A_a + M_a + P_a + V_a + A_m + M_m + P_m + V_m + A_p + M_p + P_p + V_p + A_v + M_v + P_v + V_v$ . Note that  $A_a + M_m + P_p + V_v = C$ ,  $M_a + P_a + V_a = \frac{C}{2}$ ,  $A_m + P_m + V_m = \frac{C}{2}$ ,  $A_p + M_p + V_p = \frac{C}{2}$ ,  $A_v + M_v + P_v = \frac{C}{2}$ . Adding these equalities, we get  $A_a + M_a + P_a + V_a + A_m + M_m + P_m + V_m + A_p + M_p + P_p + V_p + A_v + M_v + P_v + V_v = 3C$ . Then to consume just one tube together all four persons need the time  $\frac{1}{3}(a + m + p + v)$  that is three times less than it took for them to consume the ice cream tube one after another.