

## GRADES 7-9

1. A straight line is painted in two colors. Prove that there are three points of the same color such that one of them is located exactly at the midpoint of the interval bounded by the other two.

**Answer:** Assume the contrary. Then there are at least two points of each color (RED and BLUE). Choosing the right length unit we can assume that two points with coordinate 0 and 2 are both colored RED. Then, points with coordinate -2, 1 and 4 must be colored BLUE contradicting our assumption.

2. Find all positive integral solutions  $x, y$  of the equation  $xy = x + y + 3$ .

**Answer:** Assume  $x \leq y$ . Then, clearly,  $x \leq 3$ . We have three choices:  $x = 1, 2$ , or  $3$ . If  $x = 1$ , then  $y$  satisfies  $y = y + 4$  which has no solutions. If  $x = 2$ , then  $y$  satisfies  $2y = y + 5$  with only solution  $y = 5$ . If  $x = 3$ , then  $y$  satisfies  $3y = y + 6$  with only solution  $y = 3$ .

Therefore, equation has three solutions  $x = 2, y = 5$  or  $x = 5, y = 2$ , and  $x = y = 3$ .

3. Can one cut a square into isosceles triangles with angle  $80^\circ$  between equal sides?

**Answer:** NO. Indeed, isosceles triangle with angle between equal sides  $80^\circ$  has the second and the third angle of  $50^\circ$ . There is no way to obtain the right angle combining  $80^\circ$  and  $50^\circ$  angles.

4. 20 children are grouped into 10 pairs: one boy and one girl in each pair. In each pair the boy is taller than the girl. Later they are divided into pairs in a different way. May it happen now that

- (a) in all pairs the girl is taller than the boy;
- (b) in 9 pairs out of 10 the girl is taller than the boy?

**Answer:** (a) is impossible. Since for every girl there is a boy which is taller. The tallest person is a boy. Then no matter how we divide children into pairs there is a pair where the boy is taller.

(b) is possible. Indeed, let us order children according to their height (1 is the smallest, 20 is the tallest) and assume that all odd numbers are girls while all even numbers are boys.

Then division into (1, 2), (3, 4), (5, 6), (7, 8), (9, 10), (11, 12), (13, 14), (15, 16), (17, 18), (19, 20) satisfies the first condition, while division (2, 3), (4, 5), (6, 7), (8, 9), (10, 11), (12, 13), (14, 15), (16, 17), (18, 19), and, finally, (1, 20) satisfies condition (b).

5. Mr Mouse got to the cellar where he noticed three heads of cheese weighing 50 grams, 80 grams, and 120 grams. Mr. Mouse is allowed to cut simultaneously 10 grams from any two of the heads and eat them. He can repeat this procedure as many times as he wants. Can he make the weights of all three pieces equal?

**Answer:** YES. Mr. Mouse cuts 7 times from 80 gram and 120 gram cheese heads. Then he cuts 4 times from 50 gram and 120 gram heads. After there are three heads left each weighing 10 grams.