MID-MICHIGAN OLYMPIAD IN MATHEMATICS 2014

PROBLEMS GRADES 5-6

1. Find any integer solution of the puzzle: WE+ST+RO+NG=128 (different letters mean different digits between 1 and 9).

   Solution: there are many solutions, for instance, 15+26+38+49=128

2. A 5x6 rectangle is drawn on the piece of graph paper (see the figure below). The side of each square on the graph paper is 1 cm long. Cut the rectangle along the sides of the graph squares in two parts whose areas are equal but perimeters are different by 2cm.

   Solution:

x	x	x	x	x	x
x	x	x	x	x	x
x			x		x

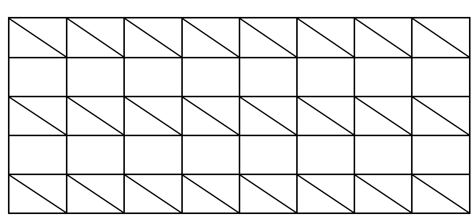
3. Three runners started simultaneously on a 1km long track. Each of them runs the whole distance at a constant speed. Runner A is the fastest. When he runs 400 meters then the total distance run by runners B and C together is 680 meters. What is the total combined distance remaining for runners B and C when runner A has 100 meters left?

   Solution: When runner A covers 900 meters the runners B and C cover together 1530 meters. Therefore they have still 470 meters left to go for the total distance of 2 km.

4. There are three people in a room. Each person is either a knight who always tells the truth or a liar who always tells lies. The first person said «We are all liars». The second replied «Only you are a liar». Is the third person a liar or a knight?

   Solution: The first person can not be knight since he claimed that all three persons in the room (including him) are liars. Then the third person is a knight. Indeed, if the third person would be a liar then the second person is the liar and the first person told the truth.

5. A 5x8 rectangle is divided into forty 1x1 square boxes (see the figure below). Choose 24 such boxes and one diagonal in each chosen box so that these diagonals don't have common points.



MID-MICHIGAN OLYMPIAD IN MATHEMATICS 2014
PROBLEMS GRADES 7-9

1. (a) Put the numbers 1 to 6 on the circle in such way that for any five consecutive numbers the sum of first three (clockwise) is larger than the sum of remaining two.
(b) Can you arrange the 6 integers so it works both clockwise and counterclockwise.

Solution: Place the numbers in the following circular order 254163.

2. A girl has a box with 1000 candies. Outside the box there is an infinite number of chocolates and muffins. A girl may replace:
   - two candies in the box with one chocolate bar;
   - two muffins in the box with one chocolate bar;
   - two chocolate bars in the box with one candy and one muffin;
   - one candy and one chocolate bar in the box with one muffin;
   - one muffin and one chocolate bar in the box with one candy.

   Is it possible that after some time it remains only one object in the box?

   Solution: note first that 1m+1c+1ch➔2c➔1ch.
   Then, 1000c➔500ch➔125c+125m➔62ch+1c+62ch+1m➔61ch+1ch=62ch➔31c+31m➔30c+1c+30m+1m➔15ch+15ch+1c+1m➔30ch➔15c+15m➔15ch➔7c+7m+1ch➔3ch+3ch+1c+1m+1ch➔7ch➔3c+3m+1ch➔1ch+1c+1ch+1m+1ch➔3ch➔1c+1m+1ch➔1ch

3. Find any integer solution of the puzzle: WE+ST+RO+NG=128 (different letters mean different digits between 1 and 9).
Solution: there are many solutions, for instance, 15+26+38+49=128

4. Two consecutive three-digit positive integer numbers are written one after the other one. Show that the six-digit number that is obtained is not divisible by 1001.

Solution: Proof by contradiction. Assume that the 6-digit number is divisible by 1001. The result is three-digit number. Multiplying any three-digit number by 1001 we get the number whose first digit coincides with the 4th one, 2nd with the 5th, and 3rd with the 6th, contradicting to the original condition.

5. There are 9 straight lines drawn in the plane. Some of them are parallel some of them intersect each other. No three lines do intersect at one point. Is it possible to have exactly 17 intersection points?

Solution: If 9 lines intersect in 17 points, then counting number of points each line intersects with the 8 remaining lines, we get 34. Now 34 divided by 4 has remainder 2. So there must be odd number of lines which meet odd number of other lines. Let L be such a line. Then there are an odd number of lines which are parallel to it. But then there are in all even number of lines which are parallel to each other. So we can out these 9 lines in different groups. First there are groups of lines in which each line is parallel to all others and then, there is a group of lines meeting all other 8 lines. The groups containing parallel lines have even number of lines among them and hence the number of lines meeting all other lines is odd. Since 8 x 5 = 40 > 34, this last group has 1 line or 3 lines. If it has 3 lines, then 8 x 3 + 6 x 3 = 42 > 34. So this group must have only 1 line. Now the lines from groups other than the last one meet other lines in 34 – 8 = 26 points. So the groups of lines must have 2 and 6 lines (since the number of groups of lines with 2 mod 4 lines must be 2 mod 4). But then total number of intersection points for these lines = 2 x 7 + 6 x 3 = 32, which is still greater than 26. So there cannot be 17 points of intersection for 9 lines.

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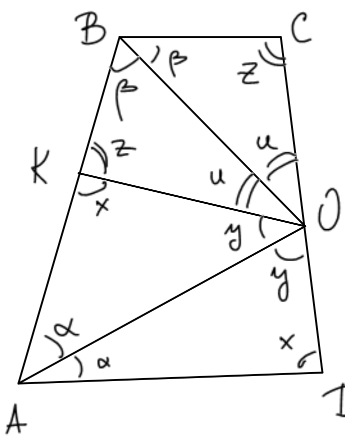
PROBLEMS GRADES 10-12

1. The length of the side AB of the trapezoid with bases AD and BC is equal to the sum of lengths |AD|+|BC|. Prove that bisectors of angles A and B do intersect at a point of the side CD.



Solution: Since $|AB| = |AD| + |BC|$ we can choose point K on side AB such that $|AK| = |AD|$ and $|BK| = |BC|$. Then, $\triangle AKO = \triangle ADO$ ($|AK| = |AD|$, $\angle OAK = \alpha = \angle OAD$, AD is common) and $\triangle BKO = \triangle BCO$ ($|BK| = |BC|$, $\angle KBO = \beta = \angle CBO$, BO is common). Therefore, $\angle BCO = \angle BKO =: z$, $\angle COB = \angle KOB =: u$, $\angle AKO = \angle ADO =: x$, $\angle AOK = \angle AOD =: y$. Note: $\angle COD = 2u + 2y = 2(\pi - \beta - z) + 2(\pi - \alpha - x) =$
$= 4\pi - 2\alpha - 2\beta - 2z - 2x \underset{\text{(since } z+x=\pi)}{=} 2\pi - 2\alpha - 2\beta =$
$= \pi$ (recall, that $2\alpha + 2\beta = \pi$ since $AD \parallel BC$). Therefore, COD form a straight segment and point O belongs to side CD.

2. Polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ and $Q(x) = x^4 + cx^3 + bx^2 + ax + 1$ have two common roots. Find these common roots of both polynomials.
   Solution: Let z be a common root. Then, P(z)-Q(z)=(a-c)z^3-(a-c)z=
   (a-c)z(z^2-1). Z=0 is neither a root of P, nor Q, because the constant term is 1. Then, z=+- 1 are common roots.

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   Then, 1000c➔500ch➔125c+125m➔62ch+1c+62ch+1m➔61ch+1ch=62ch➔
   31c+31m➔30c+1c+30m+1m➔15ch+15ch+1c+1m➔30ch➔15c+15m➔15ch➔7c+7m+1ch➔
   3ch+3ch+1c+1m+1ch➔7ch➔3c+3m+1ch➔1ch+1c+1ch+1m+1ch➔3ch➔1c+1m+1ch➔1ch

4. There are 9 straight lines drawn in the plane. Some of them are parallel some of them intersect each other. No three lines do intersect at one point. Is it possible to have exactly 17 intersection points?

Solution: If 9 lines intersect in 17 points, then counting number of points each line intersects with the 8 remaining lines, we get 34. Now 34 divided by 4 has remainder 2. So there must be odd number of lines which meet odd number of other lines. Let L be such a line. Then there are an odd number of lines which are parallel to it. But then there are in all even number of lines which are parallel to each other. So we can out these 9 lines in different groups. First there are groups of lines in which each line is parallel to all others and then, there is a group of lines meeting all other 8 lines. The groups containing parallel lines have even number of lines among them and hence the number of lines meeting all other lines is odd. Since 8 x 5 = 40 > 34, this last group has 1 line or 3 lines. If it has 3 lines, then 8 x 3 + 6 x 3 = 42 > 34. So this group must have only 1 line. Now the lines from groups other than the last one meet other lines in 34 – 8 = 26 points. So the groups of lines must have 2 and 6 lines (since the number of groups of lines with 2 mod 4 lines must be 2 mod 4). But then total number of intersection points for these lines = 2 x 7 + 6 x 3 = 32, which is still greater than 26. So there cannot be 17 points of intersection for 9 lines.

5. It is known that x is a real number such that $x+\frac{1}{x}$ is an integer. Prove that $x^n+\frac{1}{x^n}$ is an integer for any positive integer n.

Solution: P roof by induction. Note that $x^n+\frac{1}{x^n}=(x^{n-1}+\frac{1}{x^{n-1}})(x+\frac{1}{x})-(x^{n-2}+\frac{1}{x^{n-2}})$. By induction we may assume that all parentheses on the right hand side contain integer numbers.