MidMichigan Olympiad
2018

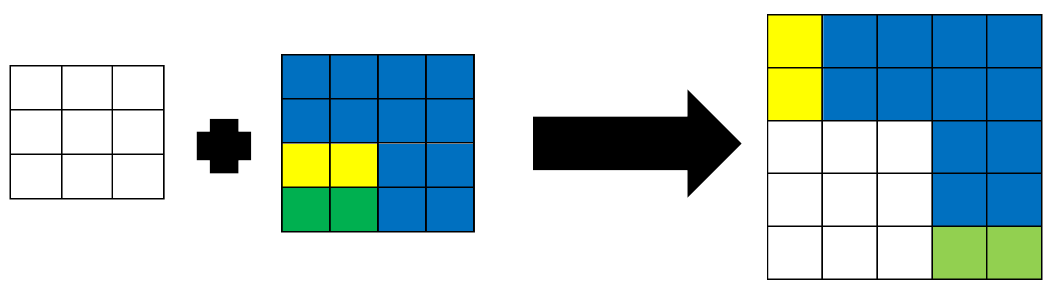
Problems
5-6

1. A Slavic dragon has three heads. A knight fights the dragon. If the knight cuts off one dragon's head three new heads immediately grow. Is it possible that the dragon has 2018 heads at some moment of the fight?

   Solution: *Removing one head and adding three heads is equivalent to adding 2 heads each time. After first decapitating the dragon will get 5 heads, after second 7 heads and so on and so forth. Hence, the number of heads is always odd and the dragon can never have 2018 heads.*

2. Peter has two squares 3x3 and 4x4. He must cut one of them or both of them in no more than four parts in total. Is Peter able to assemble a square using all these parts?

   Solution:



3. Usually, dad picks up Constantine after his music lessons and they drive home. However, today the lessons have ended earlier and Constantine started walking home. He met his dad 14 minutes later and they drove home together. They arrived home 6 minutes earlier than usually. How many minutes earlier than usual have the lessons ended? Please, explain your answer.

Solution: *Dad saved 6/2=3 minutes on the way to school and 3 minutes on the way from school. Hence dad met 3 minutes before usual time. Therefore, Constantine started to walk 14+3=17 minutes before the usual lesson's end. The lesson ended 17 minutes before the usual time.*

4. All positive integers from 1 to 2018 are written on a blackboard. First, Peter erased all numbers divisible by 7. Then, Natalie erased all remaining numbers divisible by 11. How many numbers did Natalie remove? Please, explain your answer.

Solution: *There are 183 numbers divisible by 11 between 1 and 2018. Each 7^{th} of these numbers (7,14,21,etc) is divisible by 7. Totally, 26 of 183 numbers are erased by Peter. Remaining 157 numbers are erased by Natalie.*

5. 30 students took part in a mathematical competition consisting of four problems. 25 students solved the first problem, 24 students solved the second problem, 22 students solved the third, and, finally, 21 students solved the fourth. Show that there are at least two students who solved all four problems.

Solution: *Let's count the number of pairs (student, problem solved by this student). If at most one student solved all four problems that the number of such pairs does not exceed 1x4+29x3=91. On the other hand, this number equals 25+24+22+21=92. Since 92 is larger than 91, this contradiction shows that there are at least two students which solved all four problems.*

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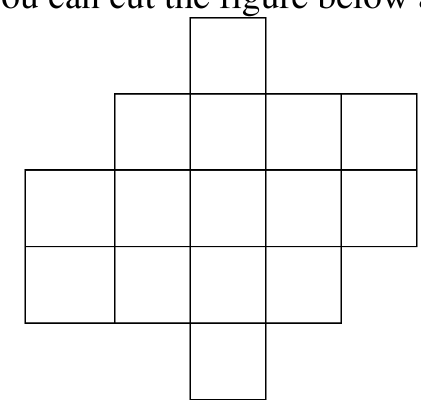
Problems
7-9

1. Is it possible to put 9 numbers 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 , 9 in a circle in a way such that the sum of any three circularly consecutive numbers is divisible by 3 and is, moreover:
   a) greater than 9 ? b) greater than 15?

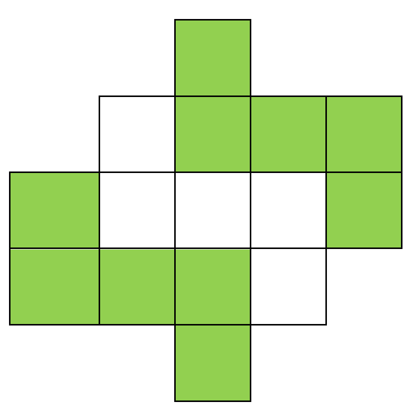
   Solution: *a) YES, Example: 2,4,6,5,7,3,8,1,9. b) NO, the sum of all these numbers is 45. All 9 sums of 3 consecutive numbers contains 27 numbers, each number enters three times. The sum of all such sums equals 45x3=9x15. Hence, not all such sums exceed 15.*

2. You can cut the figure below along the sides of the small squares into



   several (at least two) identical pieces. What is the minimal number of such equal pieces?

   Solution: *In three pieces. The figure contains 15 small squares. Hence each piece contains either 5, or 3, or 1 square. The corresponding number of pieces is 3,5, or 15. Here is the example of cutting into three identical pieces.*

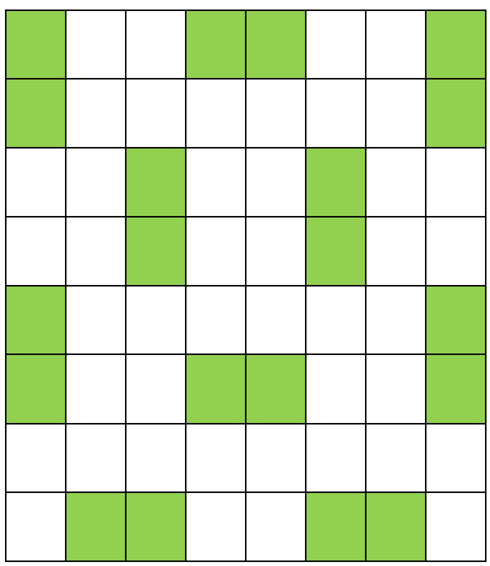


3. There are 100 colored marbles in a box. It is known that among any set of ten marbles there are at least two marbles of the same color. Show that the box contains 12 marbles of the same color.

   Solution: *Assume that there are at most 11 marbles of each color. Then, there are at least10 different colors (otherwise, there are at most 9x11=99 marbles). If we take 1 marble of each color we obtain a contradiction with the problem statement.*

4. Is it possible to color squares of a 8x8 board in white and black color in such a way that every square has exactly one black neighbor square separated by a side?

   Solution: *It is possible. See the figure.*



5. In a basket, there are more than 80 but no more than 200 white, yellow, black, and red balls. Exactly 12% are yellow, 20% are black. Is it possible that exactly 2/3 of the balls are white?

   Solution: *12%=12/100=3/25, 20%=20/100=1/5. If we take the sum of yellow, black, and white balls then the total is 3/25+1/5+2/3=9/75+15/75+50/75=74/75. Then, exactly 1/75 part of all balls is red. Hence, if, 2 balls are red, 100 balls are white, 30 balls are black, and 18 balls are yellow. Totally, there are 150 balls.*

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Problems
10-12

1. Twenty five horses participate in a competition. The competition consists of seven runs, five horse compete in each run. Each horse shows the same result in any run it takes part. No two horses will give the same result. After each run you can decide what horses participate in the next run. Could you determine the three fastest horses? (You don't have stopwatch. You can only remember the order of the horses.)

Solution: *First, we make 5 runs with exactly five horses in each, each horse runs exactly once. Then let fastest horses in each run compete in the 6^{th} run. We get the following table with columns corresponding to runs, each column lists horses top to down fastest horses on top. We write Horse A>Horse B if A is faster than B. Order columns such that Horse A1 > Horse A2 > Horse A3.*

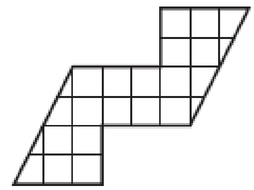
Run1	Run 2	Run 3	Run 4	Run 5
Horse A1	Horse A2	Horse A3	Horse A4	Horse A5
Horse B1	Horse B2	Horse B3	Horse B4	Horse B5
Horse C1	Horse C2	Horse C3	Horse C4	Horse C5
Horse D1	Horse D2	Horse D3	Horse D4	Horse D5
Horse E1	Horse E2	Horse E3	Horse E4	Horse E5

Then, boxes colored blue do not contain either of the three fastest horses. Horses B3 and C3 are not among the three fastest one because A1>A2>A3>B3>C3. Horse C2 is not among the three fastest one since, A1>A2>B2>C2. Therefore, green boxes do not contain fastest horses either. Let horses A2,A3,B1,B2,C1 compete in the 7^{th} run. Then, the three fastest horses are A1, and two fastest horses in the seventh run.

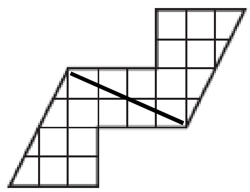
2. Prove that the equation $x^6 - 143x^5 - 917x^4 + 51x^3 + 77x^2 + 291x + 1575 = 0$ does not have solutions in integer numbers.

   Solution: *Assume that x is an odd number. Then,* $x^6 - 143x^5 - 917x^4 + 51x^3 + 77x^2 + 291x + 1575$ *is the sum of 7 odd numbers which is odd and can't be 0. If x is even then this expression is odd again as a sum of 6 even numbers and one odd which is odd and can't be 0 again.*

3. Show how we can cut the figure shown in the picture into two parts for us to be able to assemble a square out of these two parts. Show how we can assemble a square.

   Solution:

4. The city of Vyatka in Russia produces local drink, called "Vyatka Cola". «Vyatka Cola» is sold in 1, 3/4, and 1/2-gallon bottles. Ivan and John bought 4 gallons of "Vyatka Cola". Can we say **for sure**, that they can split the Cola evenly between them without opening the bottles?

   Solution: *No, they can't. Ivan and John might buy ¾+3/4+3/4+1 gallon bottles, and they are unable to get 2 gallons without opening the bottles.*

5. Positive numbers a, b and c satisfy the condition $a + bc = (a + b)(a + c)$. Prove that $b + ac = (b + a)(b + c)$.

   Solution: *Open the brackets.* $a + bc = a^2 + ba + ac + bc$. *Hence,* $a = a^2 + ba + ac$. *Since,* a *is positive, we can divide the last equation by it, and obtain that* $a + b + c = 1$. *Therefore,*
   $b = ab + b^2 + bc$. *Finally,* $b + ac = ab + b^2 + bc + ac = (b + a)(b + c)$.