

**2019 MID-MICHIGAN MATHEMATICS OLYMPIAD:
GRADES 5-6**

SOLUTIONS

- (1) It takes 12 months for Santa Claus to pack gifts. It would take 20 months for his apprentice to do the job. If they work together, how long will it take for them to pack the gifts?

Solution: In one month, Santa Claus finishes $\frac{1}{12}$ of his work of packing gifts and his apprentice finishes $\frac{1}{20}$ of it. So, if they work together, they will finish $\frac{1}{12} + \frac{1}{20} = \frac{2}{15}$ of the total work. So they will finish the work in $\frac{15}{2} = 7.5$ months.

- (2) All passengers on a bus sit in pairs. Exactly $\frac{2}{5}$ of all men sit with women, exactly $\frac{2}{3}$ of all women sit with men. What part of passengers are men?

Solution: Let x denote the number of men, y denote the number of women. Then, $\frac{2}{5}x = \frac{2}{3}y$, $x/(x+y) = x/(x+3/5x) = 5/8$.

For example: assume, there are 4 pairs of woman and man, 3 pairs of man and man, 4 pairs of woman and woman. There are totally 10 men and 12 women. Men make $10/22 = 5/11$ of the total number of all passengers.

- (3) There are 100 colored balls in a box. Every 10-tuple of balls contains at least two balls of the same color. Show that there are at least 12 balls of the same color in the box.

Solution: Assume that there are at most 11 balls of each color. Then, there are at least 10 different colors (otherwise the total number of colors does not exceed $9 \times 11 = 99$). Then, we can take 10 balls all of different colors.

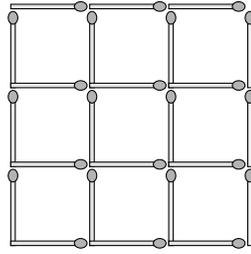
- (4) There are 81 wheels in storage marked by their two types, say first and second type. Wheels of the same type weigh equally. Any wheel of the second type is much lighter than a wheel of the first type. It is known that exactly one wheel is marked incorrectly. Show that one can determine which wheel is incorrectly marked with four measurements.

Solution: First we divide 81 wheels in three groups as follows: Divide all wheels of type 1 in three groups such that the sizes of any two groups differ at most by one wheel. Either all groups have equal number of wheels, or one or two groups have an extra wheel. Then starting with the first group that has fewer number of wheels of type 1, start adding wheels of type 2 to each group one at a time until all the wheels are finished. We obtain at least two groups that have equal number of wheels of type 1 and type 2.

Then weigh two such groups against each other. If they balance, the incorrectly marked wheel is in the remaining 27 wheels. If not, it is in one of these two groups. In the latter case, we take all the wheels of type 1 from the group that weighed lighter and combine them with all the wheels marked type 2 from the group that weighed heavier. The incorrectly marked wheel must be in this group of 27 wheels. In both cases case, at the end of the first turn, we find a group of 27 wheels that contain the incorrectly marked wheel.

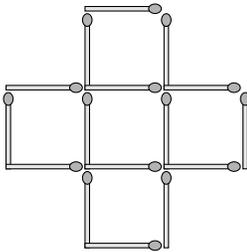
We repeat the process two more times to arrive at a group of 3 wheels that contain the incorrectly marked wheel. For the final measurement, we weigh two wheels of the same type against each other. If they weight equally, the remaining wheel is the incorrectly marked wheel. If they weigh differently, their type will tell us which one of those two was marked incorrectly.

- (5) Remove from the figure below the specified number of matches so that there are exactly 5 squares of equal size left: (a) 8 matches (b) 4 matches



Solution:

(a)



(b)

