

**2019 MID-MICHIGAN MATHEMATICS OLYMPIAD:  
GRADES 7-9**

**You must explain your answer.** The points awarded are not just for correct answer, but also for your explanation.

**SOLUTIONS**

- (1) Prove that the equation

$$x^6 - 143x^5 - 917x^4 + 51x^3 + 77x^2 + 291x + 1575 = 0$$

has no integer solutions.

*Solution: If  $x$  is an even integer then the left hand side of the equation has 6 even summands, and the last summand is odd. Hence the sum can not vanish. If  $x$  is odd, then all seven summands in the left hand side are odd. Hence the sum did not vanish again.*

- (2) There are 81 wheels in storage marked by their two types, say first and second type. Wheels of the same type weigh equally. Any wheel of the second type is much lighter than a wheel of the first type. It is known that exactly one wheel is marked incorrectly. Show that one can determine which wheel is incorrectly marked with four measurements.

*First we divide 81 wheels in three groups as follows: Divide wheels of type 1 in equal three groups as much as possible. So either all groups have equal number of wheels, or one or two groups have an extra wheel. Then starting with the first group that has fewer number of wheels of type 1, start adding wheels of type 2 to each group one at a time until all the wheels are finished. Now we have at least two groups that have equal number of wheels of type 1 and type 2. Then weigh two such groups against each other. If they balance, the incorrectly marked wheel is in the remaining 27 wheels. If not, it is in one of these two groups. In this case, we take the wheels of type 1 from the group that weighed lighter and combine them with the wheels marked type 2 from the group that weighed heavier. The incorrectly marked wheel must be in this group. In any case, at the end of the first turn, we have a group of 27 wheels that contain the incorrectly marked wheel. We repeat the process two more times to arrive at a group of 3 wheels that contain the incorrectly marked wheel. For the final measurement, we weigh two wheels of the same type against each other. If they weight equally, the remaining wheel is the incorrectly marked wheel. If they weigh differently, their type will tell us which one of those two was marked incorrectly.*

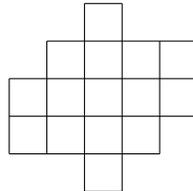
- (3) Rob and Ann multiplied the numbers from 1 to 100 and calculated the sum of digits of this product. For this sum, Rob calculated the sum of its digits as well. Then Ann kept repeating this operation until she got a one-digit number. What was this number?

*Solution: The product of numbers from 1 to 100 is clearly divisible by 9. Hence by the divisibility rule for 9, the sum is divisible by 9. Repeating this process, we will arrive at a single digit number that must be divisible by 9 as well. So Ann will get 9 as the final number.*

- (4) Rui and Jui take turns placing bishops on the squares of the  $8 \times 8$  chessboard in such a way that bishops cannot attack one another. (In this game, the color of the rooks is irrelevant.) The player who cannot place a rook loses the game. Rui takes the first turn. Who has a winning strategy, and what is it?

*Solution: Jui should make same moves as Rui that are symmetric with respect to the vertical line through the center of the board. After each move by Jui, vertical symmetry is restored. After Rui has chosen the square for her next move, none of the bishops previously on the board can attack either this new square nor the vertically symmetric square. Also the new bishop cannot attack is the vertically symmetric square to itself as it travels diagonally. So Jui will always be able to make a symmetric move and win the game.*

- (5) The following figure can be cut along sides of small squares into several (more than one) identical shapes. What is the smallest number of such identical shapes you can get?



*Solution: Since, the figure contains 15 small squares, then each of equal shapes contains the same amount of squares. Hence, the number of squares in each shape may be 1,3, or 5. The number of shapes is 15, 5 or 3. Below is an example of splitting the figure into three shapes.*

