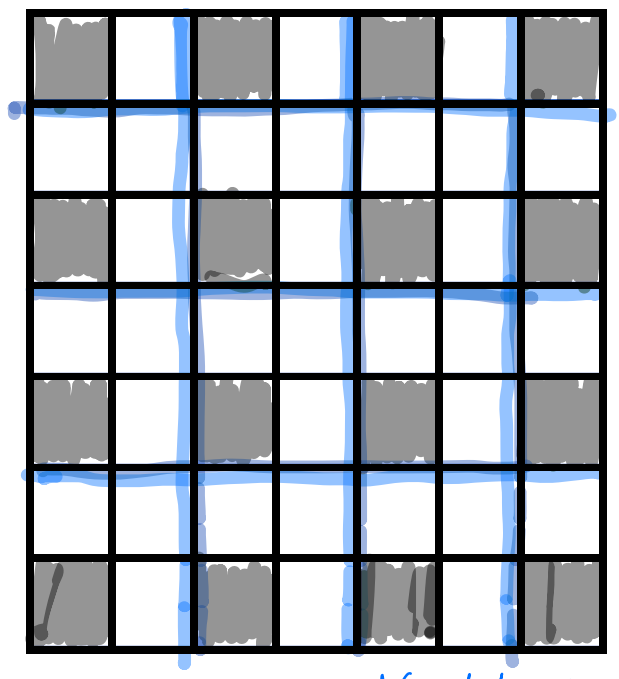
MidMichigan Mathematical Olympiad, Spring 2022
Grades 5-6

1. An animal farm has geese and pigs with a total of 30 heads and 84 legs. Find the number of pigs and geese on this farm.

Solution: Suppose all pigs will stand up on two hind legs only. Then, each animal has 2 legs on the ground and pigs have two extra front legs in air. Since we have 30 animals, there are 60 legs on the ground and $84-60=24$ legs $=12$ pairs of legs in air. Hence, there are 12 pigs and $30-12=18$ geese.

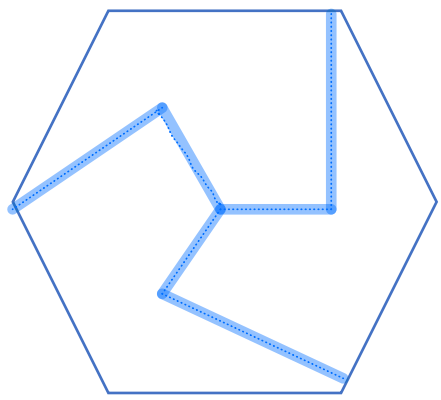
2. What is the maximum number of 1 x 1 squares of a 7 x 7 board that can be colored black in such a way that the black squares don't touch each other even at their corners? Show your answer on the figure below and explain why it is not possible to get more black squares satisfying the given conditions.



Solution. One can color 16 1×1 squares as above. Split 7×7 into 9 pieces of size 2×2, 6 pieces of 2×1, and one piece of size 1×1, totally 16 pieces as shown by blue fat lines above. Note that each piece can contain at most one colored 1×1 square otherwise colored squares will touch each other. Therefore, we can not have more than 16 colored 1×1 squares.

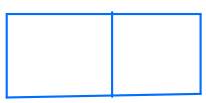
3. Decide whether it is possible to divide a regular hexagon into three equal not necessarily regular hexagons? A regular hexagon is a hexagon with equal sides and equal angles.

Solution:
See the figure.



4. A rectangle is subdivided into a number of smaller rectangles. One observes that perimeters of all smaller rectangles are whole numbers. Is it possible that the perimeter of the original rectangle is not a whole number?

Solution. Yes, it is possible. See example below.



Let a rectangle is made of two squares with sides of length $\frac{1}{4}$.

The perimeter of each square is $4 \times \frac{1}{4} = 1$

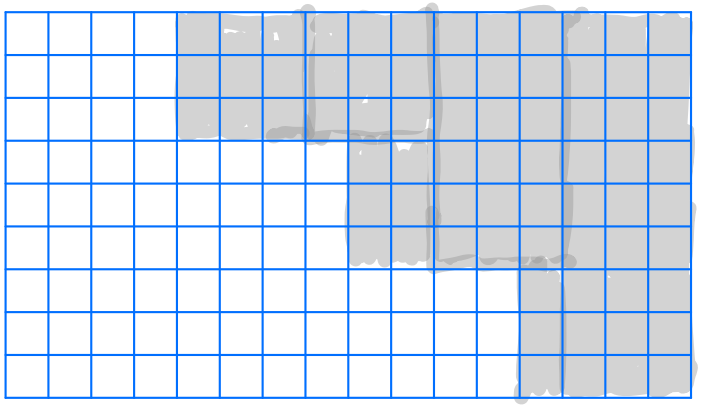
The perimeter of the large rectangle is

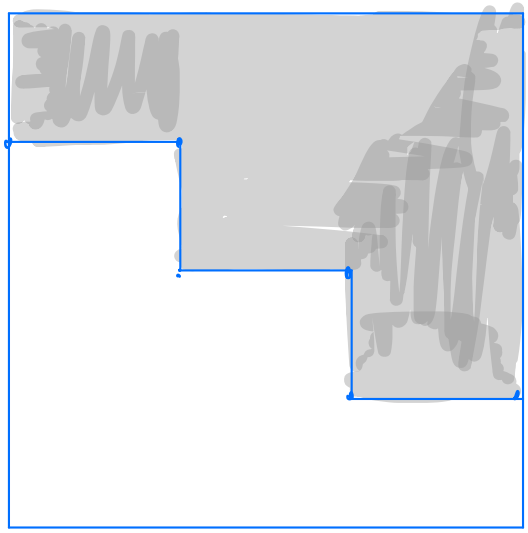
$2 \times \frac{1}{4} + 2 \times \frac{1}{2} = 1\frac{1}{2}$.

5. Place parentheses on the left hand side of the following equality to make it correct.

Solution: $4 \times 12 + (18 \div (6 + 3)) = 50$

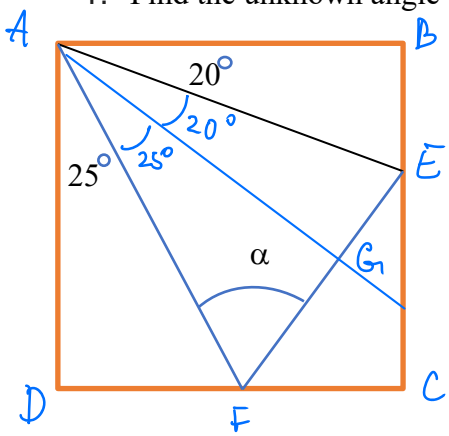
6. Is it possible to cut a 16x9 rectangle into two equal parts which can be assembled into a square?





MidMichigan Mathematical Olympiad, Spring 2022
GRADES 7-9

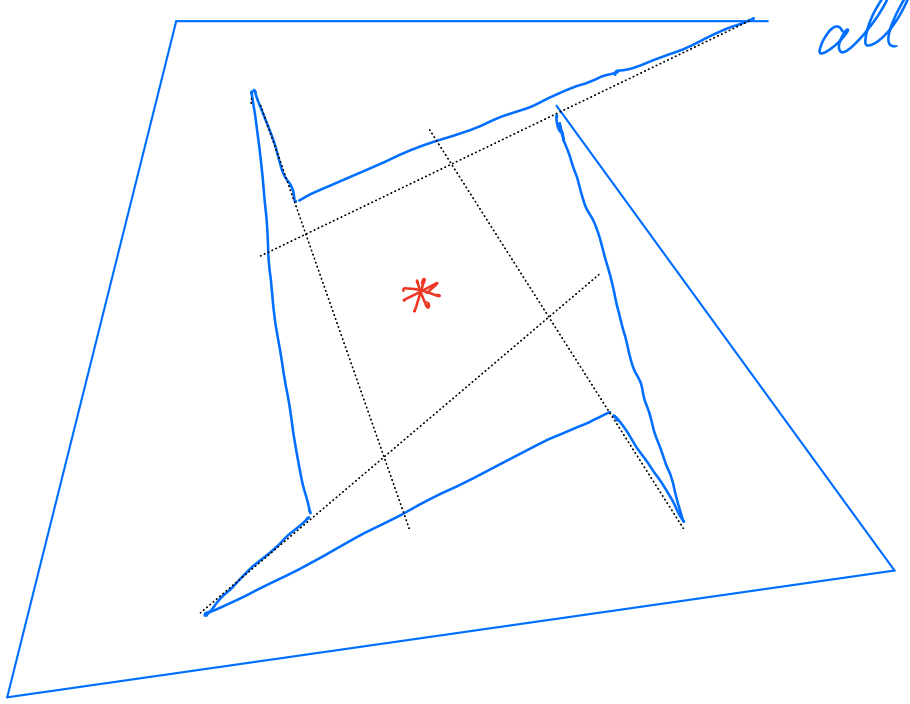
1. Find the unknown angle α of the triangle inscribed in the square.



Solution: The angle $\angle EAF = 45^\circ = 90^\circ - 20^\circ - 25^\circ$. Draw a line AG such that $\angle GAE = 20^\circ$, $\angle GAF = 20^\circ$. Flip $\triangle ABE$ about line AE to obtain $\triangle AGE$, where $|AG| = |AB|$, $\angle AGE = 90^\circ$. Similarly, flip $\triangle ADE$ about AF to obtain AGF, where $|AG| = |AD| = |AB|$ and $\angle FGA = 90^\circ \Rightarrow$ two segment FG and GE make a straight line and $\alpha = \angle AFG = 90^\circ - 25^\circ = 65^\circ$.

2. Draw a polygon in the plane and a point outside of it with the following property: no edge of the polygon is completely visible from that point (in other words, the view is obstructed by some other edge).

Solution. Blue polygon and red point satisfy all conditions.



3. This problem has two parts. In each part, 2022 real numbers are given, with some additional property.

(a) Suppose that the sum of any three of the given numbers is an integer. Show that the total sum of the 2022 numbers is also an integer.

(b) Suppose that the sum of any five of the given numbers is an integer. Show that 5 times the total sum of the 2022 numbers is also an integer, but the sum itself is not necessarily an integer.

Solution. (a) Divide all 2022 numbers into 674 disjoint triples of numbers, the sum of numbers of each triple is integer and hence the total sum is integer.

(b) Divide 2022 numbers into 337 of 6-tuples of numbers. 5 times each 6-tuple can be presented as a union $5\times\{1,2,3,4,5,6\} = \{1,2,3,4,5\}\cup\{2,3,4,5,6\}\cup\{1,3,4,5,6\}\cup\{1,2,4,5,6\}\cup\{1,2,3,5,6\}\cup\{1,2,3,4,6\}$. Then, $5\times$ total sum is an integer. Assume, all numbers are $\frac{1}{5}$, then the sum of 2022 numbers is $\frac{2022}{5}$ is $404\frac{2}{5}$.

4. Replace stars with digits so that the long multiplication in the example below is correct.

$$
\begin{array}{r}
* * * \\
\times \quad \cancel{*}\,* * \\
\hline
* 0 * * \\
* * 5 \\
* 1 * * \\
\hline
* * * * * 8
\end{array}
\qquad
\begin{array}{r}
131 \\
958 \\
\hline
1048 \\
655 \\
1179 \\
\hline
125498
\end{array}
$$

5. Five nodes of a square grid paper are marked (called marked points). Show that there are at least two marked points such that the middle point of the interval connecting them is also a node of the square grid paper.

Solution For each point there are $2^2=4$ options for parity of x, y-coordinates: (even, even), (odd, even), (even, odd) (odd, odd). By the pigeonhole principle there are at least two points with the same parity of coordinates. They give integer coordinates of the middle point.

6. Solve the system

$$\begin{cases} \dfrac{xy}{x+y} = 8/3 \\ \dfrac{yz}{y+z} = 12/5 \\ \dfrac{xz}{x+z} = 24/7 \end{cases}$$

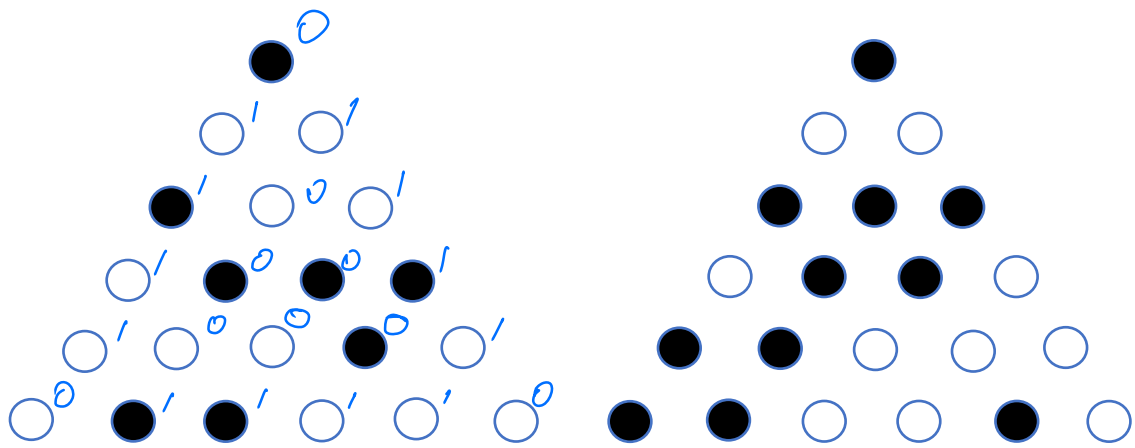
Solution: take reciprocal of both sides.

$$\begin{cases} \frac{1}{y}+\frac{1}{x} = \frac{3}{8} \\ \frac{1}{z}+\frac{1}{y} = \frac{5}{12} \\ \frac{1}{z}+\frac{1}{x} = \frac{7}{24} \end{cases} \Rightarrow \begin{cases} \frac{2}{y} = \frac{3}{8}+\frac{5}{12}-\frac{7}{24} = \frac{9+10-7}{24} = \frac{1}{2} \\ \frac{2}{z} = \frac{5}{12}+\frac{7}{24}-\frac{3}{8} = \frac{1}{3} \\ \frac{2}{x} = \frac{3}{8}+\frac{7}{24}-\frac{5}{12} = \frac{1}{4} \end{cases}$$

$$\begin{cases} \frac{1}{y} = \frac{1}{4} \\ \frac{1}{z} = \frac{1}{6} \\ \frac{1}{x} = \frac{1}{8} \end{cases} \Rightarrow \begin{cases} y=4 \\ z=6 \\ x=8 \end{cases}$$

MidMichigan Mathematical Olympiad, Spring 2022
GRADES 10-12

1. Consider a triangular grid: nodes of the grid are painted black and white. At a single step you are allowed to change colors of all nodes situated on any straight line (with the slope 0°, 60°, or 120°) going through the nodes of the grid. Can you transform the combination in the left picture into the one in the right picture in a finite number of steps?



Solution: Put weights 1 and 2 to every node as on the figure. The sum of weights of nodes on every line is even. Hence, the parity of total weight after any step does not change. However the parity changes from 0 mod 2 for the left config. to 1 mod 2 for the right configuration.

2. Find x satisfying $\sqrt{x\sqrt{x\sqrt{x}}}\ldots = \sqrt{2022}$ where it is an infinite expression on the left side.

Solution: Denote by $z = \sqrt{x\sqrt{x\sqrt{x}\ldots}} = \sqrt{2022}$

Then, $z^2 = 2022$

$$z^2 = x\sqrt{x\sqrt{x\sqrt{x}\ldots}} = x \cdot z = 2022$$

and

$$x \cdot \sqrt{2022} = 2022$$

Finally, $x = 2022/\sqrt{2022} = \sqrt{2022}$.

3. 179 glasses are placed upside down on a table. You are allowed to do the following moves. An integer number k is fixed. In one move you are allowed to turn any k glasses .
(a) Is it possible in a finite number of moves to turn all 179 glasses into "bottom-down" positions if k=3?
(b) Is it possible to do it if k=4?

Solution. (a) It is possible. The following sequence of moves turns exactly one glass:

△ △ △ △ △
▽ ▽ ▽ △ △
▽ △ △ ▽ △
△ △ △ △ ▽

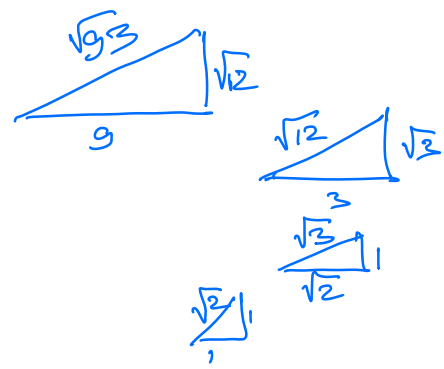
Repeating this sequence 179 times we turn all glasses "bottom-down".

(b) It is not possible because after each move the number of glasses in "bottom-down" position is even.

4. An interval of length 1 is drawn on a paper. Using a compass and a simple ruler construct an interval of length $\sqrt{93}$

Solution:

$$(\sqrt{93})^2 = 9^2 + (\sqrt{12})^2$$
$$(\sqrt{12})^2 = 3^2 + (\sqrt{3})^2$$
$$(\sqrt{3})^2 = 1^2 + (\sqrt{2})^2$$
$$(\sqrt{2})^2 = 1^2 + 1^2$$



5. Show that $5^{2n+1} + 3^{n+2}2^{n-1}$ is divisible by 19 for any positive integer n.

Solution: Prove by induction:

Base: $n=1$. $5^3 + 3^3 \cdot 2^0 = 125 + 27 = 152$

$152 \div 19 = 8$

Induction step: Assume the statement holds true for $n=m$.

For $n = m+1$ $5^{2n+1} + 3^{n+2} 2^{n-1} =$

$= 5^2 \cdot 5^{2m+1} + 3 \cdot 3^{m+2} \cdot 2 \cdot 2^{m-1} = 25 \cdot 5^{2m+1} + 6 \cdot 3^{m+2} 2^{m-1} =$

$= \underbrace{19 \cdot 5^{2m+1}}_{\text{divisible by 19}} + 6\underbrace{(5^{2m+1} + 3^{m+2} 2^{m-1})}_{\text{divisible by 19 by induction assumption.}}$

6. Solve the system

$$\begin{cases} \dfrac{xy}{x+y} = 1 - z \\ \dfrac{yz}{y+z} = 2 - x \\ \dfrac{zx}{z+x} = 2 - y \end{cases}$$

Solution:

$$\begin{cases} xy = x + y - xz - yz \\ yz = 2y + 2z - xy - xz \\ zx = 2z + 2x - xy - yz \end{cases}$$

$$\begin{cases} xy + xz + yz = x + y \\ xy + xz + yz = 2y + 2z \\ xy + xz + yz = 2x + 2z \end{cases} \qquad \begin{cases} -x + y + 2z = 0. \\ x - y + 2z = 0. \end{cases} \Rightarrow \begin{cases} yz = 0. \\ \Rightarrow z = 0, \end{cases}$$

After substitution: $\begin{cases} xy = x + y \\ xy = 2y \\ xy = 2x \end{cases} \Rightarrow \begin{cases} x = y \\ x^2 = 2x \end{cases} \Rightarrow x = 2$

$x = 0$ does not work.

Answer: $x = 2$
$y = 2$
$z = 0$