MidMichigan Mathematical Olympiad 2023

Grades 5-6

1) Solve: **INK + INK + INK + INK + INK + INK = PEN**
(INK and PEN are 3-digit numbers, and different letters stand for different digits).

Solution. Answer: INK=105, PEN=630.
We have the number of digits in INK =3 = the number of digits in PEN.
This means I = 1 (if I > 1, then the product 6*INK would have 4 digits). For the same reason, 6*N<40 implies N<7. Then K cannot be 1. If K=0,2,4,6, or 8 then N=K contradicting to condition that N and K represent different digits. If K=3 then N=8 contradicting N<7. If K=5 we get solution INK=105, PEN=630. If K=7 then N=2 producing P=N=2. Finally, if K=9 then E=K=9. We notice that the only solution is INK=105, PEN=630.

2) Two people play a game. They put 3 piles of matches on the table:
the first one contains 1 match, the second one 3 matches, and the third one 4 matches.
Then they take turns making moves. In a move, a player may take any nonzero number
of matches FROM ONE PILE. The player who takes the last match from the table loses the game.
a) The player who makes the first move can win the game. What is the winning first move? b) How can he win? (Describe his strategy.)
Solution. (a) The winning move is to take two matches from the last pile making piles 1-3-2
(b) After the first move is made, the second player has several options. Let's assume that the second player removes one pile completely. This leaves only two piles on the table. If one pile contains only one matchstick, the first player wins by removing all the matches from the other pile. If there are two piles with 2 and 3 matches, the winning move for the first player is to make them both of size 2. In the next move, the second player can either remove one of the piles completely or leave one pile with only one matchstick. In both cases, the first player wins.
Now let's consider the scenario where all three piles remain on the table after the second player's move. The piles could be 1-2-2, 1-1-2, 1-3-1, or 1-2-1. In the last three cases, the first player wins by making all piles 1-1-1 in their next move. In the only remaining case, the first player wins by making the piles 2-2 which reduces this case to the one considered above.

3) The planet Naboo is under attack by the imperial forces. Three rebellion camps are located at the vertices of a triangle. The roads connecting the camps are along the sides of the triangle. The length of the first road is less than or equal to 20 miles, the length of the second road is less than or equal to 30 miles, and the length of the third road is less than or equal to 45 miles. The Rebels have to cover the area of this triangle with a defensive field.
What is the maximal area that they may need to cover?

Solution. Denote the vertices of the triangle A, B, and C.

Then, $|AB| \le 20$, $|AC| \le 30$, $|BC| \le 45$. Let θ be the angle between AB and AC and S be the area of the triangle ABC. $S = |AB| \cdot |AC| \sin\theta /2 \le |AB| \cdot |AC|/2 = 20 \cdot 30/2 = 300$.

So, the largest area is the area of the right triangle with sides 20, 30, and $\sqrt{20^2 + 30^2} = \sqrt{1300} = 10\sqrt{13} \le 10 \cdot 4 = 40 < 45$

Answer: The maximal area is 300 square miles.

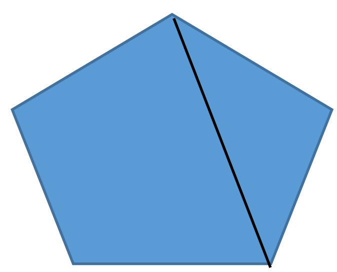
4) Money in Wonderland comes in \$5 and \$7 bills. What is the smallest amount of money you need to buy a slice of pizza that costs \$1 and get back your change in full? (The pizza man has plenty of \$5 and \$7 bills.) For example, having \$7 won't do, since the pizza man can only give you \$5 back.

Solution: Answer: \$15. The smallest amount to buy a \$1 pizza slice is \$15 (3 \$5-bills); the change will be two \$7 bills.

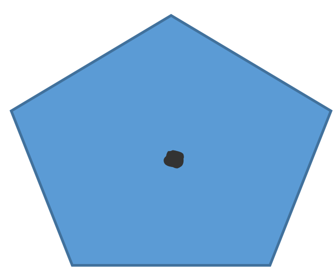
5) Put 5 points on the plane so that each 3 of them are vertices of an isosceles triangle (i.e., a triangle with two equal sides), and no three points lie on the same line.

Solution. Choose 5 vertices of a regular pentagon.

(a)



(b)

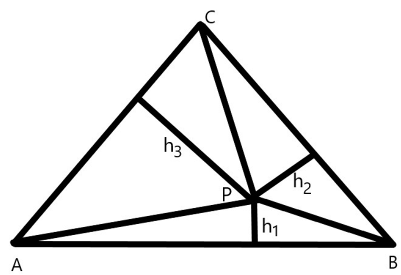


MidMichigan Mathematical Olympiad 2023

Grades 7-9

1) Three camps are located in the vertices of an equilateral triangle. The roads connecting camps are along the sides of the triangle. Captain America is inside the triangle and he needs to know the distances between camps. Being able to see the roads he has found that the sum of the shortest distances from his location to the roads is 50 miles. Can you help Captain America to evaluate the distances between the camps?

Solution. Denote the locations of cities by A, B, and C, and the location of Captain America by P. Let h_1, h_2, and h_3 be heights of the triangles APB, BPC, and APC.



$$h_1 + h_2 + h_3 = 50$$

Let |AB| = x, then Area(APB) = $x \cdot h_1/2$; Area(BPC) =$x \cdot h_2/2$; Area(APC) = $x \cdot h_3/2$. The height of h the triangle ABC equals $h = \sqrt{x^2 - (x/2)^2} = x\sqrt{3}/2$. Therefore Area(ABC)=$xh/2 = x^2\sqrt{3}/4$.

Since, Area(APB)+Area(BPC)+Area(APC)=Area(ABC)

we get $xh_1/2 + xh_2/2 + xh_3/2 == \mathrm{x}^2\sqrt{3}/4$.Therefore, $x(h_1 + h_2 + h_3)/2 = x^2\sqrt{3}/4$.

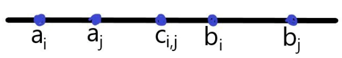
Thus, $50x/2 = x^2\sqrt{3}/4$ and $x = 100/\sqrt{3} = 100\sqrt{3}/3$.

Answer: the distance between the camps equals $100\sqrt{3}/3$ miles.

2) N regions are located in the plane, every pair of them have a non-empty overlap. Each region is a connected set, that means every two points inside the region can be connected by a curve all points of which belong to the region. Iron Man has one charge remaining to make a laser shot. Is it possible for him to make the shot that goes through all N regions?

Solution. Choose an arbitrary straight line l and project all regions on that line. The projections are intervals and each pair of intervals has a non-empty intersection. Denote the left endpoins of intervals by $a_1, a_2, \dots, a_N$, *and the right endpoints by* $b_1, b_2, \dots, b_N$. *Then for every pair of endpoints* $a_i, b_i, a_j, b_j,\ a_i \le b_j,\ a_j \le b_i$. *Indeed, since intervals* $[a_i, b_i]$ *and* $[a_j, b_j]$ *have non-empty intersection, there exists a point* $c_{i,j}$ *such that*

$a_i \le c_{ij} \le b_j,\ a_j \le c_{i,j} \le b_j$



Therefore, there exists a point c that belongs to all intervals. Thus the straight line passing through the point c perpendicular to the line l intersects all regions.

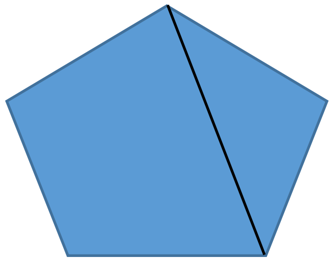
3) Money in Wonderland comes in \$5 and \$7 bills.

(a) What is the smallest amount of money you need to buy a slice of pizza that costs \$1 and get back your change in full? (The pizza man has plenty of \$5 and \$7 bills.) For example, having \$7 won't do since the pizza man can only give you \$5 back.

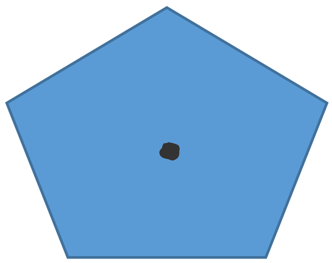
(**b**) Vending machines in Wonderland accept only exact payment (do not give back change). List all positive integer numbers which CANNOT be used as prices in such vending machines. (That is, find the sums of money that cannot be paid by exact change.)

Solution: Answer: 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23. First, any amount of at least \$28 can be paid by exact change. Indeed, let N > 27. Then one of the five numbers N, N – 7, N – 14, N – 21, N – 28 is divisible by 5. Indeed, they have the same remainders under division by 5 as N, N – 2, N – 4, N – 1, and N – 3, which are 5 consecutive numbers, and hence one of them must be divisible by 5. Thus we can pay \$N by paying N, N – 7, N – 14, N – 21, or N – 28 (whichever is divisible by 5) using \$5 bills, and paying the rest by \$7 bills. So any amount that cannot be paid by exact change is less than \$28. Thus it remains to list all numbers of the form 5a + 7b between 0 and 27, where a,b are nonnegative integers; the missing numbers are the answer to (b). It is enough to consider a < 6, b < 4 (as we are looking for numbers < 28). By direct computation, we get the following list of numbers 5a + 7b less than 28 (in increasing order): 0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24, 25, 26, 27 (*) So the missing numbers (answer to (b)) are 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23
Now, to solve (a), it is enough to find the first two consecutive numbers in the list (*). These are 14 and 15. Thus the smallest amount to buy a \$1 pizza slice is \$15 (3 \$5-bills); the change will be two \$7 bills.

4) (**a**) Put 5 points on the plane so that each 3 of them are vertices of an isosceles triangle (i.e., a triangle with two equal sides), and no three points lie on the same line.
(**b**) Do the same with 6 points.
Solution. (a)



(b)



5) Numbers 1,2,3,...,100 are randomly divided in two groups 50 numbers in each. In the first group the numbers are written in increasing order and denoted $a_1, a_2, \dots, a_{50}$. In the second group the numbers are written in decreasing order and denoted $b_1, b_2, \dots, b_{50}$. Thus, $a_1 < a_2 < \cdots < a_{50}$ and $b_1 > b_2 > \cdots > b_{50}$.
Evaluate $|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{50} - b_{50}|$.

Solution. *Divide the numbers $a_1, a_2, \dots, a_{50}$ in two groups: less than or equal to 50 and greater than 50. Let the first group contains n numbers. Thus,*
$a_1 < a_2 < \cdots < a_n \le 50 < a_{n+1} < a_{n+2} < \cdots < a_{50}$.
Now, let us divide the numbers $b_1, b_2, \dots, b_{50}$ in two groups: less than or equal to 50 and greater than 50. Then the first group contains 50-n numbers, and the second group n numbers. Thus, $b_1 > b_2 > \cdots > b_n > 50 \ge b_{n+1} > b_{n+2} > \cdots > b_{50}$.
Therefore, $|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_{50} - b_{50}|$
$= (b_1 - a_1) + \cdots + (b_n - a_n) + (a_{n+1} - b_{n+1}) + \cdots + (a_{50} - b_{50})$.
In the last sum all terms with plus are greater than 50 and all terms with minus are less than or equal to 50 (absolute values). Thus the last sum equals
(51 + 52 +…+ 100) - (1 + 2 +…+ 50)=50(51 + 100)/2-50(1 + 50)/2= 2500.
Answer: 2500.

MidMichigan Mathematical Olympiad 2023

Grades 10-12

1). There are 16 students in a class. Each month the teacher divides the class into two groups. What is the minimum number of months that must pass for any two students to be in different groups in at least one of the months?

Solution:

Example. The figure shows how to divide the class into two groups so that any two students are in different groups in at least one of four months. Each student corresponds to a column in the table, and each month corresponds to a row. A zero in a cell of the table means that the corresponding student is in the first group, and a one means that the student is in the second group. Since there are no matching columns, any two students are in different groups in at least one of the four months.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
2	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
3	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
4	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Estimate. We will prove that it is impossible to satisfy the condition in three months. Let us create a similar table of size 3x16. There are only 8 ways to fill a column with zeros and ones, so there will be two identical columns. The corresponding students in those columns will be in the same group for all three months.
Answer: 4 months.

2). Find all functions $f(x)$ defined for all real x that satisfy the equation $2f(x) + f(1 - x) = x^2$.

Solution

Substituting $1-x$ for x in the given equation, we obtain the system $2f(x) + f(1 - x) = x^2$, $f(x) + 2f(1 - x) = (1 - x)^2$, from which we find $f(x) = 1/3\ (2x^2 - (1 - x)^2) = 1/3\ (x^2 + 2x - 1)$. Checking shows that the found function satisfies the condition.

Answer

$f(x) = 1/3\ (x^2 + 2x - 1)$.

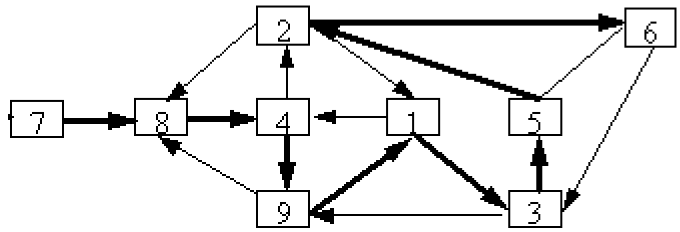
3). Arrange the digits from 1 to 9 in a row (each digit only once) so that every two consecutive digits form a two-digit number that is divisible by 7 or 13.

Answer

For example, 784913526. Note

Idea. Write down all acceptable two-digit numbers:

14, 21, 28, 35, 42, 49, 56, 63, 84, 91, 98; 13, 26, 39, 52, 65, 78, and draw a graph.



It is easy to see that we need to start with 784. Then there are a few possible options to try, but it is not difficult to find one of the possible solutions.

4). Prove that cos 1^o is irrational.

Solution.

Assume for the sake of contradiction, that $cos(1°)$ is rational.

Since, $cos(2°) = 2\,cos^2(1°) - 1$ we have that, $cos(2°)$ is also rational. Note that, we also have

$$cos(n + 1°) + cos(n - 1°) = 2 \cdot cos(1°)\, cos(n°)$$

By induction principle, this shows that cos(n∘) is rational for all integers n≥1. But, this is clearly false, as for instance, $cos(30°) = \sqrt{3}/2$ is irrational, reaching a contradiction.

5). Consider 2n distinct positive integers $a_1, a_2, \dots, a_{2n}$ not exceeding n^2 $(n > 2)$. Prove that some three of the differences $a_i - a_j$ are equal.

Solution. Without loss of generality we assume that $a_i < a_j$ for all $i < j$.

Denote by $x_{ij} = a_j - a_i,\ i < j$. We also put $b_i = x_{i,i+1}$. Assume that each value is taken by at most two b_i. Then the maximal number of a_i is at least

$1 + \sum b_i \geq 1 + 2 \cdot (1 + 2 + \cdots + n - 1) + n = 1 + n \cdot (n - 1) + n = 1 + n^2$ contradicting to the initial assumption that no a_i exceeds n^2.