Name SOLUTIONS Feb 24, 2009

Math 133 — Hourly Exam 2A

Do all problems (Total = 100 points). Show all work. Calculators are not allowed.

(1) (10 points) Which grows faster as $x \to \infty$: $x \ln x$ or $x^{3/2}$? *Explain your reasoning.*

$$\lim_{x\to\infty} \frac{x \ln x}{x^{3/2}} = \lim_{x\to\infty} \frac{\cancel{x} \ln x}{\cancel{x} \cdot x^{1/2}} = \lim_{x\to\infty} \frac{\ln x}{x^{1/2}} \quad \text{type } \frac{\infty}{\infty}$$

$$\underline{\underline{\text{L'Hop}}} \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2} x^{-1/2}}$$

$$= 2 \lim_{x\to\infty} \frac{\sqrt{x}}{x} = 2 \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

$\Rightarrow x^{3/2}$ grows faster than $x \ln x$.

(2) (18 points) Differentiate the following two functions. *Show all your steps.*

(a) $f(t) = \arcsin\left(\sqrt{t}\right) = \arcsin(u)$ where $u = t^{1/2}$

$$f'(t) = \frac{1}{\sqrt{1-u^2}} \frac{du}{dt} = \frac{1}{\sqrt{1-t}} \cdot \frac{1}{2} t^{-1/2} = \underline{\underline{\frac{1}{2\sqrt{t}\sqrt{1-t}}}}$$

(b) $h(s) = s \ln(\cosh(1+s^2)) = s \ln(\cosh(u))$ where $u = 1+s^2$.

$$h'(s) = 1 \cdot \ln(\cosh u) + s \frac{\sinh u}{\cosh u} \cdot \frac{du}{ds} \quad \text{(by product rule)}$$

$$= \underline{\underline{\ln(\cosh(1+s^2)) + 2s^2 \tanh(1+s^2)}}$$

(3) Compute the following integrals (10 points each). *Make your method and substitutions clear.*

(a) $\displaystyle\int_0^1 \frac{x^2\,dx}{x^6+1}$

$$\left[\begin{array}{c} u = x^3 \\ du = 3x^2\,dx \\ \frac{1}{3}\,du = x^2\,dx \end{array}\right] \qquad \left[\begin{array}{l} \text{When } x=0,\ u \text{ is } 0 \\ \text{When } x=1,\ u \text{ is } 1 \end{array}\right]$$

$$= \int_0^1 \frac{\frac{1}{3}\,du}{u^2+1} = \frac{1}{3}\arctan(u)\Big|_0^1 = \frac{1}{3}\left[\arctan(1) - \arctan(0)\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{4} - 0\right]$$

$$= \pi/12$$

(b) $\displaystyle\int \frac{dx}{\sqrt{8-2x-x^2}}$

Complete the square:

$$8-2x-x^2 = 8-(x^2+2x)$$
$$= 8-\left[(x+1)^2-1\right]$$
$$= 9-(x+1)^2.$$

$$= \int \frac{dx}{\sqrt{9-(x+1)^2}}.$$

$u = x+1$

$du = dx$

$$= \int \frac{du}{\sqrt{9-u^2}} = \arcsin\left(\frac{u}{3}\right) + C = \underline{\arcsin\left(\frac{x+1}{3}\right) + C}$$

(c) $\displaystyle\int x \ln x\,dx$

By parts:

$$\left|\begin{array}{ll} u = \ln x & dv = x\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^2}{2} \end{array}\right.$$

$$= \frac{x^2}{2}\ln|x| - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx.$$

$$= \frac{x^2}{2}\ln|x| - \frac{1}{2}\int x\,dx.$$

$$= \underline{\frac{x^2}{2}\ln|x| - \frac{1}{4}x^2 + C}$$

(4) (12 points) Write out the partial fractions decomposition of the following expression but ***DO NOT SOLVE for the coefficients*** $A, B, \ldots$

(a) $\dfrac{x-1}{(x-3)^2(x+5)} = \dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} + \dfrac{C}{x+5}$

(b) $\dfrac{4x^2-x-8}{(x^2-1)(x^2+6)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C\cdot 2x+D}{x^2+6}$

$\hookrightarrow$ factors as $(x-1)(x+1)$

$\hookrightarrow$ doesn't factor

(5) And to finish: three more integrals (10 points each). *Again, make your method clear.*

(a) $\displaystyle\int \frac{2x^2+1}{(x-1)(x-2)(x-3)}\,dx = \int \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}\,dx.$

$$= \int \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}\,dx.$$

so need $\quad 2x^2+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$

Take $x=1$: $\quad 3 = A(-1)(-2) + 0 + 0 \Rightarrow A = 3/2$

Take $x=2$: $\quad 5 = B(1)(-1) \Rightarrow B = -5$

Take $x=3$: $\quad 19 = C(2)(1) \Rightarrow C = 19/2.$

so this integral $\displaystyle = \frac{3}{2}\int \frac{dx}{x-1} - 5\int \frac{dx}{x-2} + \frac{19}{2}\int \frac{dx}{x-3}$

$$= \frac{3}{2}\ln|x-1| - 5\ln|x-2| + \frac{19}{2}\ln|x-3| + C$$

(b) Find $\int_0^{\pi/4} \tan x \, dx$. $= \int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx$. With $\left[\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right.$

$$\int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = -\ln |u| + C = -\ln |\cos x| + C$$

$$\Rightarrow \int_0^{\pi/4} \tan x \, dx = -\ln \cos x \Big|_0^{\pi/4} = -\left[\ln \cos(\pi/4) - \ln \cos(0) \right]$$

$$= -\left[\ln\left(\tfrac{1}{\sqrt{2}}\right) - \ln 1 \right]$$

$$= \tfrac{1}{2} \ln 2$$

(c) $I = \int x^2 e^x \, dx$. by parts : $\left[\begin{array}{ll} u = x^2 & dv = e^x \, dx \\ du = 2x \, dx & v = e^x. \end{array} \right.$

$$= x^2 e^x - \int 2x e^x \, dx$$

Again: $\left[\begin{array}{ll} u = 2x & dv = e^x \, dx \\ du = 2 \, dx & v = e^x \end{array} \right.$

$$= x^2 e^x - \left[2x e^x - \int 2 e^x \, dx \right]$$

$$= x^2 e^x - 2x e^x + 2 e^x + C$$

or $I = (x^2 - 2x + 2) e^x + C$