Name Solutions Feb 24, 2009

Math 133 — Hourly Exam 2B

Do all problems (Total = 100 points). Show all work. Calculators are not allowed.

(1) (10 points) Which grows faster as $x \to \infty$: $\frac{1}{x^2+1}$ or e^{-x} ? *Explain your reasoning.*

$$\lim_{x\to\infty} \frac{\frac{1}{x^2+1}}{e^{-x}} = \lim_{x\to\infty} \frac{e^x}{x^2+1} \quad \text{type } \frac{\infty}{\infty}$$

$$\underline{\underline{\text{L'Hop}}} \lim_{x\to\infty} \frac{e^x}{2x} \quad \text{type } \frac{\infty}{\infty}$$

$$\underline{\underline{\text{L'Hop}}} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

$\Rightarrow$ $\frac{1}{x^2+1}$ grows faster than e^{-x}.

(2) (18 points) Differentiate the following two functions. *Show all your steps.*

(a) $f(t) = \operatorname{arcsec}(e^{-t})$ $= \operatorname{arcsec}(u)$ where $u = e^{-t}$

$$f'(t) = \frac{1}{|u|\sqrt{u^2-1}} \frac{du}{dt} = \frac{1}{\cancel{e^{-t}}\sqrt{e^{-2t}-1}} \cdot (-\cancel{e^{-t}})$$

$$= \frac{-1}{\sqrt{e^{-2t}-1}}$$

(b) $h(s) = \sqrt{s}\sinh(\sqrt{s})$

$= u \sinh(u)$ where $u = s^{1/2}$

$$h'(s) = \frac{du}{ds}\sinh(u) + u \cdot \cosh(u)\frac{du}{ds}$$ by product rule

$$= \frac{1}{2\sqrt{s}}\sinh(\sqrt{s}) + \cancel{\sqrt{s}}\cosh(\sqrt{s}) \cdot \frac{1}{2\cancel{\sqrt{s}}}$$

$$= \underline{\frac{1}{2}\left[\frac{\sinh(\sqrt{s})}{\sqrt{s}} + \cosh(\sqrt{s})\right]}$$

(3) Compute the following integrals (10 points each). *Make your method and substitutions clear.*

(a) $\displaystyle\int_0^1 \frac{x\,dx}{\sqrt{4-x^4}}$

Subst. $\begin{bmatrix} u = x^2 \\ du = 2x\,dx \\ \frac{1}{2}du = x\,dx \end{bmatrix}$ $\begin{bmatrix} \text{When } x=0, & u \text{ is } 0 \\ \text{When } x=1, & u \text{ is } 1 \end{bmatrix}$

$$= \int_0^1 \frac{\frac{1}{2}\,du}{\sqrt{4-u^2}} = \frac{1}{2}\arcsin\left(\frac{u}{2}\right) + C$$

$$= \underline{\frac{1}{2}\arcsin\left(\frac{x^2}{2}\right) + C}$$

(b) $\displaystyle\int \frac{dx}{x^2-6x+13}$

$$= \int \frac{dx}{(x-3)^2+4}$$

$u = x-3$
$du = dx$

$$= \int \frac{du}{u^2+4}$$

$$= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C$$

$$= \underline{\frac{1}{2}\arctan\left(\frac{x-3}{2}\right) + C}$$

(c) Integrate by parts: $\displaystyle\int \frac{\ln x}{x^2}\,dx$

$\left| \begin{array}{ll} u = \ln x & dv = \frac{1}{x^2} = x^{-2} \\ du = \frac{dx}{x} & v = -\frac{1}{x} \end{array}\right.$

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x + \int \frac{dx}{x^2}$$

$$= -\frac{\ln x}{x} + \int x^{-2}\,dx.$$

$$= \underline{-\frac{\ln x}{x} - \frac{1}{x} + C}$$

(4) (12 points) Write out the partial fractions decomposition of the following expression but **DO NOT SOLVE** for the coefficients $A, B, \ldots$

(a) $\dfrac{x-1}{(x-3)^2(x+5)} = \dfrac{A}{x-3} + \dfrac{B}{(x-3)^2} + \dfrac{C}{x+5}$

(b) $\dfrac{4x^2-x-8}{(x^2-1)(x^2+6)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{C\cdot 2x+D}{x^2+6}$

$\to$ factors as $(x-1)(x+1)$

$\to$ Doesn't factor

(5) And to finish: three more integrals (10 points each). *Again, make your method clear.*

(a) $\displaystyle\int \frac{x+2}{x(x^2+4)}\,dx$ Write $\dfrac{x+2}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{B\cdot 2x + C}{x^2+4}$

doesn't factor

$$= \frac{A(x^2+4) + (2Bx + C)x}{x(x^2+4)}$$

Match numerators: $x+2 = A(x^2+4) + 2Bx^2 + Cx$

Take $x=0$:	$2 = 4A$	$A = \frac{1}{2}$
Match x coeff:	$1 = C$	$C = 1$
Match x^2 coeff:	$0 = A + 2B = \frac{1}{2} + 2B$	$B = -1/4$

$$\int \frac{x+2}{x(x^2+4)}\,dx = A\int \frac{dx}{x} + B\int \frac{2x\,dx}{x^2+4} + C\int \frac{dx}{x^2+4}$$

$u = x^2+4$

$$= \frac{1}{2}\ln|x| - \frac{1}{4}\int \frac{du}{u} + 1\cdot\frac{1}{2}\arctan\left(\frac{x}{2}\right)$$

$$= \underline{\frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+4) + \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C}$$

(b) Find $\int_0^{\pi/4} \tan x \, dx$. $= \int_0^{\pi/4} \frac{\sin x}{\cos x} \, dx$. With $\left[\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right.$

$$\int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C$$

$$\Rightarrow \int_0^{\pi/4} \tan x \, dx = -\ln|\cos x| \Big|_0^{\pi/4} = -\left[\ln|\cos(\tfrac{\pi}{4})| - \ln|\cos(0)| \right]$$
$$= -\left[\ln(\tfrac{1}{\sqrt{2}}) - \ln 1 \right]$$
$$= \tfrac{1}{2} \ln 2$$

(c) $I = \int e^x \sin x \, dx$. by parts: $\left[\begin{array}{ll} u = \sin x & dv = e^x \, dx \\ du = \cos x \, dx & v = e^x \end{array} \right.$

$$= \sin x \cdot e^x - \int \cos x \cdot e^x \, dx$$

Again: $\left[\begin{array}{ll} u = \cos x & dv = e^x \, dx \\ du = -\sin x \, dx & v = e^x \end{array} \right.$

$$= \sin x \cdot e^x - \left\{ \cos x \cdot e^x - \int (-\sin x) \, e^x \, dx \right\}$$

So

$$I = \sin x \, e^x - \cos x \, e^x - I.$$

Solve for I:

$$2I = (\sin x - \cos x) e^x$$

$$\underline{\underline{I = \tfrac{1}{2} (\sin x - \cos x) e^x}}$$