Name Solutions Feb 3, 2009

Math 133 — First Hourly Exam A

Do all 5 problems (Total = 100 points). Show all work. Calculators or other electronic devices are not allowed.

(1) (27 points) Compute the following derivatives.

(a) $\frac{d}{dx}\ \ln(\sec x) = \frac{1}{\sec x}(\sec x)' = \frac{\sec x \cdot \tan x}{\sec x} = \tan x$

(b) $\frac{d}{dx}\ 10^{\sin(x^2)} = \ln 10 \cdot 10^{\sin(x^2)} \cdot \cos(x^2) \cdot 2x$

since $\frac{d}{du} 10^u = \ln 10 \cdot 10^u \cdot \frac{du}{dx}$.

(c) Differentiate $y = \sqrt{\dfrac{(x^2+3)^{\frac{2}{3}}(2x+1)^3}{(x-4)^5}}$. Start by taking ln of both sides.

$$\ln y = \frac{1}{2}\left[\frac{2}{3}\ln(x^2+3) + 3\ln(2x+1) - 5\ln(x-4)\right]$$

Differentiate:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\frac{2x}{x^2+3} + \frac{3}{2}\frac{2}{2x+1} - \frac{5}{2}\frac{1}{x-4}$$

$$\frac{dy}{dx} = y\left[\frac{2x}{3(x^2+3)} + \frac{3}{2x+1} - \frac{5}{2x-8}\right]$$

where y is as above.

(2) (27 points) Compute the following integrals. Make your substitutions clear.

(a) $\displaystyle\int \frac{\cos x}{3+\sin x}\,dx.$

Set $\begin{cases} u = 3+\sin x \\ du = \cos x \; dx. \end{cases}$

$$= \int \frac{du}{u} = \ln|u| + C = \underline{\ln(3+\sin x) + C}$$

(b) $\displaystyle\int_2^3 \frac{1}{t(\ln t)^2}\,dt.$

Set $\begin{cases} u = \ln t \\ du = \frac{1}{t}\,dt. \end{cases}$

$$\int \frac{1}{t(\ln t)^2}\,dt = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{\ln t} + C$$

$$\Rightarrow \int_2^3 \frac{1}{t(\ln t)^2}\,dt = -\left(\frac{1}{\ln t}\right|_2^3 = \boxed{\frac{1}{\ln 2} - \frac{1}{\ln 3}}$$

(c) $\displaystyle\int \frac{dx}{4x-4\sqrt{x}}.$ Factoring helps!

$$= \frac{1}{4}\int \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$$

Set $u = \sqrt{x}-1 = x^{1/2}-1$

$du = \frac{1}{2}x^{-1/2}\,dx$

$= \frac{1}{2\sqrt{x}}\,dx.$

$$= \frac{1}{2}\int \frac{dx}{2\sqrt{x}(\sqrt{x}-1)}$$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \underline{\frac{1}{2}\ln\left|\sqrt{x}-1\right| + C}$$

(3) (8 points) Set up but **DO NOT EVALUATE** an integral for the arclength of the curve $y = 2\sqrt{\cos x}$ that lies between $x = 0$ and $x = \frac{\pi}{2}$.

$$\text{Arclength} = \int_0^{\pi/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = 2(\cos x)^{1/2}$$

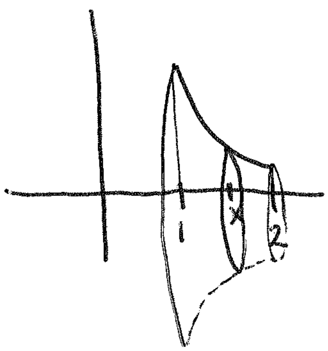
$$\frac{dy}{dx} = 2 \cdot \frac{1}{2}(\cos x)^{-\frac{1}{2}}(-\sin x)$$

$$= \frac{-\sin x}{\sqrt{\cos x}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{\sin^2 x}{\cos x}$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1 + \frac{\sin^2 x}{\cos x}}\, dx$$

(4) (18 points) Let R under the curve $y = e^{-3x}$ and above the interval $[1, 2]$ on the x-axis. Find the volume of the solid obtained by rotating R around the x-axis.



Slice at x is disk with area

$$\pi r^2 = \pi y^2 = \pi\left(e^{-3x}\right)^2 = \pi e^{-6x}$$

$$\text{Volume} = \int_1^2 \pi e^{-6x}\, dx$$

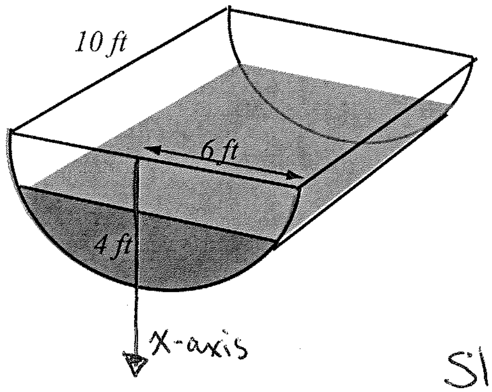
$$= \pi\left(\frac{e^{-6x}}{-6}\right|_1^2$$

$$= -\frac{\pi}{6}\left(e^{-12} - e^{-6}\right)$$

$$= \boxed{\frac{\pi}{6}\left(e^{-6} - e^{-12}\right)}$$

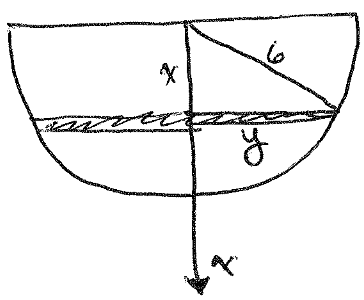
$$\text{or } \frac{\pi}{6} e^{-6}\left(1 - e^{-6}\right)$$

(5) (20 points) A water tank is 10 feet long and has a semicircular cross-section with a radius of 6 feet. It is filled to a depth of 4 feet with water weighing 62.4 lbs per cubic foot. Find the work required to pump the water over the rim of the tank. *Define your variable carefully. Express your answer as a product of 62.4 and some other numbers, and include units.*



Label slices by x = distance <u>down</u> from top of tank.

$\left[\begin{array}{l}\text{Top slice at } x = 2 \\ \text{Bottom slice at } x = 6.\end{array}\right.$



Slice at x is rectangle with

$\left[\begin{array}{l}\text{length } 10 \text{ ft.} \\ \text{width } 2y \text{ ft.}\end{array}\right.$

where $y = \sqrt{36 - x^2}$ by Pythag. Theorem

$\Rightarrow$ weight $= 10 \cdot 2\sqrt{36-x^2} \cdot 62.4 \, \Delta x$

Slice at x lifted distance x.

$$\Rightarrow \text{Work} = \int_2^6 \text{weight} \cdot \text{distance } dx$$

$$= 624 \int_2^6 \sqrt{36-x^2} \cdot 2x \, dx.$$

With $\left[\begin{array}{l} u = 36 - x^2 \\ du = -2x\,dx \end{array}\right.$ $\quad \int \sqrt{36-x^2} \cdot 2x \, dx = -\int \sqrt{u} \, dx = -\int u^{\frac{1}{2}} du = \frac{-2}{3} u^{3/2} + C$

$$\text{Work} = 624 \left(-\tfrac{2}{3}\right) \left((36-x^2)^{3/2} \Big|_2^6 \right.$$

$$= 624 \left(-\tfrac{2}{3}\right) \left[0 - 32^{3/2} \right]$$

$$= \underline{\underline{\tfrac{2}{3} \cdot 624 \cdot 32^{3/2} \text{ foot-lbs}}}$$