Name $\underline{\text{SOLUTIONS}}$ March 24, 2009

Math 133 — Hourly Exam 3A

Do all problems (Total = 100 points). Show all work. Calculators are not allowed.

(1) (20 points) Compute the following integrals. *Make your method and substitutions clear.*

(a) $\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \cdot \tan x \sec x \, dx$

$\left[\begin{array}{l} u = \sec x \\ du = \sec x \cdot \tan x \, dx \end{array}\right.$

$= \int (1-u^2) u^2 \, du$

$= \int u^2 - u^4 \, du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C = \underline{\frac{1}{3}\sec^3 x - \frac{1}{5}\sec^5 x + C}$

(b) $\int_0^\pi \cos^2 x \, dx = \int_0^\pi \frac{1+\cos(2x)}{2} dx = \frac{1}{2}\int_0^\pi 1 \cdot dx + \frac{1}{2}\int_0^\pi \cos(2x)\, dx$

$= \frac{x}{2}\Big|_0^\pi + \frac{1}{4}\sin(2x)\Big|_0^\pi$

$= \left(\frac{\pi}{2} - 0\right) + \frac{1}{4}(0-0)$

$= \pi/2$

(2) (10 points) Evaluate the improper integral $I = \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$. *Show all steps.*

With $\left.\begin{array}{l} u = x-1 \\ du = dx \end{array}\right]$ this becomes $I = \int_0^2 \frac{du}{u^{2/3}}$

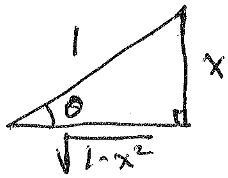
when $x = 3$, u is 2
$x = 1$ u is 0

This is improper integral, so is

$I = \lim_{a\to 0^+} \int_a^2 u^{-2/3} \, du = \lim_{a \to 0^+} 3u^{1/3}\Big|_a^2 = 3 \lim_{a\to 0^+} \left(\sqrt[3]{2} - \sqrt[3]{a}\right)$

$= \boxed{3\sqrt[3]{2}}$

(3) (10 points) Find $I = \int x^3\sqrt{1-x^2}\,dx$.



$x = \sin\theta$

$dx = \cos\theta\, d\theta.$

$\sqrt{1-x^2} = \cos\theta$

$$I = \int \sin^3\theta \cdot \cos\theta \cdot \cos\theta\, d\theta$$

$$= \int \sin^2\theta \cdot \cos^2\theta \cdot (\sin\theta\, d\theta)$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta.$

$$= -\int (1-u^2)\, u^2\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C$$

$$= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C$$

$$= \underline{\frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C}$$

(3) Determine whether the following integrals converge or diverge. DO NOT EVALUATE. Justify your answers by the Direct Comparison Test (DCT) or Limit Comparision Test (LCT).

(a)(8 pts) $I = \int_1^\infty \frac{dx}{x\sqrt{x+3}}$

Method 1 Since $x+3 > x$ we have $\frac{1}{x\sqrt{x+3}} < \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}}$. But $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges by the p-test, so I converges by the DCT.

Method 2 With $f(x) = \frac{1}{x\sqrt{x+3}}$ and $g(x) = \frac{1}{x\sqrt{x}}$ we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\cancel{x}\sqrt{x}}{\cancel{x}\sqrt{x+3}} = \lim_{x\to\infty}\sqrt{\frac{x}{x+3}} = \sqrt{\lim_{x\to\infty}\frac{1}{1+3/x}} = 1.$$

Also, $\int_1^\infty g(x) = \int_1^\infty \frac{1}{x^{3/2}}$ converges by the p-test, so I converges by the LCT.

(b)(6 pts) $I = \int_{10}^{\infty} \frac{\ln x}{x}\, dx.$ Because $\ln x \geq 1$ for all $x \geq 10$ we have $\frac{\ln x}{x} \geq \frac{1}{x}$ and $\int_{10}^{\infty} \frac{dx}{x}$ diverges by the p-test.

$\Rightarrow$ I diverges by the DCT.

(4) (14 points) Solve the differential equation $\frac{dy}{dx} = \sqrt{xe^y}$ for $x \geq 0$ with the initial condition $y(0) = 0$.

Separate variables: $dy = \sqrt{x}\, e^{y/2}\, dx \Rightarrow \int e^{-y/2}\, dy = \int x^{1/2}\, dx$

integrate: $-2e^{-y/2} = \frac{2}{3}x^{3/2} + C$

Plug in $x=0$, $y=0$: $-2e^0 = C \Rightarrow \boxed{C=-2}$

Solve for y: $e^{-y/2} = -\frac{1}{2}\left[\frac{2}{3}x^{3/2} - 2\right]$

$= -\frac{1}{3}x^{3/2} + 1$

take $\ln$: $\frac{-y}{2} = \ln\left(1 - \frac{1}{3}x^{3/2}\right)$

$$\boxed{y = -2\ln\left(1 - \frac{1}{3}x^{3/2}\right)}$$

(5) (9+9+6 points) Calculate the limit of the following sequences. *Show all steps clearly.*

(a) $\lim_{n\to\infty} \frac{\sqrt{3n^2-5}}{2n} = \lim_{n\to\infty} \frac{\sqrt{3n^2-5}}{\sqrt{4n^2}} = \sqrt{\lim_{n\to\infty} \frac{3n^2-5}{4n^2}}$

$= \sqrt{\lim_{n\to\infty} \frac{3 - 5/n^2}{4}}$

$= \sqrt{\frac{3}{4}} = \boxed{\frac{\sqrt{3}}{2}}$

(b) $\lim_{n\to\infty} \sqrt[n]{2n-1} = \lim_{n\to\infty} (2n-1)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln(2n-1)\cdot\frac{1}{n}}$

$$= e^{\lim_{x\to\infty} \frac{\ln(2x-1)}{x}} \quad \leftarrow \text{type } \frac{\infty}{\infty}$$

$$\overset{\text{L'Hop}}{=} e^{\lim_{x\to\infty} \frac{\frac{2}{2x-1}}{1}}$$

$$= e^0 = 1$$

(c) If $a_n = \dfrac{\sin n}{n^2}$ what is $\lim_{n\to\infty} a_n$? Justify your answer.

Because $-1 \le \sin n \le 1$ for all n, we have

$$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$

But $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} \frac{-1}{n} = 0$, so $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ by the Sandwich Theorem.

(6) (8 pts) For the series below, write down the first 3 or 4 terms and calculate the sum:

$$\sum_{n=2}^{\infty} \left(\frac{-2}{5}\right)^n = \left(\frac{-2}{5}\right)^2 + \left(\frac{-2}{5}\right)^3 + \left(\frac{-2}{5}\right)^4 + \left(\frac{-2}{5}\right)^5 + \cdots$$

Starts with $n=2$

geometric series with $A = \frac{4}{25}$, $r = \frac{-2}{5}$

$$= \frac{4}{25} \cdot \frac{1}{1-(-2/5)}$$

$$= \frac{4}{25} \quad \frac{1}{7/5}$$

$$= \frac{4}{5\cdot\cancel{5}} \cdot \frac{\cancel{5}}{7}$$

$$= \frac{4}{35}$$